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Inegalităţi geometrice de tipul Erdös- Mordell într-un poligon convex

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Abstract. Some Erdös -Mordel type inequalities for general convex poligins are presented. The main tool in the proofs is the Cauchy-Buniakowski-Schwarz inequality.

Keywords: Erdös-Mordell type inequality, convex polygon

MSC : 51Mxx, 26D15.

Noţiuni introductive

Fie $A_1, A_2, ..., A_n, n \geq 3$, vârfurile unui poligon convex şi M un punct în interiorul său. Notăm cu R_k distanţa de la punctul M la vârful A_k şi cu r_k distanţa de la punctul M la latura $[A_k A_{k+1}]$ de lungime $A_k A_{k+1} = a_k$, unde $k = \overline{1, n}$ şi $A_{n+1} \equiv A_1$. De asemenea, vom nota prin w_k lungimea bisectoarei duse din M în triunghiul $A_k M A_{k+1}, (\forall)\, k = \overline{1, n}$ şi $A_{n+1} \equiv A_1$.

Plecând de la inegalitatea lui *Erdös-Mordell*,

$$R_1 + R_2 + R_3 \geq 2\,(r_1 + r_2 + r_3)\,,$$

pentru triunghi, *L. Fejes Tóth* conjecturează o inegalitate asemănătoare, referitoare la poligonul convex, aceasta fiind amintită în [1] şi [3], astfel

$$\sum_{k=1}^{n} r_k \leq \cos\left(\frac{\pi}{n}\right) \sum_{k=1}^{n} R_k. \tag{1}$$

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În anul 1961 *H.-C. Lenhard* demonstrează inegalitatea (1), întrebuinţând inegalitatea

$$\sum_{k=1}^{n} w_k \leq \cos\left(\frac{\pi}{n}\right) \sum_{k=1}^{n} R_k, \tag{2}$$

pe care o stabileşte în lucrarea [5], unele dintre rezultate fiind menţionate şi în [7].

De asemenea, o altă soluţie pentru inegalitatea (1) a fost dată de *M. Dincă* în revista *Gazeta Matematică – Seria* B în anul 1998, vezi [4].

Altă inegalitate de tipul *Erdös-Mordell* pentru poligoane este dată de *N. Ozeki* în [8] în anul 1957 şi anume,

$$\prod_{k=1}^{n} R_k \geq \left(\sec\frac{\pi}{n}\right)^n \prod_{k=1}^{n} w_k, \tag{3}$$

inegalitate ce demonstrează inegalitatea 16.8 din [3], adică:

$$\prod_{k=1}^{n} R_k \geq \left(\sec\frac{\pi}{n}\right)^n \prod_{k=1}^{n} r_k. \tag{4}$$

D. Buşneag propune în G.M.-B nr. 1/1971 problema 10876, care este o inegalitate de tipul *Erdös-Mordell* pentru poligoane, astfel:

$$\sum_{k=1}^{n} \frac{a_k}{r_k} \geq \frac{2p^2}{S}, \tag{5}$$

unde p este semiperimetrul poligonului $A_1A_2...A_n$, iar S este aria poligonului.

În legătură cu inegalitatea (5), *D. M. Bătineţu* stabileşte în [2] inegalitatea

$$\sum_{k=1}^{n} \frac{a_k}{r_k} \geq \frac{2p}{r}, \tag{6}$$

dacă poligonul $A_1A_2...A_n$ este circumscris unui cerc de rază r.

Printre relaţiile care se stabilesc între elementele poligonului $A_1A_2...A_n$ remarcăm relaţia următoare:

$$2S = a_1r_1 + a_2r_2 + ... + a_nr_n. \tag{7}$$

Procedee de obţinere a unor inegalităţi geometrice de tipul *Erdös- Mordell* într-un poligon convex

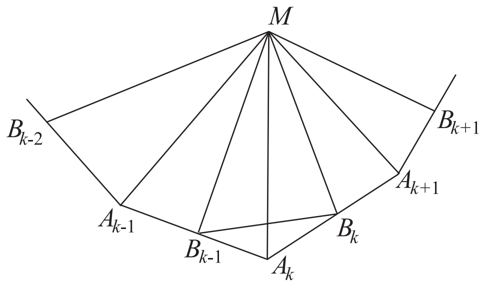
În continuare vom urmări câteva procedee utilizate în lucrarea [6], prin care se vor obţine unele inegalităţi geometrice de tipul *Erdös-Mordell* pentru poligoane convexe. Vom pune accentul pe utilizarea inegalităţii *Cauchy-Buniakowski-Schwarz.*

Lema 1.

$$R_k^2 = \frac{r_{k-1}^2 + r_k^2 + 2r_{k-1}r_k \cdot \cos A_k}{\sin^2 A_k}, (\forall)\, k = \overline{1,n},\ \textit{cu}\ r_0 = r_n. \tag{8}$$

Demonstraţie. Fie B_k proiecţia punctului M pe latura $[A_kA_{k+1}]$, unde $k=\overline{1,n}$ şi $A_{n+1}\equiv A_1$, deci $MB_k=r_k$, $(\forall)\,k=\overline{1,n}$. Aplicăm teorema cosinusului în triunghiul $B_{k-1}MB_k$ (vezi figura 1) şi avem:

$$B_{k-1}B_k^2=MB_{k-1}^2+MB_k^2-2MB_{k-1}\cdot MB_k\cdot\cos B_{k-1}MB_k,$$



dar $MB_{k-1}=r_{k-1}$, $MB_k=r_k$ $B_{k-1}B_k=MA_k\sin A_k$, iar $\mathrm{m}\,(\sphericalangle B_{k-1}MB_k)=$ $=180^\circ-\mathrm{m}\,(\sphericalangle A_k)$, în consecinţă

$$MA_k^2\sin^2 A_k=r_{k-1}^2+r_k^2+2r_{k-1}r_k\cdot\cos A_k,$$

ceea ce implică

$$R_k^2=MA_k^2=\frac{r_{k-1}^2+r_k^2+2r_{k-1}r_k\cdot\cos A_k}{\sin^2 A_k}.$$

Lema 2. *Să se arate că în poligonul convex $A_1A_2...A_n$ au loc inegalităţile*

$$R_k\geq\frac{r_{k-1}+r_k}{2\sin\dfrac{A_k}{2}},\,(\forall)\,k=\overline{1,n},\ \text{cu } r_0=r_n. \tag{9}$$

Demonstraţie. Plecând de la lema 1, găsim o altă formulă de exprimare a lui MA_k^2, astfel,

$$MA_k^2-\left(\frac{r_{k-1}+r_k}{2\sin\dfrac{A_k}{2}}\right)^2=\frac{r_{k-1}^2+r_k^2+2r_{k-1}r_k\cdot\cos A_k}{\sin^2 A_k}-\left(\frac{r_{k-1}+r_k}{2\sin\dfrac{A_k}{2}}\right)^2=$$

$$=\frac{r_{k-1}^2+r_k^2+2r_{k-1}r_k\cdot\cos A_k}{4\sin^2\dfrac{A_k}{2}\cos^2\dfrac{A_k}{2}}-\frac{r_{k-1}^2+r_k^2+2r_{k-1}r_k}{4\sin^2\dfrac{A_k}{2}}=$$

$$=\frac{r_{k-1}^2\left(1-\cos^2\dfrac{A_k}{2}\right)+r_k^2\left(1-\cos^2\dfrac{A_k}{2}\right)+2r_{k-1}r_k\cdot\left(\cos A_k-\cos^2\dfrac{A_k}{2}\right)}{4\sin^2\dfrac{A_k}{2}\cos^2\dfrac{A_k}{2}}=$$

$$= \frac{r_{k-1}^2\left(\sin^2\dfrac{A_k}{2}\right)+r_k\left(\sin^2\dfrac{A_k}{2}\right)-2r_{k-1}r_k\cdot\left(\sin^2\dfrac{A_k}{2}\right)}{4\sin^2\dfrac{A_k}{2}\cos^2\dfrac{A_k}{2}} =$$

$$= \frac{(r_{k-1}-r_k)^2}{4\cos^2\dfrac{A_k}{2}} \geq 0, \quad \forall\, k=\overline{1,n},\ \text{cu } r_0=r_n,$$

deci:

$$R_k = MA_k \geq \frac{r_{k-1}+r_k}{2\sin\dfrac{A_k}{2}}, \quad \forall\, k=\overline{1,n},\ \text{cu } r_0=r_n.$$

Teorema 1.

$$\left(2\cos\frac{\pi}{n}\right)^n \prod_{k=1}^{n} R_k \geq \prod_{k=1}^{n}\left(r_{k-1}+r_k\right), (r_0=r_n). \tag{10}$$

Demonstraţie. Din lema 2 avem relaţiile, $R_k \geq \dfrac{r_{k-1}+r_k}{2\sin\dfrac{A_k}{2}}$, $\forall\, k=\overline{1,n}$, cu $r_0=r_n$, ceea ce ne arată că, prin trecere la produs, vom obţine relaţia:

$$2^n\prod_{k=1}^{n}\sin\frac{A_k}{2}\cdot\prod_{k=1}^{n}R_k \geq \prod_{k=1}^{n}\left(r_{k-1}+r_k\right), \tag{11}$$

dar, cum funcţia $f:\left(0,\dfrac{\pi}{2}\right)\to\mathbb{R}$, definită de $f(x)=\ln\sin x$, este concavă, vom aplica inegalitatea lui *Jensen*, astfel:

$$\frac{\ln\sin\dfrac{A_1}{2}+\ln\sin\dfrac{A_2}{2}+...+\ln\sin\dfrac{A_n}{2}}{n} \leq \ln\sin\frac{\dfrac{A_1}{2}+\dfrac{A_2}{2}+...+\dfrac{A_n}{2}}{n} =$$

$$= \ln\sin\frac{(n-2)\pi}{2n} = \ln\left(\sin\frac{(n-2)\pi}{2n}\right) = \ln\left(\sin\left(\frac{\pi}{2}-\frac{\pi}{n}\right)\right) = \ln\left(\cos\frac{\pi}{n}\right),$$

deoarece $A_1+A_2+...+A_n=(n-2)\pi$, prin urmare:

$$\prod_{k=1}^{n}\sin\frac{A_k}{2} \leq \left(\cos\frac{\pi}{n}\right)^n. \tag{12}$$

Din inegalităţile (11) şi (12), rezultă că:

$$\left(2\cos\frac{\pi}{n}\right)^n \prod_{k=1}^{n} R_k \geq \prod_{k=1}^{n}\left(r_{k-1}+r_k\right), (r_0=r_n).$$

Observaţie. Datorită faptului că media aritmetică este mai mare decât media geometrică, rezultă că

$$r_{k-1}+r_k \geq 2\sqrt{r_{k-1}r_k}, \quad \forall\, k=\overline{1,n},\ \text{cu } r_0=r_n,$$

ceea ce înseamnă că

$$\prod_{k=1}^{n}\left(r_{k-1}+r_k\right)\geq 2^n\prod_{k=1}^{n}r_k, \tag{13}$$

iar prin utilizarea inegalităţilor (10) şi (13) se deduce uşor inegalitatea (4).

Teorema 2.

$$\sum_{k=1}^{n}R_k\sin\frac{A_k}{2}\geq\sum_{k=1}^{n}r_k. \tag{14}$$

Demonstraţie. De la lema 2 avem relaţiile, $2R_k\sin\dfrac{A_k}{2}\geq r_{k-1}+r_k$, $\forall\, k=\overline{1,n}$, cu $r_0=r_n$, deci

$$\sum_{k=1}^{n}2R_k\sin\frac{A_k}{2}\geq\sum_{k=1}^{n}\left(r_{k-1}+r_k\right),$$

adică

$$2\sum_{k=1}^{n}R_k\sin\frac{A_k}{2}\geq 2\sum_{k=1}^{n}r_k,$$

ceea ce implică

$$\sum_{k=1}^{n}R_k\sin\frac{A_k}{2}\geq\sum_{k=1}^{n}r_k.$$

Teorema 3 (*R. R. Janić* [3]).

$$\sum_{k=1}^{n}R_kr_k\sin\frac{A_k}{2}\geq\sum_{k=1}^{n}r_{k-1}r_k. \tag{15}$$

Demonstraţie. Cum $R_k=MA_k\geq\dfrac{r_{k-1}+r_k}{2\sin\dfrac{A_k}{2}}$, $\forall\, k=\overline{1,n}$, cu $r_0=r_n$, prin înmulţire cu $2r_k\sin\dfrac{A_k}{2}$, rezultă

$$2R_kr_k\sin\frac{A_k}{2}\geq r_{k-1}r_k+r_k^2,$$

de unde, prin trecere la sumă, obţinem

$$\sum_{k=1}^{n}2R_kr_k\sin\frac{A_k}{2}\geq\sum_{k=1}^{n}\left(r_{k-1}r_k+r_k^2\right),$$

ceea ce implică

$$2\sum_{k=1}^{n}R_kr_k\sin\frac{A_k}{2}\geq\sum_{k=1}^{n}r_{k-1}r_k+\sum_{k=1}^{n}r_k^2.$$

Aplicând inegalitatea lui *Cauchy-Buniakowski-Schwarz* în modul următor:

$$\left(r_1^2+r_2^2+...+r_{n-1}^2+r_n^2\right)\left(r_2^2+r_3^2+...+r_n^2+r_1^2\right)\geq$$
$$\geq\left(r_1r_2+r_2r_3+...+r_nr_1\right)^2,$$

deducem inegalitatea

$$r_1^2+r_2^2+...+r_n^2\geq r_1r_2+r_2r_3+...+r_nr_1.$$

Prin urmare,

$$2\sum_{k=1}^{n}R_kr_k\sin\frac{A_k}{2}\geq\sum_{k=1}^{n}r_{k-1}r_k+\sum_{k=1}^{n}r_k^2\geq2\sum_{k=1}^{n}r_{k-1}r_k,$$

adică

$$\sum_{k=1}^{n}R_kr_k\sin\frac{A_k}{2}\geq\sum_{k=1}^{n}r_{k-1}r_k.$$

Teorema 4.

$$\sum_{k=1}^{n}\frac{r_{k-1}+r_k}{R_k}\leq2n\cos\frac{\pi}{n}.\tag{16}$$

Demonstraţie. Datorită lemei 2 avem inegalitatea

$$R_k=MA_k\geq\frac{r_{k-1}+r_k}{2\sin\dfrac{A_k}{2}},\qquad\forall\,k=\overline{1,n},$$

cu $r_0=r_n$, iar aceasta se poate rescrie astfel

$$\frac{r_{k-1}+r_k}{R_k}\leq2\sin\frac{A_k}{2},$$

iar prin trecere la sumă, obţinem

$$\sum_{k=1}^{n}\frac{r_{k-1}+r_k}{R_k}\leq2\sum_{k=1}^{n}\sin\frac{A_k}{2}.$$

Întrucât funcţia $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}$, definită de $f(x)=\sin x$, este concavă, vom aplica inegalitatea lui *Jensen*, astfel

$$\frac{\sin\dfrac{A_1}{2}+\sin\dfrac{A_2}{2}+...+\sin\dfrac{A_n}{2}}{n}\leq\sin\frac{\dfrac{A_1}{2}+\dfrac{A_2}{2}+...+\dfrac{A_n}{2}}{n}=$$
$$=\sin\frac{(n-2)\pi}{2n}=\sin\left(\frac{\pi}{2}-\frac{\pi}{n}\right)=\cos\frac{\pi}{n},$$

deoarece $A_1,A_2,...,A_n=(n-2)\,\pi$, prin urmare

$$\sin\frac{A_1}{2}+\sin\frac{A_2}{2}+...+\sin\frac{A_n}{2}\leq n\cos\frac{\pi}{n},$$

ceea ce implică inegalitatea

$$\sum_{k=1}^{n}\frac{r_{k-1}+r_k}{R_k}\leq 2n\cos\frac{\pi}{n}.$$

Consecinţă.

$$\sum_{k=1}^{n}\frac{\sqrt{r_{k-1}r_k}}{R_k}\leq 2n\cos\frac{\pi}{n}. \tag{17}$$

Datorită faptului că media aritmetică este mai mare decât media geometrică, rezultă simplu din inegalitatea (16).

Teorema 5.

$$\sum_{k=1}^{n}\frac{R_k^2}{r_k}\geq \sec\frac{\pi}{n}\sum_{k=1}^{n}R_k. \tag{18}$$

Demonstraţie. În inegalitatea *Cauchy-Buniakowski-Schwarz*

$$\sum_{k=1}^{n}x_k^2\sum_{k=1}^{n}y_k^2\geq\left(\sum_{k=1}^{n}x_ky_k\right)^2$$

vom lua $x_k=\dfrac{R_k}{\sqrt{r_k}}$ şi $y_k=\sqrt{r_k}$, iar inegalitatea devine

$$\sum_{k=1}^{n}\frac{R_k^2}{r_k}\sum_{k=1}^{n}r_k\geq\left(\sum_{k=1}^{n}R_k\right)^2,$$

deci

$$\sum_{k=1}^{n}\frac{R_k^2}{r_k}\geq$$

$$\geq\frac{\left(\sum_{k=1}^{n}R_k\right)^2}{\sum_{k=1}^{n}r_k}=\frac{\left(\sum_{k=1}^{n}R_k\right)\left(\sum_{k=1}^{n}R_k\right)}{\sum_{k=1}^{n}r_k}\geq\frac{\left(\sum_{k=1}^{n}R_k\right)\sec\frac{\pi}{n}\left(\sum_{k=1}^{n}r_k\right)}{\sum_{k=1}^{n}r_k}=\sec\frac{\pi}{n}\sum_{k=1}^{n}R_k,$$

de unde, deducem inegalitatea

$$\sum_{k=1}^{n}\frac{R_k^2}{r_k}\geq \sec\frac{\pi}{n}\sum_{k=1}^{n}R_k.$$

Observaţii.

a) Egalităţile au loc în teoremele de mai sus când poligonul este regulat.

b) Ţinând cont de egalitatea (7),

$$2S=a_1r_1+a_2r_2+...+a_nr_n=\sum_{k=1}^{n}a_kr_k,$$

şi de inegalitatea lui *Cauchy-Buniakowski-Schwarz*, unde vom lua $x_k = \sqrt{a_k r_k}$ şi $y_k = \sqrt{\frac{a_k}{r_k}}$, atunci

$$2S\sum_{k=1}^{n}\frac{a_k}{r_k} = \sum_{k=1}^{n}a_k r_k \sum_{k=1}^{n}\frac{a_k}{r_k} \geq \left(\sum_{k=1}^{n}a_k\right)^2 = 4p^2,$$

ceea ce demonstrează inegalitatea (5).

BIBLIOGRAFIE

[1] F. Abi-Khuzam, *A Trigonometric Inequality and Its Geometric Applications*, Mathematical Inequalities & Applications, Vol. 3, no.3(2000), 437-442.

[2] D. M. Bătineţu, *O inegalitate între medii ponderate şi aplicaţii*, G.M.-B nr. 7/1982.

[3] O. Botema, R. Z. Djordjević, R. R. Janić, D. S. Mitrinović, P. M. Vasić, *Geometric Inequalities*, Groningen,1969.

[4] M. Dincă, *Generalizarea inegalităţii lui Erdös-Mordell*, G.M.-B nr. 7-8/1998.

[5] H.-C. Lenhard, *Verallgemeinerung und Verscharfung der Erdös-Mordellschen Ungleichung für Polygone*, Arch. Math. Vol. XII (1961), 311-314.

[6] N. Minculete, *Teoreme şi probleme specifice de geometrie*, Editura Eurocarpatica, Sfântu Gheorghe, 2007.

[7] D. S. Mitrinović, J. Pečarić, V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989.

[8] N. Ozeki, *On P. Erdös Inequality for the Triangle*, J. College Arts Sci. Chiba Univ., 2(1957), 247-250.

[9] V. Gh. Vodă, *Vraja geometriei demodate*, Editura Albatros,1983.

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Hadamard Type Inequalities For Near Convex Functions

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Abstract. In this paper we give an estimate, from below and from above, of the mean value of the function $f : [a, b] \to \mathbb{R}$ such that f is continuous on $[a, b]$ and twice differentiable on (a, b).

Keywords: Convex functions, Hermite-Hadamard integral inequality, Twice differentiable functions, Open question.

MSC : 52A40, 52A41.

1. Introduction and main results

Throughout this note, we write I and $\mathring{I}$ for the intervals $[a, b]$ and (a, b) respectively. A real-valued function f is said to be convex on I if

$$\lambda f(x) + (1 - \lambda)f(y) \geq f(\lambda x + (1 - \lambda)y)$$

for all $x, y \in I$ and $0 \leq \lambda \leq 1$. Conversely, if the opposite inequality holds, the function is said to be concave on I. A function f that is continuous on I and twice differentiable on $\mathring{I}$ is convex on I if and only if $f''(x) \geq 0$ for all $x \in \mathring{I}$ (f is concave if and only if $f''(x) \leq 0$ for all $x \in \mathring{I}$).

The classical *Hermite-Hadamard* inequality which was first published in [5] gives us an estimate, from below and from above, of the mean value of a convex function $f : I \to \mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

An account on the history of this inequality can be found in [6]. Surveys on various generalizations and developments can be found in [7] and [3]. The description of best possible inequalities of *Hadamard-Hermite* type are due to *Fink* [4]. A generalization to higher-order convex functions can be found in [1], while [2] offers a generalization for functions that are *Beckenbach*-convex with respect to a two dimensional linear space of continuous functions. For some related results on convex functions and their applications, we refer the reader to [8], [9].

In this paper, we give an estimate, from below and from above, of the mean value of $f : I \to \mathbb{R}$ such that f is continuous on I, twice differentiable

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on $\mathring{I}$ and there exists $m = \inf\limits_{x\in\mathring{I}} f''(x)$ or $M = \sup\limits_{x\in\mathring{I}} f''(x)$. We obtain the following results:

Theorem 1.1. *Assume that $f : I \to \mathbb{R}$ is continuous on I, twice differentiable on $\mathring{I}$.*

(i) *If there exists $m = \inf\limits_{x\in\mathring{I}} f''(x)$, then we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) + \frac{m}{6}\left(\frac{b-a}{2}\right)^2 &\le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \\ &\le \frac{f(a)+f(b)}{2} - \frac{m}{3}\left(\frac{b-a}{2}\right)^2. \end{aligned} \tag{1.2}$$

Equality in (1.2) *holds if $f(x) = \alpha x^2 + \beta x + \gamma$, $\alpha, \beta, \gamma \in \mathbb{R}$.*

(ii) *If there exists $M = \sup\limits_{x\in\mathring{I}} f''(x)$, then we have*

$$\begin{aligned} \frac{f(a)+f(b)}{2} - \frac{M}{3}\left(\frac{b-a}{2}\right)^2 &\le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \\ &\le f\left(\frac{a+b}{2}\right) + \frac{M}{6}\left(\frac{b-a}{2}\right)^2. \end{aligned} \tag{1.3}$$

Equality in (1.3) *holds if $f(x) = \alpha x^2 + \beta x + \gamma$, $\alpha, \beta, \gamma \in \mathbb{R}$.*

From Theorem 1.1, we obtain the following corollaries.

Corollary 1.1. *Assume that $f : I \to \mathbb{R}$ is continuous on I, twice differentiable on $\mathring{I}$ and there exist $m = \inf\limits_{x\in\mathring{I}} f''(x)$ and $M = \sup\limits_{x\in\mathring{I}} f''(x)$. Then we have*

$$\frac{m}{3}\left(\frac{b-a}{2}\right)^2 \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{M}{3}\left(\frac{b-a}{2}\right)^2 \tag{1.4}$$

and

$$\frac{m}{6}\left(\frac{b-a}{2}\right)^2 \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right) \le \frac{M}{6}\left(\frac{b-a}{2}\right)^2. \tag{1.5}$$

Equality in (1.4) *and* (1.5) *hold if $f(x) = \alpha x^2 + \beta x + \gamma$, $\alpha, \beta, \gamma \in \mathbb{R}$.*

Corollary 1.2 *Assume that $f : I \to \mathbb{R}$ is continuous on I, twice differentiable on $\mathring{I}$ and there exist $m = \inf\limits_{x\in\mathring{I}} f''(x)$ and $M = \sup\limits_{x\in\mathring{I}} f''(x)$. Then we* *have*

$$\frac{1}{2}\left(\frac{m}{6}-\frac{M}{3}\right)\left(\frac{b-a}{2}\right)^{2}\leq\frac{1}{b-a}\int_{a}^{b}f(x)\mathrm{d}x-$$

$$\frac{1}{2}\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\leq\frac{1}{2}\left(\frac{M}{6}-\frac{m}{3}\right)\left(\frac{b-a}{2}\right)^{2} \tag{1.6}$$

and

$$\frac{m}{2}\left(\frac{b-a}{2}\right)^{2}\leq\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\leq\frac{M}{2}\left(\frac{b-a}{2}\right)^{2}. \tag{1.7}$$

Equality in (1.6) *holds if* $f(x)=\alpha x^{2}+\beta x+\gamma \quad \alpha,\beta,\gamma\in\mathbb{R}$.

In the following corollary, if $f : I \to \mathbb{R}$ is continuous, convex or concave on I and twice differentiable on $\mathring{I}$, then we obtain an estimation better than (1.1) in [5].

Corollary 1.3 *Assume that* $f : I \to \mathbb{R}$ *is continuous on* I, *twice differentiable on* $\mathring{I}$.

(i) *If there exists* $m=\inf\limits_{x\in\mathring{I}} f''(x)$ *and* f *is convex on* I, *then we have*

$$f\left(\frac{a+b}{2}\right)\leq l\leq\frac{1}{b-a}\int_{a}^{b}f(x)\,\mathrm{d}x\leq L\leq\frac{f(a)+f(b)}{2}, \tag{1.8}$$

where $l=f\left(\frac{a+b}{2}\right)+\frac{m}{6}\left(\frac{b-a}{2}\right)^{2}$, $L=\frac{f(a)+frac(b)}{2}-\frac{m}{3}\left(\frac{b-a}{2}\right)^{2}$.

(ii) *If there exists* $M=\inf\limits_{x\in\mathring{I}} f''(x)$ *and* f *is concave on* I, *then we have*

$$\frac{f(a)+f(b)}{2}\leq\lambda\leq\frac{1}{b-a}\int_{a}^{b}f(x)\,\mathrm{d}x\leq\mu\leq f\left(\frac{a+b}{2}\right), \tag{1.9}$$

where $\lambda=\frac{f(a)+frac(b)}{2}-\frac{M}{3}\left(\frac{b-a}{2}\right)^{2}$, $\mu=f\left(\frac{a+b}{2}\right)+\frac{M}{6}\left(\frac{b-a}{2}\right)^{2}$.

Corollary 1.4 *Assume that* $f : I \to \mathbb{R}$ *is continuous on* I, *twice differentiable on* $\mathring{I}$ *and there exist* $m=\inf\limits_{x\in\mathring{I}} f''(x)$ *and* $M=\sup\limits_{x\in\mathring{I}} f''(x)$. *Then we have*

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)\mathrm{d}x-\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{3}\left(\frac{b-a}{2}\right)^{2}\max\{|m|,|M|\} \tag{1.10}$$

and

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right) \right| \le \frac{1}{6}\left(\frac{b-a}{2}\right)^2 \max\{|m|,|M|\}. \qquad (1.11)$$

Remark 1.1 In the above if $f \in C^2([a,b])$, then we can replace inf and sup by min and max respectively.

2. Proof of Theorem and Corollaries

Proof of Theorem 1.1. (i) Suppose that $f : I \to \mathbb{R}$ is continuous on I, twice differentiable on $\mathring{I}$. Set $g(x) = f(x) - \frac{m}{2}x^2$. Differentiating twice times both sides of g we get $g''(x) = f''(x) - m \ge 0$ for all $x \in \mathring{I}$, then g is a convex function on I. By formula (1.1), we have

$$g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x \le \frac{g(a)+g(b)}{2}. \qquad (2.1)$$

Substituting $g(x) = f(x) - \frac{m}{2}x^2$ into (2.1), we get

$$f\left(\frac{a+b}{2}\right) - \frac{m}{2}\left(\frac{b+a}{2}\right)^2 + \frac{1}{b-a}\int_a^b \frac{m}{2}x^2\mathrm{d}x \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le$$

$$\le \frac{f(a)+f(b)}{2} - m\frac{a^2+b^2}{4} + \frac{1}{b-a}\int_a^b \frac{m}{2}x^2\mathrm{d}x. \qquad (2.2)$$

By simple calculus from (2.2), we get (1.2).

(ii) Suppose that $f : I \to \mathbb{R}$ is continuous on I, twice differentiable on $\mathring{I}$. Set $h(x) = -f(x) + \frac{M}{2}x^2$. Differentiating twice times both sides of h we get $h''(x) = -f''(x) + M \ge 0$ for all $x \in \mathring{I}$, then h is a convex function on I. By formula (1.1), we have

$$h\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b h(x)\,\mathrm{d}x \le \frac{h(a)+h(b)}{2}. \qquad (2.3)$$

Substituting $h(x) = -f(x) + \frac{M}{2}x^2$ into (2.3) we get

$$-f\left(\frac{a+b}{2}\right) + \frac{M}{2}\left(\frac{a+b}{2}\right)^2 - \frac{1}{b-a}\int_a^b \frac{M}{2}x^2\mathrm{d}x \le -\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$

$$\leq -\frac{f(a)+f(b)}{2} + M\frac{a^2+b^2}{4} - \frac{1}{b-a}\int_a^b \frac{M}{2}x^2\mathrm{d}x. \tag{2.4}$$

By simple calculus from (2.4), we get (1.3).

Proof of Corollary 1.1. This can be concluded by using Theorem 1.1.

Proof of Corollary 1.2. By Corollary 1.1, we have

$$-\frac{M}{3}\left(\frac{b-a}{2}\right)^2 \leq \frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2} \leq -\frac{m}{3}\left(\frac{b-a}{2}\right)^2 \tag{2.5}$$

and

$$\frac{m}{6}\left(\frac{b-a}{2}\right)^2 \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right) \leq \frac{M}{6}\left(\frac{b-a}{2}\right)^2. \tag{2.6}$$

By addition from (2.5) and (2.6), we get (1.6). Now we prove (1.7).
By using Corollary 1.1, we have

$$\frac{m}{3}\left(\frac{b-a}{2}\right)^2 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{M}{3}\left(\frac{b-a}{2}\right)^2 \tag{2.7}$$

and

$$\frac{m}{6}\left(\frac{b-a}{2}\right)^2 \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right) \leq \frac{M}{6}\left(\frac{b-a}{2}\right)^2. \tag{2.8}$$

By addition from (2.7) and (2.8), we get (1.7).

Proof of Corollary 1.3. (i) By f is convex function, we have $m \geq 0$. Then by (1.2), we get (1.8).

(ii) By f is concave function we obtain $M \leq 0$. Then by (1.3, we get (1.9).

Proof of Corollary 1.4. This can be concluded by using Corollary 1.1.

Open question. If f is only convex function on I, does there exist a real numbers l, L such that

$$f\left(\frac{a+b}{2}\right) \leq l \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq L \leq \frac{f(a)+f(b)}{2}?$$

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REFERENCES

[1] M. Bessenyei, Z. Páles, *Higher-order generalizations of Hadamard's inequality*, Publ. Math. Debrecen 61 (2002), no. 3-4, 623–643.

[2] M. Bessenyei, Z. Páles, *Hadamard-type inequalities for generalized convex functions.* Math. Inequal. Appl. 6 (2003), no. 3, 379–392.

[3] S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities*, (RGMIA Monographs http://rgmia.vu.edu.au/monographs/hermite_hadamard.html), Victoria University, 2000.

[4] A. M. Fink, *A best possible Hadamard inequality*, Math. Inequal. Appl. 1 (1998), no. 2, 223–230.

[5] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., 58 (1893), 171–215.

[6] D. S. Mitrinović, I. B. Lacković, *Hermite and convexity*, Aequationes Math. 28 (1985), no. 3, 229–232.

[7] C. P. Niculescu, L.-E. Persson, *Old and new on the Hermite–Hadamard inequality*, Real Analysis Exchange, (2004).

[8] C. P. Niculescu, L.-E. Persson, *Convex Functions and Their Applications*, A Contemporary Approach, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23. Springer, New York, 2006.

[9] J. E. Pečarić, F. Proschan,Y. C. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.

Asupra unor inegalităţi geometrice în tetraedre ortocentrice şi oarecare

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Abstract. Some inequalities connected with quantities in ortogonal tetrahedron are discussed.

Keywords: tetrahedron ortogonal, Euler's sphere, ortocenter, center of gravity.

MSC : 51xx, 51F20, 20H05.

Scopul acestui articol matematic este acela de a prezenta o serie de inegalităţi geometrice în tetraedre ortocentrice în care intervin distanţele de la anumite puncte remarcabile la vârfurile tetraedrului, precum şi unele extinderi ale acestor inegalităţi la tetraedre oarecare.

Vom folosi următoarele notaţii pentru elementele unui tetraedru $[ABCD]$:

• $a = BC$, $b = AC$, $c = AB$, $l = AD$, $m = BD$, $n = CD$;

• h_x– lungimea înălţimii tetraedrului ce conţine vârful $X \in \{A, B, C, D\}$, $x \in \{a, b, c, d\}$;

• m_x– lungimea medianei tetraedrului ce conţine vârful $X \in \{A, B, C, D\}$, $x \in \{a, b, c, d\}$;

• r_x- raza sferei exînscrise tetraedrului, de speţa întâi, asociată vârfului $X \in \{A, B, C, D\}$, $x \in \{a, b, c, d\}$;

• H_X, G_X- ortocentrul, respectiv centrul de greutate al feţei opuse vârfului $X \in \{A, B, C, D\}$;

• S_X- aria feţei opuse vârfului $X \in \{A, B, C, D\}$;

• $S-$ aria totală;

• $V-$ volumul tetraedrului;

• $s(I; r)$, $\mathcal{S}(O; R)$– sfera înscrisă, respectiv circumscrisă tetraedrului;

• $G-$ centrul de greutate al tetraedrului;

• $H-$ ortocentrul tetraedrului ortocentric $[ABCD]$;

• $\Omega-$ centrul sferei *Euler* asociată tetraedrului;

• $\Gamma-$ simetricul punctului Ω faţă de G;

• $K-$ anticentrul sau punctul lui *Monge* al tetraedrului.

Vom reaminti câteva dintre proprietăţile tetraedrului ortocentric $[ABCD]$, proprietăţi demonstrate în [3], la pag. 225-227:

(1) un tetraedru $[ABCD]$ este ortocentric dacă şi numai dacă

$$a^2 + l^2 = b^2 + m^2 = c^2 + n^2;$$

(2) într-un tetraedru ortocentric cele patru picioare ale înălţimilor acestuia coincid cu ortocentrele feţelor corespunzătoare lor;

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(3) tetraedrul ortocentric având ortocentrul situat în interiorul acestuia are toate feţele triunghiuri ascuţitunghice (sau dreptunghice, cel mult trei feţe).

Propoziţia 1. *Fie* $[ABCD]$ *un tetraedru ortocentric având ortocentrul* $H \in \operatorname{int}(ABCD)$. *Atunci avem*:

$$m_a \le m_b \le m_c \le m_d \Leftrightarrow S_A \ge S_B \ge S_C \ge S_D. \tag{4}$$

Demonstraţie. Pentru implicaţia directă se poate consulta [9], pag. 236 - 237, iar pentru reciprocă se poate consulta [9], pag. 236 - 237 sau [7], pag. 32, problema 128.

Comentariu. Implicaţia reciprocă a fost stabilită de autorul prezentului articol mai întâi în soluţionarea problemelor C:1243 din G.M.-B nr. 2-3/1992 şi O:685 din G.M.-B nr. 5/1992 (autor *Marius Olteanu*), ulterior această implicaţie apărând ca problemă de sine stătătoare sub nr. 128, pag. 32 din [7].

Prin problemele deschise, „open questions" (O.Q) nr. 1324, pag. 859 din [5] şi nr. 1379, pag. 454 din [6] (autor *Marius Olteanu*) se pune în discuţie existenţa unor asemenea relaţii şi în cazul tetraedrelor oarecare (prin analogie cu triunghiul). Răspunsul, care este negativ, a fost dat în mod independent de către domnii profesori *Jósef Sándor* în [9], pag. 236 - 239 şi *Ovidiu Bagdasar* în [1], pag. 302 - 304 şi în [2], pag. 1060.

Consecinţa 1. Deoarece $AG = \frac{3}{4}m_a$ (şi analoagele), atunci într-un tetraedru ortocentric $[ABCD]$, având ortocentrul $H = \operatorname{int}(ABCD)$ avem

$$AG \le BG \le GC \le DG \Leftrightarrow S_A \ge S_B \ge S_C \ge S_D. \tag{5}$$

Totodată, pentru un tetraedru oarecare $[ABCD]$ se cunosc, conform celor stabilite în [8], pag. 203, inegalităţile:

$$16r \le h_a + h_b + h_c + h_d \le m_a + m_b + m_c + m_d \le 16\frac{R}{3}, \tag{6}$$

$$\frac{1}{r} \ge \frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} + \frac{1}{m_d} \ge \frac{3}{R}, \tag{7}$$

$$64r^2 \le h_a^2 + h_b^2 + h_c^2 + h_d^2 \le m_a^2 + m_b^2 + m_c^2 + m_d^2 \le \frac{64}{9}R^2. \tag{8}$$

Este evident că avem şi inegalităţile:

$$12r \le AG + BG + CG + DG + \le 4R, \tag{6'}$$

$$\frac{4}{3r} \ge \frac{1}{AG} + \frac{1}{BG} + \frac{1}{CG} + \frac{1}{DG} \ge \frac{4}{R}, \tag{7'}$$

$$36r^2 \le AG^2 + BG^2 + CG^2 + DG^2 \le 4R^2. \tag{8'}$$

Propoziţia 2. *Fie* $[ABCD]$ *un tetraedru oarecare. Atunci avem*

$$h_a \le h_b \le h_c \le h_d \Leftrightarrow S_A \ge S_B \ge S_C \ge S_D. \tag{9}$$

Demonstraţia este imediată folosind relaţiile $h_x = \frac{3V}{S_X}$, unde $x \in \in \{a, b, c, d\}$, iar $X \in \{A, B, C, D\}$.

Propoziţia 3. *Fie* $[ABCD]$ *un tetraedru ortocentric având ortocentrul* $H \in \mathrm{int}(ABCD)$. *Atunci avem*

$$AH \leq BH \leq CH \leq DH \Leftrightarrow S_A \geq S_B \geq S_C \geq S_D. \tag{10}$$

Demonstraţie. Se utilizează soluţia I a problemei 132, pag. 129 din [7] împreună cu relaţia (4) din prezentul articol.

Propoziţia 4. *Fie* $[ABCD]$ *un tetraedru ortocentric având ortocentrul* $H \in \mathrm{int}(ABCD)$. *Atunci avem*

$$HH_A \geq HH_B \geq HH_C \geq HH_D \Leftrightarrow S_A \geq S_B \geq S_C \geq S_D. \tag{11}$$

Demonstraţie. Se ştie că într-un tetraedru ortocentric, perpendicularele comune ale celor trei perechi de muchii opuse sunt concurente în ortocentrul H al tetraedrului, ortocentru care divide aceste perpendiculare, precum şi înălţimile tetraedrului, în segmente orientate al căror produs este constant (*Jacobi*, 1834) – a se consulta problema 45, pag. 19 din [7].

Deci

$$AH \cdot HH_A = BH \cdot HH_B = CH \cdot HH_C = DH \cdot HH_D = q. \tag{12}$$

Dacă $S_A \geq S_B \geq S_C \geq S_D$, atunci, conform Propoziţiei 3, rezultă că

$$AH \leq BH \leq CH \leq DH. \tag{13}$$

Din relaţiile (12) şi (13) obţinem

$$HH_A = \frac{q}{AH} \geq \frac{q}{BH} = HH_B \quad \text{(şi analoagele).}$$

Reciproca este imediată.

Propoziţia 5. *Fie* $[ABCD]$ *un tetraedru ortocentric având ortocentrul* H. *Atunci avem*

$$AH + BH + CH + DH \leq 2 \cdot \sqrt{AH^2 + BH^2 + CH^2 + DH^2} = 4R. \tag{14}$$

Demonstraţie. Din relaţia lui *Leibniz* în spaţiu avem

$$AH^2 + BH^2 + CH^2 + DH^2 = 4GH^2 + GA^2 + GB^2 + GC^2 + GD^2,$$

$$4R = 4OG^2 + \sum GA^2.$$

Deoarece punctele G, H, O aparţin dreptei *Euler* a tetraedrului $[ABCD]$, iar $OG = GH$, din cele de mai sus rezultă imediat că

$$AH^2 + BH^2 + CH^2 + DH^2 = 4R^2.$$

Aplicând în continuare inegalitatea dintre media aritmetică şi media pătratică a patru numere pozitive se obţine relaţia (14), q.e.d.

Propoziţia 6. *Fie* $[ABCD]$ *un tetraedru ortocentric având ortocentrul* $H \in int(ABCD)$. *Atunci au loc inegaliăţile*

a) $HH_A + HH_B + HH_C + HH_D \leq 4r,$ (15)

b) $12r \leq AH + BH + CH + DH \leq 4R,$ (16)

c) $36r^2 \leq AH^2 + BH^2 + CH^2 + DH^2 = 4R^2.$ (17)

Demonstraţie. a) Presupunem $S_A \geq S_B \geq S_C \geq S_D$. Atunci, conform Propoziţiei 4 avem $HH_A \geq HH_B \geq HH_C \geq HH_D$ şi, în plus $h_a \leq h_b \leq h_c \leq h_d$, conform Propoziţiei 2. Aplicând inegalitatea *Cebâşev* şi ţinând seama de identitatea *Gergonne* în spaţiu avem

$$1 = \frac{HH_A}{h_a} + \frac{HH_B}{h_b} + \frac{HH_C}{h_c} + \frac{HH_D}{hd} \geq$$

$$\geq \frac{1}{4}\left(HH_A + HH_B + HH_C + HH_D\right)\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} + \frac{1}{h_d}\right).$$

Deoarece $\sum \frac{1}{h_a} = \frac{1}{r}$ rezultă imediat că

$$4r \geq HH_A + HH_B + HH_C + HH_D,$$

adică (15), q.e.d.

b) $AH+BH+CH+DH = (h_a - HH_A)+(h_b - HH_B)+(h_c - HH_C)+ + (h_d - HH_D) = \sum h_a - \sum HH_A \geq 16r - 4r = 12$, unde, evident, am utilizat inegalitatea (15) şi clasica inegalitate $\sum h_a \geq 16r$ (valabilă în orice tetraedru). Pentru inegalitatea din partea dreaptă se aplică relaţia (14).

c) Rezultă imediat, aplicând inegalitatea dintre media aritmetică şi media pătratică a patru numere pozitive, ţinându-se seama de pct. b) şi relaţia (14).

Propoziţia 7. *În tetraedrul ortocentric* $[ABCD]$ *având ortocentrul* $H \in \operatorname{int}(ABCD)$, *ortocentrul* H *divide înălţimile şi perpendicularele comune ale muchiilor opuse în segmente orientate al căror produs este constant şi cel mult egal cu* $3r^2$.

Demonstraţie. Pentru prima parte a demonstraţiei se utilizează ideea din demonstraţia Propoziţiei 4. În continuare, se ştie conform problemei 318 de la pag. 61 din [7] că

$$\sum (h_a \cdot AH) = \frac{1}{3}\sum a^2 \Leftrightarrow \sum (AH + HH_A)\, AH =$$

$$= \sum AH^2 + \sum AH \cdot HH_A = 4R^2 + 4q = \frac{1}{3}\sum a^2 \Rightarrow$$

$$\Rightarrow q = \frac{1}{12} \cdot \sum a^2 - R^2. \tag{18}$$

Însă, conform teoremei 5, pag. 165 din [4], avem relaţia

$$0 \leq HI^2 = R^2 + 3r^2 - \frac{1}{12}\sum a^2, \tag{19}$$

astfel că

$$\frac{1}{12} \cdot \sum a^2 \leq R^2 + 3r^2. \tag{20}$$

Din relaţiile (18) şi (20) obţinem

$$q \leq 3r^2, \tag{21}$$

ceea ce trebuia demonstrat.

Observaţie. Presupunând că $S_A \geq S_B \geq S_C \geq S_D$, atunci, conform Propoziţiilor 3 şi 4, avem $AH \leq BH \leq CH \leq DH$ şi respectiv $HH_A \geq \geq HH_B \geq HH_C \geq HH_D$. Aplicând inegalitatea *Cebâşev* şi ţinând cont de inegalităţile (15) şi (16) obţinem succesiv:

$$4q = \sum AH \cdot HH_A \leq \frac{1}{4}\left(\sum AH\right)\cdot\left(\sum HH_A\right) \leq \frac{1}{4}\cdot 4R\cdot 4r \Rightarrow q \leq Rr. \tag{21}$$

Este evident faptul că această abordare permite găsirea inegalităţii (22) în mod direct şi nu ca o consecinţă a inegalităţii *Euler* în tetraedru ($R \geq 3r$), aşa cm se observă din relaţia (21).

Propoziţia 8. *În orice tetraedru ortocentric* $[ABCD]$ *are loc inegalitatea*

$$OH^2 \geq OI^2 + 3 \cdot HI^2.$$

Demonstraţie. Conform relaţiei (19) (care este valabilă în orice tetraedru ortocentric) precum şi a formulelor consacrate din geometria tetraedrului avem:

$$\begin{aligned}
&HI^2 = R^2 + 3r^2 - \frac{1}{12}\cdot\sum a^2,\\
&OG^2 = R^2 - \frac{1}{16}\cdot\sum a^2,\\
&2\cdot OG = OH\\
&\Rightarrow 16\cdot OG^2 - 12\cdot HI^2 = 4\left(R^2 - 9r^2\right).
\end{aligned}$$

Dar, conform problemei 333, pag. 63 din [7], avem

$$R^2 - 9r^2 \geq OI^2 \Rightarrow 16\cdot OG^2 - 12\cdot HI^2 \geq 4\cdot OI^2 \Leftrightarrow$$

$$\Leftrightarrow 4\cdot OG^2 \geq 3\cdot HI^2 + OI^2 \Leftrightarrow OH^2 \geq OI^2 + 3HI^2,$$

ceea ce trebuia demonstrat.

Consecinţa 2. Din $OH^2 \geq OI^2 + 3HI^2$ rezultă că

$$OH^2 \geq OI^2 + HI^2. \tag{23}$$

Propoziţia 9. *Fie* $[ABCD]$ *un tetraedru ortocentric. Atunci au loc inegalităţile*

$$12r \leq AI + BI + CI + DI \leq 2\sqrt{AI^2 + BI^2 + CI^2 + DI^2} \leq 4R.$$

Demonstraţie. Din teorema medianei aplicată în triunghiul HIO avem (unde G este mijlocul lui $[OH]$)

$$4GI^2 = 2\left(OI^2 + HI^2\right) - OH^2 = 2\left(OI^2 + HI^2\right) - 4OG^2 \Rightarrow$$

$$\Rightarrow 2GI^2 = OI^2 + HI^2 - 2OG^2 \leq OH^2 - 2OG^2 = 2OG^2,$$

unde am utilizat şi relaţia (23).

$$\Rightarrow GI^2 \leq OG^2. \tag{24}$$

Din relaţia lui *Leibniz* avem

$$AI^2 + BI^2 + CI^2 + DI^2 = 4GI^2 + GA^2 + GB^2 + GC^2 + GD^2 =$$

$$= 4\left(GI^2 + R^2 - OG^2\right) \leq 4R^2,$$

conform relaţiei (24).

Aplicând inegalitatea dintre media aritmetică şi media pătratică rezultă că

$$AI + BI + CI + DI \leq 2\sqrt{AI^2 + BI^2 + CI^2 + DI^2} \leq 2\sqrt{4R^2} = 4R.$$

În continuare, se ştie că dacă P este un punct în interiorul tetraedrului oarecare $[ABCD]$, iar $[A'B'C'D']$ este tetaedrul său pedal, unde $\{A'\} = (AP \cap [BCD]$, $\{B'\} = (BP \cap [ACD]$ etc., atunci, conform problemei 92-d), pag. 26 din [7], avem

$$PA \cdot PB \cdot PC \cdot PD \geq 81 \cdot PA' \cdot PB' \cdot PC' \cdot PD'.$$

Dacă $P \equiv I$, atunci $PA' = IA' \geq r$ (şi analoagele) şi rezultă că

$$AI \cdot BI \cdot CI \cdot DI \geq 81 \cdot r^4.$$

Cum $AI + BI + CI + DI \geq 4\sqrt[4]{AI \cdot BI \cdot CI \cdot DI}$, obţinem că

$$AI + BI + CI + DI \geq 4\sqrt[4]{81 \cdot r^4} = 12r,$$

ceea ce trebuia demonstrat.

Propoziţia 10. *Fie* $[ABCD]$ *un tetraedru ortocentric cu centrul* $H \in \operatorname{int}(ABCD)$. *Atunci au loc inegalităţile*

$$\frac{16r}{3q} \geq \sum\left(\frac{1}{AH} + \frac{1}{HH_A}\right) \geq \frac{16}{3r} \geq 4\left(\frac{1}{r} + \frac{1}{R}\right),$$

unde q *are semnificaţia din relaţia* (12).

Demonstraţie.

$$\sum\left(\frac{1}{AH} + \frac{1}{HH_A}\right) = \sum \frac{H_a}{q} = \frac{1}{q} \cdot \sum h_a. \tag{25}$$

Din relaţiile (6) şi (21), asociate relaţiei (25), obţinem că

$$\frac{16R}{3q} \geq \sum\left(\frac{1}{AH} + \frac{1}{HH_A}\right) \geq \frac{16}{3r}, \tag{26}$$

iar din inegalitatea lui *Euler* $R \geq 3r$, găsim $\frac{16}{3r} \geq 4\left(\frac{1}{r} + \frac{1}{R}\right)$, q.e.d.

Observaţie. Inegalitatea $\sum\left(\frac{1}{AH}+\frac{1}{HH_A}\right)\geq 4\left(\frac{1}{r}+\frac{1}{R}\right)$ se poate demonstra direct, aplicând inegalitatea dintre media armonică şi media aritmetică şi ţinând cont de relaţiile (15) şi (16) după cum urmează:

$$\left.\begin{array}{l}\sum\frac{1}{AH}\geq\frac{16}{\sum AH}\geq\frac{16}{4R}=\frac{4}{R}\\ \sum\frac{1}{HH_A}\geq\frac{16}{\sum HH_A}\geq\frac{16}{4r}=\frac{4}{r}\end{array}\right\}\Rightarrow\sum\left(\frac{1}{AH}+\frac{1}{HH_A}\right)\geq 4\left(\frac{1}{r}+\frac{1}{R}\right).$$

Egalităţile au loc numai dacă $[ABCD]$ este tetraedru regulat.

Propoziţia 11. *Fie $[ABCD]$ un tetraedru ortocentric având ortocentrul $H\in\operatorname{int}(ABCD)$. Atunci are loc următoarea rafinare a inegalităţii $R\geq 3r$ (Euler-Durrande):*

$$12r\leq A\Omega+B\Omega+C\Omega+D\Omega+\leq 4R. \tag{27}$$

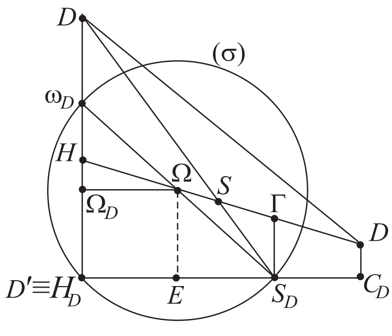
Demonstraţie. Se cunoaşte faptul că punctele G, H, O, Ω şi Γ se bucură de proprietatea

$$\overrightarrow{H\Omega}=2\overrightarrow{\Omega G}=2\overrightarrow{G\Gamma}=\overrightarrow{\Gamma O}. \tag{28}$$

Pentru demonstraţia părţii drepte a inegalităţii (27) se poate consulta, de exemplu, [8], pag. 201-202.

Conform teoremei 7, pag. 229 din [3], sfera *Euler* $\sigma\left(\Omega;\frac{R}{3}\right)$ împarte segmentele $|HA|$, $|HB|$, $|HC|$, $|HD|$ în raportul $1:2$. Fie deci $W_D\in|HD|$, astfel încât $\frac{W_DH}{W_DD}=\frac{1}{2}$. Notăm cu O_D centrul cercului circumscris triunghiului ABC, cu Ω_D proiecţia ortogonală a punctului Ω pe înălţimea DH_D, unde $\Omega_D\in(DH_D)$ şi cu E mijlocul segmentului $D'G_D$ ($D'\equiv H_D$).

Proiecţia ortogonală a punctului Γ pe dreapta $D'O_D$ (vezi fig. 1) coincide cu punctul G_D (centrul de greutate al triunghiului ABC).



Punctele D', W_D, G_D aparţin sferei *Euler* (σ). Evident $OO_D\perp D'O_D$ şi $\Omega E\perp D'G_D$. Din triunghiul dreptunghic $D\Omega\Omega_D$ avem $D\Omega=\sqrt{D\Omega_D^2+\Omega\Omega_D^2}$.

Dar, $\Omega\Omega_D=EH_D=O_DG_D$, iar $D\Omega_D=DH_D-H_D\Omega_D=h_d-\Omega E$.

Cum ΩE este linie mijlocie în triunghiul dreptunghic $W_DG_DH_D$, rezultă

$$\Omega E=\frac{1}{2}W_DH_D=\frac{1}{2}\left(h_d-DW_D\right)=$$
$$=\frac{1}{2}\left(h_d-\frac{2}{3}HD\right)=\frac{h_d}{2}-\frac{1}{3}HD.$$

Atunci avem

$$D\Omega_D=h_d-\left(\frac{h_d}{2}-\frac{1}{3}HD\right)=\frac{h_d}{2}+\frac{1}{3}HD.$$

Deci
$$D\Omega=\sqrt{\left(\frac{h_d}{2}+\frac{1}{3}HD\right)^2+O_DG_D^2}\ \text{(şi analoagele).}$$
Rezultă că,
$$\sum A\Omega=\sum\sqrt{\left(\frac{h_a}{2}+\frac{1}{3}AH\right)^2+O_AG_A^2}. \tag{29}$$

Dacă $a_1, a_2, b_1b_2 \in \mathbb{R}^*$ se ştie, conform inegalităţii lui *Minkovski*, că $\sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}\geq\sqrt{(a_1+a_2)^2+\left(b_1^2+b_2\right)^2}$, cu egalitate dacă şi numai dacă $\frac{a_1}{b_1}=\frac{a_2}{b_2}$.

Prin inducţie matematică completă se poate demonstra că are loc inegalitatea mai generală
$$\sum_{i=1}^{n}\sqrt{a_i^2+b_i^2}\geq\sqrt{\left(\sum_{i=1}^{n}a_i\right)^2+\left(\sum_{i=1}^{n}b_i\right)^2}, \tag{30}$$
unde $a_i\geq 0$, $b_i>0$, $i=\overline{1,n}$, $n\geq 2$, $n\in\mathbb{N}$, cu egalitate dacă şi numai dacă $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots=\frac{a_n}{b_n}$.

Considerând în (30) $n=4$, $a_=O_AG_A$, $a_2=O_BG_B$, $a_3=O_CG_C$, $a_4=$ $=O_DG_D$, $b_1=\frac{h_a}{2}+\frac{1}{3}A_H$, $b_3=\frac{h_c}{2}+\dfrac{1}{3}CH$, $b_4=\frac{h_d}{2}+\frac{1}{3}DH$ obţinem
$$\sum\sqrt{\left(\frac{h_a}{2}+\frac{1}{3}AH\right)^2+O_AG_A^2}=\sqrt{\left(\sum O_AG_A\right)^2+\left(\sum\left(\frac{h_a}{2}+\frac{1}{3}AH\right)\right)^2}\geq$$
$$\geq\frac{1}{2}\cdot\sum h_a+\frac{1}{3}\sum AH\geq 8r+4r=12r, \tag{31}$$
unde s-a ţinut seama de inegalităţile (6) şi (16).

În fine, din relaţiile (29) şi (31) rezultă că $\sum A\Omega\geq 12r$, ceea ce trebuia demonstrat.

Propoziţia 12. *Fie $[ABCD]$ un tetraedru oarecare şi Γ simetricul punctului Ω faţă de G. Atunci are loc următoarea rafinare a inegalităţii $R\geq 3r$:*
$$12r\leq A\Gamma+B\Gamma+C\Gamma+D\Gamma\leq 4R. \tag{27*}$$

Demonstraţie. Partea dreaptă constituie inegalitatea (12), pag. 202 din [8]. Însă, conform relaţiei (15) pag. 203 din [8], avem
$$D\Gamma^2=\frac{3}{8}m_d^2+\frac{R^2}{3}-\frac{2}{9}OG^2\quad\text{(şi analoagele)}\Rightarrow$$
$$\Rightarrow D\Gamma^2=\frac{3}{8}m_d^2+\frac{3R^2-2OG^2}{9}=\frac{3}{8}m_d^2+\frac{R^2-OG^2}{3}+\frac{OG^2}{9}\ \text{(şi analoagele)}\Rightarrow$$

$$\Rightarrow D\Gamma = \sqrt{\frac{3}{8}m_d^2 + \frac{R^2 - OG^2}{3}} \quad \text{(şi analoagele).}$$

Particularizând din nou inegalitatea (30) avem

$$\begin{aligned} \sum A\Gamma \geq \sum \sqrt{\frac{3}{8}m_a^2 + \frac{R^2 - OG^2}{3}} \geq \\ \geq \sqrt{\frac{3}{8}\left(\sum m_a\right)^2 + 16 \cdot \frac{R^2 - OG^2}{3}}. \end{aligned} \tag{32}$$

Însă, din geometria tetraedrului se cunosc relaţiile

$$16\left(R^2 - OG^2\right) = a^2 + b^2 + c^2 + l^2 + m^2 + n^2 \geq 144r^2$$

şi

$$\sum m_a \geq 16r \qquad \text{(conform (6)).}$$

Ţinând cont de relaţia (32) obţinem

$$\sum A\Gamma \geq \sqrt{\frac{3}{8} \cdot 256r^2 + \frac{144r^2}{3}} = \sqrt{3 \cdot 32r^2 + 48r^2} = 12r \Leftrightarrow \sum A\Gamma \geq 12r,$$

ceea ce trebuia demonstrat.

Egalitatea se atinge dacă şi numai dacă $[ABCD]$ este tetraedru regulat.

Propoziţia 13. *În tetraedrul oarecare* $[ABCD]$ *au loc inegalităţile*:

a) $27 \cdot \dfrac{r^2}{R} \leq \dfrac{AX^2}{m_a} + \dfrac{BX^2}{m_b} + \dfrac{CX^2}{m_c} + \dfrac{DX^2}{m_d} \leq \dfrac{R^2}{r}$, *unde* $X \in \{G, O, \Gamma\}$;

b) $\dfrac{A\Omega^4}{A\Gamma^2} + \dfrac{B\Omega^4}{B\Gamma^2} + \dfrac{C\Omega^4}{C\Gamma^2} + \dfrac{D\Omega^4}{D\Gamma^2} \geq 36r^2$;

c) $\dfrac{A\Gamma^4}{A\Omega^2} + \dfrac{B\Gamma^4}{B\Omega^2} + \dfrac{C\Gamma^4}{C\Omega^2} + \dfrac{D\Gamma^4}{D\Omega^2} \geq 36r^2$.

Demonstraţie. a) Partea dreaptă a inegalităţii reprezintă un caz particular al inegalităţii a) din Propoziţia 1 demonstrată în [8] la pag. 203-204.

Se ştie că, dacă $x_i \in \mathbb{R}$ şi $\alpha_i > 0$, $i = \overline{1,p}$, $p \in \mathbb{N}^*$, atunci

$$\frac{x_1^2}{\alpha_1} + \frac{x_2^2}{\alpha_2} + \ldots + \frac{x_p^2}{\alpha_p} \geq \frac{\left(x_1 + x_2 + \ldots + x_p\right)^2}{\alpha_1 + \alpha_2 + \ldots + \alpha_p} \tag{33}$$

(a se consulta [8], pag. 204).

Ţinând seama de relaţia (33) avem

$$\frac{AX^2}{m_a} + \frac{BX^2}{m_b} + \frac{CX^2}{m_c} + \frac{DX^2}{m_d} \geq \frac{(AX + BX + CX + DX)^2}{m_a + m_b + m_c + m_d}, \tag{34}$$

unde $X \in \{G, O, \Gamma\}$. Dar

$$\begin{cases} A\Gamma + B\Gamma + C\Gamma + D\Gamma \geq 12r, \text{ conform relaţiei } (27^*), \\ AO + BO + CO + DO = 4R \geq 12r, \text{ conf. inegalit. } \textit{Euler}\ , \\ AG + BG + CG + DG \geq 12R, \text{ conform relaţiei } (6'), \\ m_a + m_b + m_c + m_d \leq \dfrac{16R}{3}, \text{ conform relaţiei } (6). \end{cases} \tag{35}$$

Aplicând relaţiile (35) coroborat cu inegalitatea (34) obţinem

$$\frac{AX^2}{m_a} + \frac{BX^2}{mb} + \frac{CX^2}{m_c} + \frac{DX^2}{m_d} \geq \frac{144r^2}{16} \cdot \frac{3}{R} = 27 \cdot \frac{r^2}{R},$$

ceea ce trebuia demonstrat.

b) Conform relaţiei (10), pag. 202 din [8], se cunoaşte că

$$\sum A\Gamma^2 = \sum A\Omega^2 \geq 36r^2. \tag{36}$$

În continuare se aplică din nou inegalitatea (34), obţinând

$$\sum \frac{A\Omega^4}{A\Gamma^2} \geq \frac{\left(\sum A\Omega^2\right)^2}{\sum A\Gamma^2} = \sum A\Omega^2 \geq 36r^2,$$

ceea ce trebuia demonstrat.

c) Idem punctul b).

Propoziţia 14. *În orice tetraedru $[ABCD]$ au loc următoarele rafinări ale inegalităţii Euler-Durrande, $R \geq 3r$:*

a) $\left(\dfrac{3}{2R}\right)^2 \leq \sum \dfrac{1}{m_a^2} \leq \sum \dfrac{1}{h_a^2} \leq \dfrac{1}{r^2} - \dfrac{27}{4} \cdot \dfrac{1}{R^2}$,

b) $\dfrac{1}{r^2} \leq \sum \dfrac{1}{r_a^2} \leq \dfrac{4}{r^2} - \dfrac{27}{R^2}$,

c) $4^5 \cdot r^4 \leq \sum h_a^4 \leq \sum m_a^4 \leq \dfrac{4^5}{3^4} \cdot \left(4 \cdot R^4 - 243 \cdot r^4\right)$,

d) $\dfrac{R}{r} \geq \dfrac{AX}{m_a} + \dfrac{BX}{m_b} + \dfrac{CX}{m_c} + \dfrac{DX}{m_d} \geq 27 \cdot \dfrac{r^2}{R^2}$, unde $X \in \{G, O, \Gamma\}$,

e) $\dfrac{R}{r^2} - \dfrac{27}{4} \cdot \dfrac{1}{R} \geq \dfrac{AX}{m_a^2} + \dfrac{BX}{m_b^2} + \dfrac{CX}{m_c^2} + \dfrac{DX}{m_d^2} \geq \dfrac{81}{4} \cdot \dfrac{r^2}{R} \cdot \dfrac{1}{\sqrt{4R^4 - 243r^4}}$, unde $X \in \{G, O, \Gamma\}$,

f) $\dfrac{R^2}{r^2} - \dfrac{27}{4} \geq \dfrac{AX^2}{m_a^2} + \dfrac{BX^2}{m_b^2} + \dfrac{CX^2}{m_c^2} + \dfrac{DX^2}{m_d^2} \geq \dfrac{3^6}{4} \cdot \left(\dfrac{r}{R}\right)^4$, unde $X \in \{G, O, \Gamma\}$.

Demonstraţie. a) $m_a \geq h_a$ (şi analoagele)

$$\Rightarrow \sum \frac{1}{h_a^2} \geq \sum \frac{1}{m_a^2} \geq \frac{1}{4}\left(\sum \frac{1}{m_a}\right)^2 \geq \frac{1}{4} \cdot \left(\frac{3}{R}\right)^2 = \left(\frac{3}{2R}\right)^2,$$

unde am utilizat inegalitatea (7). Mai departe,

$$\sum \frac{1}{h_a^2} = \left(\sum \frac{1}{h_a}\right)^2 - 2\sum \frac{1}{h_a \cdot h_b}.$$

În plus,

$$\sum \frac{1}{h_a \cdot h_b} \geq \frac{36}{\sum h_a \cdot h_b} \geq \frac{36}{\frac{3}{2}\left(\sum h_a^2\right)} = \frac{24}{\sum h_a^2} \geq \frac{24}{\frac{64}{9}R^2} = \frac{27}{8R^2},$$

unde am utilizat şi inegalitatea (8).

Rezultă că

$$2 \cdot \sum \frac{1}{h_a \cdot h_b} \geq \frac{27}{4R^2} \Rightarrow \sum \frac{1}{h_a^2} \leq \frac{1}{r^2} - \frac{27}{4R^2},$$

deoarece $\sum \frac{1}{h_a} = \frac{1}{r}$. rac b) Se ştie că $\sum \frac{1}{r_a} = \frac{2}{r}$, iar

$$\sum \frac{1}{r_a^2} = 4 \cdot \sum \frac{1}{h_a^2} \tag{36*}$$

(conform relaţiei (33), pag. 205, din [8]).

Avem $\sum \frac{1}{r_a^2} \geq \frac{1}{4}\left(\sum \frac{1}{r_a}\right)^2 = \frac{1}{r^2}$; ţinând seama de pct. a) obţinem

$$\sum \frac{1}{r_a^2} = 4\sum \frac{1}{h_a^2} \leq \frac{4}{r^2} - \frac{27}{R^2},$$

ceea ce trebuia demonstrat.

c) $\sum m_a^4 \geq \sum h_a^4 \geq \dfrac{1}{4}\left(\sum h_a^2\right)^2 \geq \dfrac{1}{4} \cdot \left(64r^2\right)^2 = 4^5 \cdot r^4,$
unde am ţinut seama de inegalitatea (8).

Mai departe, $\sum m_a^4 = \left(\sum m_a^2\right)^2 - 2 \cdot \left(\sum m_a^2 \cdot m_b^2\right)$.

Însă $\sum m_a^2 \cdot m_b^2 \geq 6 \cdot \sqrt[6]{(m_a m_b m_c m_d)^6} = 6 m_a m_b m_c m_d \geq 6 h_a h_b h_c h_d \geq$
$\geq 6 \cdot \left(\dfrac{4}{\sum \frac{1}{h_a}}\right)^4 = 6 \cdot 4^4 \cdot r^4 \Rightarrow -2\sum m_a^2 \cdot m_b^2 \leq -3 \cdot 4^5 \cdot r.$

Dar, $\sum m_a^2 \leq \frac{64}{9}R^2$ (conform inegalităţii (8)) $\Rightarrow \left(\sum m_a^2\right)^2 \leq \frac{64^2}{81} \cdot R^4$.
Atunci,

$$\sum m_a^4 \leq \frac{4^6}{3^4} \cdot R^4 - 3 \cdot 4^5 \cdot r^4 = \frac{4^5}{3^4} \cdot \left(4R^2 - 243 \cdot r^4\right),$$

ceea ce trebuia demonstrat.

$$\text{d) } \sum \frac{AX}{m_a} = \sum \frac{AX^2}{m_a \cdot AX} \geq \frac{\left(\sum AX\right)^2}{\sum (m_a \cdot AX)}, \tag{37}$$

conform inegalităţii (33).

Însă, conform inegalităţii *Cauchy-Buniakovski-Schwarz* (C.B.S.) avem

$$\sum (m_a \cdot AX) \leq \sqrt{\left(\sum m_a^2\right) \cdot \left(\sum AX^2\right)}. \tag{38}$$

Dar

$$\begin{cases} \sum AG^2 \leq 4R^2, & \text{conform inegalităţii } (8'), \\ \sum A\Gamma^2 \leq 4R^2 = \sum AO^2, & \text{conform relaţiei (10), pag. 202 din [8].} \end{cases} \tag{39}$$

Din (38) şi (39) rezultă

$$\sum m_a \cdot AX \leq \sqrt{\frac{64}{9} \cdot R^2 \cdot 4R^2} = \frac{16}{9} \cdot R^2. \tag{40}$$

Acum din relaţiile (35), (37), (40) obţinem

$$\sum \frac{AX}{m_a} \geq 144r^2 \cdot \frac{3}{16R^2} = 27 \cdot \frac{r^2}{R^2},$$

ceea ce trebuia demonstrat.

Partea stângă a inegalităţii constituie un caz particular ($n = 1$) al inegalităţii Propoziţiei 1, a), pag. 203 din [8].

e) - f) Putem presupune că $m_a \geq m_b \geq m_c \geq m_d \Rightarrow$

$$\frac{1}{m_a^2} \leq \frac{1}{m_b^2} \leq \frac{1}{m_c^2} \leq \frac{1}{m_d^2}. \tag{41}$$

Însă, conform inegalităţii (23), pag. 204 din [8], rezultă că

$$AX^n \geq BX^n \geq CX^n \geq DX^n,\ X \in \{G, O.\Gamma\},\ n \in \{0, 1, 2\}. \tag{42}$$

În baza relaţiilor (41) şi (42) putem aplica inegalitatea *Cebâşev*, obţinând

$$\begin{cases} \sum \frac{AX}{m_a^2} \leq \frac{1}{4}\left(\sum AX\right) \cdot \left(\sum \frac{1}{m_a^2}\right), \\ \sum \frac{AX^2}{m_a^2} \leq \frac{1}{4}\left(\sum AX^2\right)\left(\sum \frac{1}{m_a^2}\right). \end{cases} \tag{43}$$

Ţinând seama de relaţiile (35), (39) şi punctul a) al prezentei propoziţii avem, în conformitate cu relaţiile (43),

$$\sum \frac{AX}{m_a^2} \leq \frac{1}{4} \cdot 4R\left(\frac{1}{r^2} - \frac{27}{4} \cdot \frac{1}{R^2}\right) = \frac{R}{r^2} - \frac{27}{4} \cdot \frac{1}{R}$$

şi

$$\sum \frac{AX^2}{m_a^2} \leq \frac{1}{4} \cdot 4R^2 \left(\frac{1}{r^2} - \frac{27}{4} \cdot \frac{1}{R^2} \right) = \frac{R^2}{r^2} - \frac{27}{4},$$

ceea ce trebuia demonstrat.

În continuare, pentru partea dreaptă a inegalităţii e) putem scrie, conform inegalităţii (33), că

$$\sum \frac{AX}{m_a^2} = \sum \frac{AX^2}{AX \cdot m_a^2} \geq \frac{\left(\sum AX\right)^2}{\sum \left(AX \cdot m_a^2\right)}. \tag{44}$$

În plus

$$\sum \left(AX \cdot m_a^2\right) \leq \sqrt{\left(\sum AX^2\right) \cdot \left(\sum m_a^4\right)} \leq \sqrt{4R^2 \cdot \frac{4^5}{3^4} \cdot (4R^4 - 243 \cdot r^4)} =$$

$$= \frac{4^3}{3^2} \cdot R \cdot \sqrt{4R^4 - 243 \cdot r^4}.$$

Ţinând cont de inegalităţile (35), din (44) rezultă că

$$\sum \frac{AX}{m_a^2} \geq \frac{144 \cdot r^2 \cdot 9}{64 \cdot R \cdot \sqrt{4R^4 - 243r^4}} = \frac{81}{4} \cdot \frac{r^2}{R} \cdot \frac{1}{\sqrt{4R^4 - 243r^4}}.$$

Totodată, la punctul f), pentru inegalitatea din dreapta, aplicăm inegalitatea dintre media pătratică şi media aritmetică la care ţinem apoi cont de rezultatul stabilit la punctul d).

Observaţie. Partea stângă a inegalităţilor punctelor d), e) şi f) rămâne valabilă şi pentru $X \in \{K, \Omega\}$.

Propoziţia 15. *Fie $[ABCD]$ un tetraedru ortocentric având ortocentrul $H \in \mathrm{int}(ABCD)$. Atunci au loc inegalităţile*:

a) $\dfrac{R}{r} \geq \dfrac{AX}{m_a} + \dfrac{BX}{m_b} + \dfrac{CX}{m_c} + \dfrac{DX}{m_d} \geq 27 \cdot \dfrac{r^2}{R^2}$;

b) $\dfrac{R^2}{r} \geq \dfrac{AX^2}{m_a} + \dfrac{BX^2}{m_b} + \dfrac{CX^2}{m_c} + \dfrac{DX^2}{m_d} \geq 27 \cdot \dfrac{r^2}{R}$;

c) $\dfrac{1}{324} \cdot \dfrac{R^5}{r^6} \geq \dfrac{AX}{m_a^2} + \dfrac{BX}{m_b^2} + \dfrac{CX}{m_c^2} + \dfrac{DX}{m_d^2} \geq \dfrac{81}{4} \cdot \dfrac{r^2}{R} \cdot \dfrac{1}{\sqrt{4R^4 - 243r^4}}$;

d) $\dfrac{1}{324} \cdot \dfrac{R^6}{r^6} \geq \dfrac{AX^2}{m_a^2} + \dfrac{BX^2}{m_b^2} + \dfrac{CX^2}{m_c^2} + \dfrac{DX^2}{m_d^2} \geq \dfrac{3^6}{4} \cdot \left(\dfrac{r}{R}\right)^4$;

e) $\dfrac{2}{9} \cdot \dfrac{R^3}{r^3} \geq \dfrac{AX}{r_a} + \dfrac{BX}{r_b} + \dfrac{CX}{r_c} + \dfrac{DX}{r_d} \geq 6$,

unde $X \in \{G, H, O, \Gamma, \Omega\}$.

Demonstraţie. Partea stângă a inegalităţilor a) – e) constituie cazuri particulare ale inegalităţilor Propoziţiilor 1 şi 3 din [8], pag. 203 - 207.

Pentru inegalităţile din partea dreaptă vom analiza, pe rând, fiecare caz în parte după cum urmează:

Presupunem $S_A \geq S_B \geq S_C \geq S_D$. Atunci, conform relaţiei (46), pag. 208 din [8], avem $AX \leq BX \leq CX \leq DX$, pecum şi $\frac{1}{r_a} \leq \frac{1}{r_b} \leq \frac{1}{r_c} \leq \frac{1}{r_d}$, conform relaţiei (45), pag. 208 din [8], (44*),

$$\text{a)} \sum \frac{AX}{m_a} = \sum \frac{AX^2}{m_a \cdot AX} \geq \frac{\left(\sum AX\right)^2}{\sum (m_a \cdot AX)}. \tag{45}$$

$$\sum (m_a \cdot AX) \leq \sqrt{\left(\sum m_a^2\right) \cdot \left(\sum AX^2\right)} \leq \sqrt{\frac{64}{9} R^2 \cdot 4R^2} = \frac{16}{3} R^2, \tag{46}$$

unde am utilizat inegalităţile (8), (17), precum şi $\sum A\Omega^2 \leq 4R$ ([8], pag. 202, relaţia (10)).

Ţinând seama acum de inegalităţile (16) şi (27) coroborate cu (45) şi (46), obţinem

$$\sum \frac{AX}{m_a} \geq \frac{144r^2}{R^2} \cdot \frac{3}{16} = 27 \cdot \frac{r^2}{R^2},$$

ceea ce trebuia demonstrat.

$$\text{b)} \sum \frac{AX^2}{m_a} \geq \frac{\left(\sum AX\right)^2}{\sum m_a} \geq \frac{144r^2}{R^2} \cdot \frac{3}{16} = 27 \cdot \frac{r^2}{R^2},$$

ceea ce trebuia demonstrat (am utilizat şi inegalităţile (6), (27), (27*), (16)).

c) se utilizează metoda folosită la demonstrarea punctului e) al Propoziţiei (14), împreună cu inegalităţile suplimenetare (16) şi (27).

$$\text{d)} \sum \frac{AX^2}{m_a^2} \geq \frac{\left(\sum AX\right)^2}{\sum m_a^2} \geq \frac{3^6}{4} \cdot \frac{r^4}{R^4}$$

(conform pct. f) Propoziţi (14) şi a relaţiilor (16) şi (27).

e) Ţinând seama de relaţiile (16), (27), (27*), (36*), (44*), prin aplicarea iniţială a inegalităţii *Cebâşev* obţinem succesiv

$$\sum \frac{AX}{r^2} \geq \frac{1}{4} \cdot \left(\sum AX\right) \cdot \left(\sum \frac{1}{r_a}\right) \geq \frac{1}{4} \cdot 12 \cdot r \cdot \frac{2}{r} = 6,$$

ceea ce trebuia demonstrat.

Egalităţile se ating dacă şi numai dacă $[ABCD]$ este tetraedru regulat.

În baza celor prezentate până acum, propunem cititorului interesat să arate că în orice tetraedru $[ABCD]$ are loc dubla inegalitate

$$\frac{R^2}{r} \geq \frac{AX^2}{m_a} + \frac{BX^2}{m_b} + \frac{CX^2}{m_c} + \frac{DX^2}{m_d} \geq 27 \cdot \frac{r^2}{R}, \quad \forall X \in \{G, O, I\}.$$

Observaţii finale. În Propoziţiile 13 şi 14 egalităţile se ating numai dacă $[ABCD]$ este tetraedru regulat.

Având în vedere inegalitatea (20) şi formulele de calcul ale medianelor unui tetraedru funcţie de lungimile muchiilor acestuia se pot rafina pentru tetraedrele ortocentrice inegalităţile (6), (7), (8) şi a), b), c) ale Propoziţiei 14, lucru ce permite mai departe rafinarea tuturor inegalităţilor prezentate în acest articol (pentru tetraedrele ortocentrice) în care intervin lungimile medianelor tetraedrului. Lăsăm în seama cititorului dezvoltarea acestor rafinări.

La încheierea acestui articol enunţăm următoarea:

Problemă deschisă. *Fie $ABCD$ un tetraedru ortocentric având ortocentrul $H \in \mathrm{int}(ABCD)$. Notăm cu $\{I_A\} = (AI \cap (BCD)$, $\{I_B\} = (BI \cap \cap(ACD)$, $\{I_C\} = (CI \cap (ABD)$, $\{I_D\} = (DI \cap (ABC)$, unde I este centrul sferei înscrisă în tetraedrul $[ABCD]$. Atunci avem*:

a) $AI \leq BI \leq CI \leq DI$;

b) $AI_A \leq BI_B \leq CI_C \leq DI_D$;

dacă şi numai dacă $S_A \geq S_B \geq S_C \geq S_D$

BIBLIOGRAFIE

[1] O. Bagdasar, *A new class of non similarities the triangle and the tetrahedron*, Octogon M. M. vol. 12, no. 1, april 2004, 302-303.

[2] O. Bagdasar, *On O.Q.1379*, Octogon M. M. vol. 12, no. 2B, october 2004, 1060.

[3] D. Brânzei, S. Aniţa, C. Cocea, *Planul şi spaţiul euclidian*, Editura Academiei R.S.R., Bucureşti, 1986.

[4] L. Nicolescu, A. Bumbăcea, A. Catană, P. Horje, G.G.Niculescu, N. Oprea, C. Zara, *Metode de rezolvare a problemelor de geometrie*, Editura Universităţii din Bucureşti, 1998.

[5] M. Olteanu, *Open question (O.Q.) no. 1324*, Octogon M. M. vol. 11, no. 2, october 2003, 859.

[6] M. Olteanu, *Open question (O.Q.) no. 1379*, Octogon M. M. vol. 12, no. 1, april 2004, 454.

[7] M. Olteanu, *Inegalităţi în tetraedru – Culegere de probleme"*, Editura Universitară Conspress, Bucureşti, 2003.

[8] M. Olteanu, *Asupra unor inegalităţi în tetraedru*, G.M.-A, nr. 3, 2006, 200-208.

[9] J. Sándor, *On certain inequalities in a tetrahedron*, Octogon M. M. vol. 12, no. 1, april 2004, 236-239.

A problem suggested by the Atiyah conjecture

Wladimir G. Boskoff[1)], Laurenţiu Homentcovschi[2)]

Abstract. In [2] is stated that Atiyah conjecture is a problem in elementary space geometry that has arisen from M.V.Berry and J.M. Robbins investigations on the spin-statistics theorem in quantum mechanics. Our aim is to state and prove a variation of Atiyah conjecture in the Euclidean plane.

Keywords: Atiyah's conjecture, linear independence over a field.

MSC : 51M04, 51M16, 70G25, 11J72.

1. Introduction.

The *Atiyah*'s conjecture connection with physics via geometric energies is explained in [1], [3] and [6]. Some interesting particular cases are investigated in [4] and a complete solution in the case $n = 4$ one can find in [5]. Some other points of view, strengthening of the conjecture and other solutions for particular configurations can be seen in [4] and [6]. In the case $n = 3$ *Atiyah* showed that the problem is equivalent to the following statement.

Consider the unit circle centered in O and three different lines passing through O which intersect the circle in the six points of affixes $\alpha, -\alpha$, $\beta, -\beta$, γ, $-\gamma$. Then the polynomials

$$P_1 = (X - \gamma)(X + \beta), \quad P_2 = (X + \gamma)(X - \alpha), \quad P_3 = (X + \alpha)(X - \beta)$$

are linearly independent over $\mathbb{C}$.

2. A problem arised from the Atiyah's conjecture.

The original *Atiah*'s conjecture in the case $n = 3$ leads to a plane configuration obtained by the author starting from the space problem after the study of some transformations which leave invariant a given determinant. The previous statement showed that one particular point, the center of the circle denoted by O, and the diameters determine the three previous linear independent polynomials. Can we change the center of the circle with somne other point O_1 and the diameters with lines passing through O_1? The affixes of the endpoints will change. What can we say about the new polynomials created in such way? Or suppose A_1, A_2, A_3 be three distinct and noncollinear points in the interior of the unit circle. The lines A_iA_j intersect the circle in six distinct points. It is known from [2] that for collinear points we still have linear independence over $\mathbb{C}$. Can we assert something about the linear independence of the polynomials created as in *Atiyah*'s conjecture when the three initial points are distinct and noncollinear? At this two questions

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we ask in the present article. The following Lemma allows us to answer to both questions.

Lemma. *Let $z_1, z_2, z_3, z_4, z_5, z_6$ be the affixes of six complex numbers on the unit circle. Consider the polynomials $f, g, h \in \mathbb{C}[X]$,*

$$f = (X - z_1)(X - z_2), g = (X - z_3)(X - z_4), h = (X - z_5)(X - z_6).$$

If the three polynomials are linearly dependent over $\mathbb{R}$, then at most four out of the six initial roots are distinct.

Proof. Let us denote $S_1 = z_1 + z_2$, $S_2 = z_3 + z_4$, $S_3 = z_5 + z_6$; $P_1 = z_1 \cdot z_2$, $P_2 = z_3 \cdot z_4$, $P_3 = z_5 \cdot z_6$. Since $|z_i| = 1$, for $i = \overline{1,6}$ we have $|P_1| = |P_2| = |P_3| = 1$. Suppose it exists $\alpha, \beta, \gamma \in \mathbb{R}$ (at least one nonzero) such that

$$\alpha \cdot f + \beta \cdot g + \gamma \cdot h = 0.$$

It means

$$\alpha(X^2 - S_1X + P_1) + \beta(X^2 - S_2X + P_2) + \gamma(X^2 - S_3X + P_3) = 0.$$

Without to loose the generality we may consider $\gamma = -1$. It results

$$\begin{cases} \alpha + \beta = 1 \\ \alpha S_1 + \beta S_2 = S_3 \\ \alpha P_1 + \beta P_2 = P_3. \end{cases}$$

The first and the last line lead to $\alpha P_1 + (1 - \alpha)P_2 = P_3$. The geometric interpretation of the last relation shows that the points M_1, M_2, M_3 of affixes P_1, P_2 and P_3 are collinear. That is, M_3 belongs to the line M_1, M_2 if $M_1 \neq M_2$. In the case when $M_1 = M_2$ we obtain $P_1 = P_2$, therefore $\alpha P_1 + (1-\alpha)P_1 = P_3$ which implies $P_3 = P_1$. Taking into accout the enounce M_1, M_2, M_3 belong to the unit circle, so we have or $M_3 = M_1$ or $M_3 = M_2$.

Therefore or $P_3 = P_1$ or $P_3 = P_2$. If $P_3 = P_1$ we have $\alpha = 1$, which implies $S_3 = S_1$, that is $\{z_1, z_2\} = \{z_5, z_6\}$. Analogously, if $P_3 = P_2$ we obtain $\{z_3, z_4\} = \{z_5, z_6\}$.

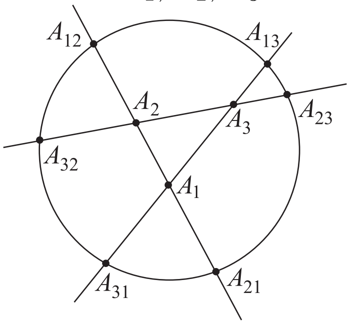
As a conclusion, the linear dependence of the polynomials implies that at most four out of the six initial complex numbers are distinct, at least one of the following set equalities is valid

$$\{z_1, z_2\} = \{z_3, z_4\},\ \{z_1, z_2\} = \{z_5, z_6\},\ \{z_3, z_4\} = \{z_5, z_6\}.$$

The first conclusion we have according to Lemma 1 is that if we have six distinct points on the circle determined by three (distinct) lines passing through O_1, the above polynomials are linerely independent over $\mathbb{R}$.

The main result

Let A_1, A_2, A_3 be three distinct and noncollinear points in the interior of the unit circle. For any $i, j = \overline{1,3}$, $i \neq j$, the halfline $(A_iA_j$ intersects the circle in the point A_{ij} with the affix z_{ij}. Then the polynomials $f_1, f_2, f_3 \in \mathbb{C}[X]$,



$$f_1 = (X - z_{12})(X - z_{13}),$$
$$f_2 = (X - z_{21})(X - z_{23}),$$
$$f_3 = (X - z_{31})(X - z_{32}),$$

are linearly independent over $\mathbb{R}$.

The proof. By contrary, let us suppose that the polynomials are linearly dependent. According to the previous lemma we have or $\{z_{12}, z_{13}\} = \{z_{21}, z_{23}\}$ or $\{z_{12}, z_{13}\} = \{z_{31}, z_{32}\}$ or $\{z_{21}, z_{23}\} = \{z_{31}, z_{32}\}$.

We analyze only the first case, the other two can be analyzed in the same manner. So, let us start from $\{z_{12}, z_{13}\} = \{z_{21}, z_{23}\}$. It may happen or (I) $z_{12} = z_{21}$ and $z_{13} = z_{23}$ or (II) $z_{12} = z_{23}$ and $z_{13} = z_{21}$.

If (I) then $z_{12} = z_{21}$ and $z_{13} = z_{23}$. So, we have both $A_{12} = A_{21}$ and $A_{13} = A_{23}$. It results $A_{12} = A_{21} = A_{13} = A_{23}$ and this means that $A_1 = A_2 = A_3$ and this point belongs to the cicle. The other possible situation is $A_{12} = A_{21} = A_{32} = A_{31}$ and $A_{13} = A_{23}$ that is $A_1 = A_2$.

If (II) we have $z_{12} = z_{23}$ and $z_{13} = z_{21}$. So, we have both $A_{12} = A_{23}$ and $A_{13} = A_{21}$. It results $A_{12} = A_{23} = A_{13} = A_{23}$ that is $A_1 = A_2 = A_3$. The other possible situation is $A_{12} = A_{21} = A_{13} = A_{31} = A_{23} = A_{32}$, who leads to $A_1 = A_2 = A_3$ and this point belongs to the cicle. Therefore in all the cases the three points A_1, A_2, A_3 are not distinct, in collision with the hypothesis. It results that the polynomials are linearly independent over $\mathbb{R}$.

References

[1] M. Atiyah, *The Geometry of classical particles*, In Surveys in Differential Geometry, Vol. VII,S-T.Yau, International Press, 2000, 1-15.

[2] M. Atiyah, *Configurations of points*, Philosophical Transactions of the Royal Society of London, Series A, 359: 1375 - 1387, 2001.

[3] M. Atiyah and P. Sutcliffe, *The Geometry of point particles*, arXiv: hep-th/0105179 v2, 15 October 2001.

[4] D.Z. Dokovisc, *Proof of Atiyah's conjective for two special types of configurations*, The Electronic Journal of Linear Algebra, 9, (2002), 132-137.

[5] M. Eastwood, P. Norbury, *A prof of Atiyah's conjecture on configurations of four points in Euclidean's three-space*, Geometry and Topology, 5, 885-893, 2001.

[6] D. Svrtan, I. Urbiha, *Verification and Streigthening of the Atiyah-Sutcliffe conjecture for Several Types of configurations*, arXiv: math/0609174v1 [math.MG], 6 Sep 2006.

Despre vehicule cu roţi poligoane regulate

Wladimir Boskoff[1)], Cătălin Ghinea[2)]

Abstract. For a vehicle having as wheels regular polygon, we compute explicitelyy the path on which it should move so that the passengers will not be shaken.

Keywords: regular polygon, geometrical path, periodicity.
MSC : 51Nxx, 51N20, 51N99.

Să considerăm un poligon regulat cu latura de lungime 1 pe care să-l gândim ca pe una din roţile unui vehicul. Ne putem întreba cum arată drumul pe care trebuie să se deplaseze respectivul vehicul fără ca pasagerii săi să fie zdruncinaţi, adică astfel încât platforma vehiculului să se mişte rectiliniu şi uniform.

Drumul se va prezenta ca reuniunea unor arce de curbă de lungime 1, având capetele pe axa Ox, pentru că fiecare latură a poligonului trebuie să parcurgă un arc de curbă de lungime egală cu lungimea lui. Vom presupune că vehiculul nu alunecă pe respectiva curbă. Să notăm cu $\{x, f(x)|x \in \mathbb{R}\}$ reuniunea arcelor care reprezintă drumul. Este suficient să determinăm primul arc, celelalte rezultând din periodicitatea impusă de mişcarea roţilor vehiculului.

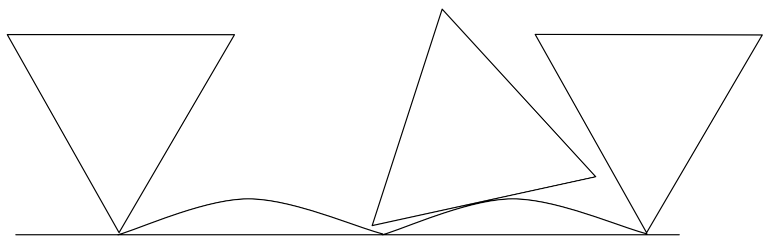


Figura 1

Roata poligon în poziţia iniţială (vezi figura 1) o presupunem având unul din vârfuri în originea sistemului de coordonate şi centrul de greutate pe axa Oy. Din presupunerea noastră, după parcurgerea primului arc, când vârful următor al roţii poligon se va afla pe axa Ox, roata poligon va avea laturile respectiv paralele cu laturile roţii poligon aflată în poziţia iniţială. Să trecem la determinarea drumului.

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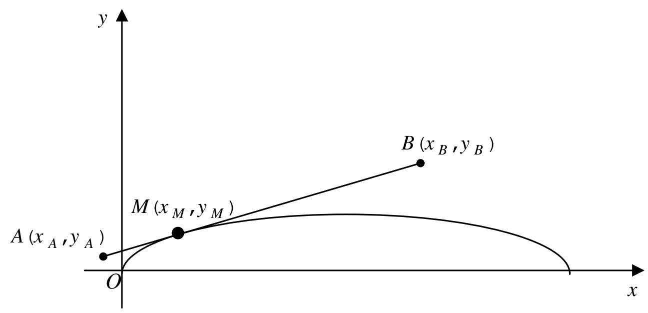

Figura 2

Fie $M\,(x, f(x))$ punctul curent aflat pe primul arc de curbă. Să considerăm latura AB a poligonului regulat, $A(x_A, y_A)$ şi $B(x_B, y_B)$, şi presupunem că dreapta AB este tangentă la primul arc de curbă în punctul curent $M(x, f(x))$ (vezi figura 2). Avem îndeplinite următoarele condiţii:

$$\begin{cases} y_A - f(x) = f'(x)\,(x_A - x) \\ y_B - f(x) = f'(x)\,(x_B - x) \\ y_B - y_A = f'(x)\,(x_B - x_A) \\ (y_B - y_A)^2 + (x_B - x_A)^2 = 1, \end{cases} \tag{1}$$

de unde

$$x_B - x_A = \frac{1}{\sqrt{1 + (f'(x))^2}}\,, \tag{2}$$

pentru că $x_B - x_A > 0$ şi

$$y_B - y_A = \frac{f'(x)}{\sqrt{1 + (f'(x))^2}}\,. \tag{3}$$

Lungimea arcului de curbă OM o notăm cu $L_{OM}^{f(x)}$,

$$L_{OM}^{f(x)} = \int\limits_0^x \sqrt{1 + (f'\,(t))^2}\mathrm{d}t.$$

Impunând condiţia $\|AM\| = L_{OM}^{f(x)}$ (egalitatea dintre AM şi lungimea drumului din poziţia iniţială până în poziţia curentă) obţinem:

$$(f(x) - y_A)^2 + (x - x_A)^2 = \left(L_{OM}^{f(x)}\right)^2. \tag{4}$$

Din (1) şi (4) avem:

$$(x - x_A)^2 \left(1 + \left(f'\,(t)\right)^2\right) = \left(L_{OM}^{f(x)}\right)^2,$$

de unde

$$x_A = x - \frac{L_{OM}^{f(x)}}{\sqrt{1+(f'(t))^2}}. \tag{5}$$

Folosind (1), (2), (3) şi (5) obţinem succesiv:

$$y_A = f(x) - \frac{f'(x)L_{OM}^{f(x)}}{\sqrt{1+(f'(t))^2}}, \tag{6}$$

$$x_B = x + \frac{1-L_{OM}^{f(x)}}{\sqrt{1+(f'(t))^2}}, \tag{7}$$

$$y_B = f(x) + \frac{f'(x)\left(1-L_{OM}^{f(x)}\right)}{\sqrt{1+(f'(t))^2}}. \tag{8}$$

Dacă centrul de greutate G al poligonului este de afix $z_G = x_G + \mathrm{i}y_G$ atunci condiţia de rostogolire va fi: coordonata y_G este constantă, mai precis:

$$y_G = \frac{1}{2\sin\left(\frac{\pi}{n}\right)}. \tag{9}$$

Pentru A de afix $z_A = x_A + \mathrm{i}y_A$ şi B de afix $z_B = x_B + \mathrm{i}y_B$ avem

$$\begin{aligned} z_G &= \frac{1}{2\sin\left(\frac{\pi}{n}\right)}(z_B - z_A)\left[\cos\left(\frac{\pi}{2}-\frac{\pi}{n}\right) + \mathrm{i}\sin\left(\frac{\pi}{2}-\frac{\pi}{n}\right)\right] + z_A = \\ &= \frac{1}{2\sin\left(\frac{\pi}{n}\right)}(z_B - z_A)\left[\sin\left(\frac{\pi}{n}\right) + \mathrm{i}\cos\left(\frac{\pi}{n}\right)\right] + z_A, \end{aligned}$$

de unde, folosind condiţia de rostogolire (9), obţinem ecuaţia:

$$\frac{1}{2}(y_B - y_A) + y_A + \frac{\cos\left(\frac{\pi}{n}\right)}{2\sin\left(\frac{\pi}{n}\right)}(x_B - x_A) = \frac{1}{2\sin\left(\frac{\pi}{n}\right)}. \tag{10}$$

Înlocuind în (10) cantităţile cunoscute şi efectuând reducerile, rezultă ecuaţia diferenţială a drumului:

$$\frac{f'(x)\left(\sin\left(\frac{\pi}{n}\right) - 2L_{OM}^{f(x)}\sin\left(\frac{\pi}{n}\right)\right) + \cos\left(\frac{\pi}{n}\right)}{\sqrt{1+(f'(t))^2}} + 2f(x)\sin\left(\frac{\pi}{n}\right) = 1.$$

Pentru rezolvarea acestei ecuaţii diferenţiale, să notăm cu $y = f(x)$. Ecuaţia devine

$$\frac{y'\left(\sin\left(\frac{\pi}{n}\right) - 2L_{OM}^{y}\sin\left(\frac{\pi}{n}\right)\right) + \cos\left(\frac{\pi}{n}\right)}{\sqrt{1+(y')^2}} + 2y\sin\left(\frac{\pi}{n}\right) = 1,$$

adică

$$y = \frac{1}{2\sin\left(\frac{\pi}{n}\right)}\left(1 - \frac{y'\left(\sin\left(\frac{\pi}{n}\right) - 2L_{OM}^{y}\sin\left(\frac{\pi}{n}\right)\right) + \cos\left(\frac{\pi}{n}\right)}{\sqrt{1+(y')^2}}\right).$$

Derivând ultima relaţie şi reducând termenii obţinem:

$$y''\left(\sin\left(\frac{\pi}{n}\right) - 2L_{OM}^{y}\sin\left(\frac{\pi}{n}\right) - y'\cos\left(\frac{\pi}{n}\right)\right) = 0.$$

Dacă $y'' = 0$ rezultă $y = ax + b$, soluţie ce nu convine pentru că roata nu alunecă. Rezultă că:

$$\sin\left(\frac{\pi}{n}\right) - 2L_{OM}^{y}\sin\left(\frac{\pi}{n}\right) - y'\cos\left(\frac{\pi}{n}\right) = 0$$

şi, prin derivare, ajungem la:

$$\frac{y''}{\sqrt{1+(y')^2}} = -2\tan\left(\frac{\pi}{n}\right).$$

Integrând relaţia, rezultă:

$$y' + \sqrt{1+(y')^2} = c'\mathrm{e}^{-2\tan\left(\frac{\pi}{n}\right)x},\ c' > 0,$$

deci:

$$y' = \frac{(c')^2\,\mathrm{e}^{-4\tan\left(\frac{\pi}{n}\right)x} - 1}{2c'\mathrm{e}^{-2\tan\left(\frac{\pi}{n}\right)x}} = \frac{c'}{2}\,\mathrm{e}^{-2\tan\left(\frac{\pi}{n}\right)x} - \frac{1}{2c'}\,\mathrm{e}^{2\tan\left(\frac{\pi}{n}\right)x}.$$

Pentru $c' = \mathrm{e}^{c}$, ajungem la forma mai simplă:

$$y' = \sinh\left(c - 2x\tan\left(\frac{\pi}{n}\right)\right)$$

care, prin integrare, ne conduce la:

$$y = -\frac{1}{2\tan\left(\frac{\pi}{n}\right)}\cosh\left(c - 2x\tan\left(\frac{\pi}{n}\right)\right) + c_1.$$

Folosind condiţiile iniţiale $y(0) = 0$ şi $y'(0) = \tan\left(\frac{\pi}{n}\right)$ determinăm constantele:

$$c = \ln\left(\frac{1+\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)}\right), \qquad c_1 = \frac{1}{\sin\left(\frac{\pi}{n}\right)}.$$

Curba de rostogolire pentru poligonul regulat cu n laturi este:

$$f(x) = -\frac{1}{2\tan\left(\frac{\pi}{n}\right)} \cosh\left(\ln\left(\frac{1+\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)}\right) - 2x\tan\left(\frac{\pi}{n}\right)\right) + \frac{1}{\sin\left(\frac{\pi}{n}\right)}.$$

Se observă că:

$$\lim_{n\to\infty}\left(-\frac{1}{2\tan\left(\frac{\pi}{n}\right)} \cosh\left(\ln\left(\frac{1+\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)}\right) - 2x\tan\left(\frac{\pi}{n}\right)\right) + \frac{1}{2\sin\left(\frac{\pi}{n}\right)}\right) = 0,$$

adică curba de rostogolire a cercului (poziţia limită a poligoanelor regulate) este $y = 0$.

Se poate observa că metoda de determinare este elementară şi diferă de abordarea din [1]. Mai mult, în [1], cazul poligonul regulat cu n laturi este doar schiţat.

Bibliografie

[1] Leon Hall and Stan Wagon, *Roads and wheels*, Mathematics Magazine 65 (1992), 283-301.

Olimpiada de Matematică a studenţilor din sud-estul Europei, SEEMOUS 2010

Cornel Băeţica[1], Gabriel Mincu[2]

Abstract. This note deals with the problems of the 4th South Eastern European Mathematical Olympiad for University Students, SEEMOUS 2010, organized by the Union of Bulgarian Mathematicians in Plovdiv, Bulgaria, between March 8 and March 13, 2010.

Keywords: Series of functions; Lebesgue integral; Representations of matrices; Egyptian fractions.

MSC : 15A24; 15A36; 26A42; 40A30; 51D20.

Cea de-a patra ediţie a Olimpiadei de Matematică a studenţilor din sud-estul Europei, SEEMOUS 2010, a fost organizată de Uniunea Matematicienilor din Bulgaria şi de Societatea de Matematică din Sud-Estul Europei în localitatea Plovdiv din Bulgaria în perioada 8-13 martie 2010. Au participat 66 de studenţi de la universităţi din Bulgaria, Grecia, Israel, Macedonia şi România. Studenţii români au avut o comportare remarcabilă reuşind să obţină 6 medalii de aur din cele 8 acordate, precum şi medalii de argint şi de bronz. Andrei Deneanu, student la Facultatea de Matematică şi Informatică a Universităţii din Bucureşti a obţinut, pe lângă medalia de aur, premiul special pentru o soluţie deosebită.

Concursul a avut o singură probă constând în patru probleme. Prezentăm mai jos aceste probleme însoţite de soluţii care în mare parte au apărut în lucrările concurenţilor. Pentru soluţiile oficiale facem trimitere la http://seemous2010.fmi-plovdiv.org.

Problema 1. Fie $f_0\colon [0,1] \to \mathbb{R}$ o funcţie continuă. Definim şirul de funcţii $f_n\colon [0,1] \to \mathbb{R}$ prin

$$f_n(x) = \int\limits_0^x f_{n-1}(t)\mathrm{d}t$$

pentru fiecare număr natural nenul n.

a) Arătaţi că pentru orice $x \in [0,1]$ seria $\sum\limits_{n=1}^{\infty} f_n(x)$ este convergentă.

b) Găsiţi o formulă explicită pentru suma seriei $\sum\limits_{n=1}^{\infty} f_n(x)$, $x \in [0,1]$.

* * *

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Aceasta a fost considerată de juriu drept o problemă uşoară. Studenţii care au rezolvat problema au dat soluţii în spiritul primei soluţii oficiale. Prezentăm mai jos această soluţie.

Soluţia 1. a) Cum funcţia f_0 este continuă şi definită pe un interval compact, ea este mărginită. Există deci $M > 0$ astfel încât $|f_0(x)| \leq M$ pentru orice $x \in [0,1]$. Se obţine

$$|f_1(x)| \leq \int_0^x |f_0(t)|\mathrm{d}t \leq Mx, \text{ pentru orice } x \in [0,1],$$

$$|f_2(x)| \leq \int_0^x |f_1(t)|\mathrm{d}t \leq M\frac{x^2}{2}, \text{ pentru orice } x \in [0,1].$$

Se demonstrează prin inducţie că pentru orice $n \in \mathbb{N}$ şi orice $x \in [0,1]$

$$|f_n(x)| \leq M\frac{x^n}{n!},$$

de unde

$$\sup_{x\in[0,1]} |f_n(x)| \leq \frac{M}{n!}.$$

Cum $\sum\limits_{n=1}^{\infty} \frac{1}{n!}$ este convergentă, rezultă că seria $\sum\limits_{n=1}^{\infty} f_n$ este uniform convergentă pe $[0,1]$, deci ea este convergentă şi punctual pe acest interval.

b) Fie $F\colon [0,1] \to \mathbb{R}$, unde F este suma seriei $\sum\limits_{n=1}^{\infty} f_n$. Cum pentru fiecare $n \in \mathbb{N}^*$ are loc relaţia $f_n' = f_{n-1}$, deducem că

$$\sum_{n=1}^{\infty} f_n' = \sum_{n=0}^{\infty} f_n.$$

Conform punctului a), seria din membrul drept este uniform convergentă, deci şi seria din membrul stâng are aceeaşi proprietate. Prin urmare, seria $\sum\limits_{n=1}^{\infty} f_n$ poate fi derivată termen cu termen şi obţinem relaţia $F' = F + f_0$. Această relaţie este echivalentă cu $(F(x)\mathrm{e}^{-x})' = f_0(x)\mathrm{e}^{-x}$ $\forall x \in [0,1]$. Cum $F(0) = 0$, deducem că

$$F(x) = \mathrm{e}^x \int_0^x f_0(t)\mathrm{e}^{-t}\mathrm{d}t.$$

(*Facem menţiunea că tehnici de această natură se folosesc, de pildă, la rezolvarea ecuaţiilor integrale de tip Volterra.*) □

Soluţia a 2-a.

$$f_n(x)=\int\limits_0^x\left(\int\limits_0^{t_1}\cdots\left(\int\limits_0^{t_{n-2}}\left(\int\limits_0^{t_{n-1}}f_0(t)\mathrm{d}t\right)\mathrm{d}t_{n-1}\right)\ldots\mathrm{d}t_2\right)\mathrm{d}t_1=$$

$$=\underset{0\le t\le t_{n-1}\le\cdots\le t_1\le x}{\int\int\ldots\int}f_0(t)\mathrm{d}t\mathrm{d}t_{n-1}\ldots\mathrm{d}t_1=\underset{0\le t_{n-1}\le t_{n-2}\le\cdots\le t_1\le t\le x}{\int\int\ldots\int}f_0(t_{n-1})\mathrm{d}t\mathrm{d}t_{n-1}\ldots\mathrm{d}t_1=$$

$$=\int\limits_0^x f_0(t_{n-1})\mathrm{d}t_{n-1}\int\limits_{t_{n-1}}^x\mathrm{d}t_{n-2}\cdots\int\limits_{t_2}^x\mathrm{d}t_1\int\limits_{t_1}^x\mathrm{d}t=\int\limits_0^x f_0(t)\frac{(x-t)^{n-1}}{(n-1)!}\mathrm{d}t.$$

Prin urmare,

$$\sum_{n=1}^N f_n(x)=\int\limits_0^x f_0(t)\left(\sum_{n=1}^N\frac{(x-t)^{n-1}}{(n-1)!}\right)\mathrm{d}t.$$

Se obţine

$$\left|\sum_{n=1}^N f_n(x)-\int\limits_0^x f_0(t)\mathrm{e}^{x-t}\mathrm{d}t\right|=\left|\int\limits_0^x f_0(t)\left(\sum_{n=1}^N\frac{(x-t)^{n-1}}{(n-1)!}\right)\mathrm{d}t-\int\limits_0^x f_0(t)\mathrm{e}^{x-t}\mathrm{d}t\right|.$$

Există însă $u\in(0,x-t)$ astfel încât

$$\mathrm{e}^{x-t}=\sum_{n=0}^N\frac{(x-t)^n}{n!}+\mathrm{e}^u\frac{(x-t)^{N+1}}{(N+1)!},$$

deci

$$\mathrm{e}^{x-t}-\sum_{n=0}^N\frac{(x-t)^n}{n!}=\mathrm{e}^u\frac{(x-t)^{N+1}}{(N+1)!}\le\mathrm{e}^{x-t}\frac{(x-t)^{N+1}}{(N+1)!}.$$

Obţinem aşadar

$$\left|\sum_{n=1}^N f_n(x)-\int\limits_0^x f_0(t)\mathrm{e}^{x-t}\mathrm{d}t\right|\le\int\limits_0^x|f_0(t)|\mathrm{e}^{x-t}\frac{(x-t)^{N+1}}{(N+1)!}\mathrm{d}t\le$$

$$\le\frac{1}{(N+1)!}\int\limits_0^x|f_0(t)|\,\mathrm{e}^{x-t}\mathrm{d}t.$$

Cum

$$\lim_{N\to\infty}\frac{1}{(N+1)!}\int\limits_0^x|f_0(t)|\mathrm{e}^{x-t}\mathrm{d}t=0,$$

obţinem de aici atât convergenţa cerută la punctul a), cât şi faptul că

$$\sum_{n=1}^{\infty} f_n(x) = \mathrm{e}^x \int_0^x f_0(t)\mathrm{e}^{-t}\mathrm{d}t.$$

(*Această soluţie interesantă a fost propusă de către Dmitri Mitin, membru al juriului, şi a fost adoptată ca a doua soluţie oficială.*) $\square$

Problema 2. În interiorul unui pătrat considerăm o mulţime de cercuri. Suma circumferinţelor cercurilor din mulţime este egală cu dublul perimetrului pătratului.

a) Găsiţi numărul minim de cercuri care satisfac această condiţie.

b) Arătaţi că există o infinitate de drepte care intersectează cel puţin 3 dintre cercuri.

Marcel Roman, România

Această problemă a fost considerată de juriu drept o problemă de dificultate medie. Studenţii care au rezolvat această problemă, cu o singură excepţie, au dat soluţii similare soluţiei oficiale. Prezentăm mai întâi această soluţie.

Soluţia 1. Fără a restrânge generalitatea, putem considera că latura pătratului este de lungime 1. Notăm diametrele cercurilor date cu d_1, d_2, $\ldots$, d_n, iar centrul pătratului cu C.

a) Condiţia din enunţ se rescrie $\pi \sum\limits_{k=1}^{n} d_k = 8$. Cum pentru fiecare k avem $d_k \leq 1$, din relaţia anterioară rezultă $8 \leq n\pi$, de unde $n \geq 3$. Considerând acum cercurile de centru C şi diametre $d_1 = \frac{7}{3\pi}$, $d_2 = \frac{8}{3\pi}$ şi $d_3 = \frac{3}{\pi}$, constatăm că $d_1 < d_2 < d_3 < 1$ şi $\pi(d_1 + d_2 + d_3) = 8$. Prin urmare, numărul minim de cercuri cu proprietatea din enunţ este 3.

b) Proiectăm cercurile pe una din laturile pătratului, pe care o considerăm orizontală şi o notăm cu $[AB]$. Notăm proiecţia cercului de diametru d_i cu S_i.

Presupunem că există doar un număr finit de verticale care taie cel puţin trei cercuri. Aceasta revine la a spune că oricum am alege trei segmente din mulţimea $\{S_1, S_2, \ldots, S_n\}$, intersecţia lor conţine cel mult un punct. Colorăm inductiv segmentele $S_1, S_2, \ldots, S_n$ în roşu sau verde după cum urmează: colorăm pe S_1 în verde. Pentru $k > 1$, presupunem colorate $S_1, S_2, \ldots, S_{k-1}$ şi avem două variante:

Dacă S_k intersectează ,,semnificativ" (adică, în mai mult de un punct) vreun segment $S_j, j < k$, atunci (acesta este unic şi) colorăm S_k folosind culoarea diferită de cea a lui S_j.

Dacă S_k nu intersectează ,,semnificativ" niciun segment $S_j, j < k$, atunci colorăm S_k în verde.

Conform construcţiei, orice două segmente verzi au cel mult un punct comun; prin urmare, $\bigcup\limits_{S_j \text{ e verde}} S_j = M \cup \bigcup\limits_{S_j \text{ e verde}} \mathring{S}_j$, unde M este o mulţime finită. (Precizăm că $\mathring{S}_j$ reprezintă interiorul mulţimii S_j.)

Cum însă $\mathring{S}_1, \mathring{S}_2, \ldots, \mathring{S}_n$ sunt disjuncte două câte două, rezultă că măsura *Lebesgue* a mulţimii $\bigcup\limits_{S_j \text{ e verde}} S_j \subset [AB]$ este egală cu $\sum\limits_{S_j \text{ e verde}} d_j$. Rezultă că $\sum\limits_{S_j \text{ e verde}} d_j \leq l$. Analog, $\sum\limits_{S_j \text{ e roşu}} d_j \leq l$. Prin urmare, $\sum\limits_{k=1}^{n} d_k < 2l$. Pe de altă parte, enunţul implică relaţia $\pi \sum\limits_{k=1}^{n} d_k = 8l$. Din aceste relaţii rezultă $\frac{4}{\pi} < 1$, contradicţie.

Rămâne aşadar că există o infinitate de verticale care intersectează cel puţin trei cercuri. $\square$

Soluţia a 2-a. b) Considerăm sistemul de coordonate carteziene în care vârfurile pătratului au coordonatele $(0,0), (0,1), (1,0)$ şi $(1,1)$. Fie funcţia $f\colon [0,1] \to \mathbb{R}$ definită prin $f(x) =$ numărul de cercuri pe care le taie verticala prin x. Deoarece f are un număr finit de puncte de discontinuitate, ea este integrabilă Riemann. $d_1, d_2, \ldots, d_n$ fiind, ca şi mai sus, diametrele cercurilor date, se arată prin inducţie după n că

$$\int\limits_0^1 f(x)\mathrm{d}x = \sum_{k=1}^{n} d_k.$$

Prin urmare, $\int\limits_0^1 f(x)\mathrm{d}x = \frac{8}{\pi}$.

Presupunem acum că mulţimea verticalelor care taie cel puţin 3 cercuri este finită. Atunci, $f(x) \leq 2$ pentru orice $x \in [0,1] \setminus M$, M fiind o mulţime finită. Rezultă de aici că

$$\frac{8}{\pi} = \int\limits_0^1 f(x)\mathrm{d}x \leq 2,$$

deci $\pi \geq 4$, contradicţie.

Rămâne aşadar că există o infinitate de verticale care taie cel puţin 3 cercuri din mulţimea dată.

(*Pentru această soluţie Andrei Deneanu a primit premiul special al juriului.*) $\square$

Remarcăm faptul că s-a considerat în mod tacit (atât în abordarea oficială, cât şi în cele ale concurenţilor) că mulţimea de cercuri aflate în interiorul pătratului este finită. Enunţul problemei are însă sens şi dacă mulţimea de cercuri este cel mult numărabilă (condiţia din enunţ, reformulată explicit pentru acest caz, cere ca seria circumferinţelor cercurilor date să fie convergentă, iar suma sa să fie dublul perimetrului pătratului).

Raţionamentul de la soluţia 1 rămâne valabil şi în noile condiţii cu două precauţii:

• Mulţimea M care diferenţiază pe $\bigcup\limits_{S_j \text{ e verde}} S_j$ de $\bigcup\limits_{S_j \text{ e verde}} \mathring{S}_j$ este cel mult numărabilă, deci măsura ei este tot zero, iar raţionamentul rămâne valabil.

• Seria diametrelor cercurilor date fiind absolut convergentă, ea este şi necondiţionat convergentă, deci

$$\sum_{S_j \text{ e verde}} d_j + \sum_{S_j \text{ e roşu}} d_j = \sum_{j=1}^{\infty} d_j.$$

Şi raţionamentul lui *Andrei Deneanu* funcţionează, cu precauţiile de rigoare, în cazul unei mulţimi numărabile de cercuri. Funcţia utilizată va fi de această dată $f\colon [0,1] \to \overline{\mathbb{R}}$ definită prin

$$f(x) = \begin{cases} \text{numărul de cercuri tăiate de verticala prin } x, \\ \qquad\qquad \text{dacă acest număr este finit} \\ +\infty, \quad \text{altfel.} \end{cases}$$

Constatăm că funcţia f este finită a.p.t. astfel: notăm

$$N = \{x \in [0,1] \,|\, x \text{ este pe proiecţia unei infinităţi de cercuri}\}.$$

Oricum am alege $n \in \mathbb{N}^*$, din definiţia lui N avem că orice $x \in N$ este pe proiecţiile a cel puţin n cercuri, deci aparţine unei mulţimi de tipul $S_{j_1} \cap S_{j_2} \cap \ldots \cap S_{j_n}$. Rezultă că $N \subset \bigcup\limits_{1 \le j_1 < j_2 < \ldots < j_n} S_{j_1} \cap S_{j_2} \cap \ldots \cap S_{j_n} \subset \bigcup\limits_{k \ge n} S_k$. De aici rezultă că măsura lui N este inferioară lui $\sum\limits_{k \ge n} d_k$. Dar $\lim\limits_{n\to\infty} \sum\limits_{k \ge n} d_k = 0$, deci N este neglijabilă *Lebesgue*.

Definim acum pentru fiecare $n \in \mathbb{N}^*$ funcţia $f_n\colon [0,1] \to \mathbb{R}$, $f_n(x) =$ = numărul de cercuri de diametre $\{d_1, d_2, \ldots, d_n\}$ pe care le taie verticala prin x. Observăm următoarele:

• Funcţiile f_n sunt, după cum s-a arătat în soluţia iniţială, integrabile *Riemann* (deci şi Lebesgue) pe $[0,1]$.

• Pentru orice $x \in [0,1] \setminus N$ avem $\lim\limits_{n\to\infty} f_n(x) = f(x)$. Cum N este neglijabilă, rezultă că $\lim\limits_{n\to\infty} f_n(x) = f(x)$ a.p.t.

Putem prin urmare aplica şirului $(f_n)_n$ teorema de convergenţă monotonă (*Beppo Levi*). Obţinem

$$\int\limits_{[0,1]} f = \int\limits_{[0,1]} \lim_{n\to\infty} f_n \overset{\text{Beppo Levi}}{=\!=} \lim_{n\to\infty} \int\limits_{[0,1]} f_n = \lim_{n\to\infty} \int_0^1 f_n(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} d_k = \frac{8}{\pi},$$

finitudinea rezultatului confirmând şi faptul că f este integrabilă *Lebesgue*.

Presupunem acum că mulţimea verticalelor care taie cel puţin 3 cercuri este finită. Atunci $f(x) \leq 2$ a.p.t. în [0,1]. Rezultă de aici că

$$\frac{8}{\pi} = \int_{[0,1]} f \leq 2,$$

deci $\pi \geq 4$, contradicţie.

Rămâne aşadar că există o infinitate de verticale care taie cel puţin 3 cercuri din mulţimea dată.

Problema 3. Notăm cu $\mathcal{M}_2(\mathbb{R})$ mulţimea matricelor 2×2 cu elemente reale. Demonstraţi că:
a) pentru orice $A \in \mathcal{M}_2(\mathbb{R})$ există $B, C \in \mathcal{M}_2(\mathbb{R})$ astfel încât $A = B^2 + C^2$;
b) nu există $B, C \in \mathcal{M}_2(\mathbb{R})$ astfel încât $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B^2 + C^2$ şi $BC = CB$.

Vladimir Babev, Bulgaria

Aceasta a fost considerată de juriu drept o problemă de dificultate medie. Concurenţii au dat mai multe soluţii, dar în linii mari s-a mers pe două idei: scrierea efectivă a matricei A ca sumă de două pătrate sau folosirea relaţiei Hamilton-Cayley.

a) **Soluţia 1.** Considerăm $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ şi scriem

$$A = \begin{pmatrix} m & b \\ -b & m \end{pmatrix} + \begin{pmatrix} a-m & 0 \\ b+c & d-m \end{pmatrix},$$

unde $m \in \mathbb{R}$ este ales astfel încât $a - m > 0$ şi $d - m > 0$. Ideea acestei scrieri a matricei A este aceea că prima matrice este o matrice corespunzătoare unei rotaţii înmulţită cu un număr real pozitiv (mai precis, $\sqrt{m^2+b^2}$), deci este în mod clar pătratul unei matrice reale de aceeaşi formă, iar cea de-a doua matrice este inferior triunghiulară cu valorile proprii strict pozitive, deci şi ea este pătratul unei matrice reale inferior triunghiulare. (*Soluţia aceasta a fost dată în concurs de către Ohad Livne din Israel.*) $\square$

Soluţia a 2-a. Această soluţie, găsită de cei mai mulţi dintre concurenţii care au rezolvat problema, s-a bazat pe observaţia că orice matrice scalară reală este pătratul unei matrice reale. Mai precis, avem $\lambda I_2 = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}^2$. (Să notăm că această scriere este necesară pentru cazul în care $\lambda < 0$, pentru $\lambda \geq 0$ având scrierea mult mai evidentă $\lambda I_2 = (\sqrt{\lambda} I_2)^2$). Plecând de la această observaţie se adună la matricea A o matrice scalară în aşa fel încât matricea $X = A + \lambda I_2$ să aibă urma şi determinantul pozitive. De aici, folosind relaţia *Hamilton-Cayley* pentru matricea X, obţinem că $X = \left(\frac{1}{\sqrt{\mathrm{tr} X}} X\right)^2 + \left(\frac{\det X}{\mathrm{tr} X}\right) I_2$. Rezultă imediat acum că A este sumă de două pătrate de matrice reale. $\square$

Mergând tot pe această idee, putem aduna la matricea A o matrice scalară în aşa fel încât matricea $X = A + \lambda I_2$ să aibă elementele de pe diagonala principală pozitive şi determinantul pozitiv, caz în care se arată, prin calcul direct (folosind un sistem de ecuaţii despre care se demonstrează că are soluţii reale), că matricea X este pătratul unei matrice reale.

Au mai existat şi alte abordări, bazate pe observaţia că proprietatea matricei A de a se scrie ca sumă de pătrate de două matrice reale se păstrează prin asemănare, aceasta permiţând folosirea de forme simplificate ale matricei A, de exemplu forma Jordan.

b) Soluţia cea mai simplă, găsită de majoritatea concurenţilor care au rezolvat problema, este cea în care se observă că deoarece $BC = CB$ avem $\det(B^2 + C^2) \geq 0$, pe când $\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$.

Se mai putea folosi faptul că dacă există o astfel de scriere, atunci matricele B, C comută cu matricea $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, fiind de forma $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$. Ridicând la pătrat o astfel de matrice vom obţine $\begin{pmatrix} x^2+y^2 & 2xy \\ 2xy & x^2+y^2 \end{pmatrix}$ şi adunându-le va rezulta că B şi C sunt matrice nule, contradicţie. $\square$

Problema 4. Presupunem că A şi B sunt matrice $n \times n$ cu elemente întregi şi $\det B \neq 0$. Arătaţi că există $m \in \mathbb{N}^*$ astfel încât

$$AB^{-1} = \sum_{k=1}^{m} N_k^{-1},$$

unde N_k sunt matrice $n \times n$ cu elemente întregi pentru orice $k = 1, \ldots, m$ şi $N_i \neq N_j$ pentru orice $i \neq j$.

Vladimir Todorov, Bulgaria

Problema a fost considerată de juriu drept o problemă dificilă şi rezultatele au arătat că această apreciere a fost corectă, doar şase concurenţi reuşind un punctaj apropiat de punctajul maxim. Nici unul dintre aceştia însă nu a mers pe ideea soluţiei oficiale.

Această problemă s-a dorit a fi o generalizare la cazul matricelor a reprezentării numerelor raţionale ca fracţii egiptene, i.e. ca sumă de fracţii distincte cu numărătorul 1 şi probabil că un enunţ mai sugestiv ar fi fost acela că orice matrice A cu elemente raţionale se scrie ca o sumă de inverse de matrice cu elemente întregi şi diferite două câte două. Faptul că în cazul $n = 1$ se obţine chiar reprezentarea numerelor raţionale ca fracţii egiptene este un indiciu important despre cum se poate aborda problema în cazul general.

Începem cu o observaţie pe care o vom folosi în primele două soluţii: putem presupune $B = I_n$, reducând astfel problema la a arăta că orice matrice cu elemente întregi se scrie ca o sumă ca cea din enunţ. De fapt $AB^{-1} = (\det B)^{-1} AB^*$ şi dacă ştim că matricele cu elemente întregi se scriu ca o sumă de inverse de matrice cu elemente întregi şi diferite două câte două, avem în particular că $AB^* = \sum\limits_{k=1}^{m} M_k^{-1}$ cu $M_i \neq M_j$ pentru $i \neq j$, de unde $AB^{-1} = \sum\limits_{k=1}^{m} ((\det B)M_k)^{-1}$.

Soluţia 1. Se consideră $A \in \mathcal{M}_n(\mathbb{Z})$. Se ştie că o astfel de matrice se poate aduce prin transformări elementare la o formă diagonală. Mai precis, există $U, V \in \mathcal{M}_n(\mathbb{Z})$ inversabile astfel încât $UAV = \mathrm{diag}(d_1, \ldots, d_r, 0, \ldots, 0)$ cu $d_1, \ldots, d_r \in \mathbb{N}^*$ şi $d_1 | \ldots | d_r$, unde prin $\mathrm{diag}(d_1, \ldots, d_r, 0, \ldots, 0)$ am notat matricea diagonală $n \times n$ care are pe diagonala principală elementele $d_1, \ldots, d_r, 0, \ldots, 0$ (în această ordine).

Dacă arătăm că putem scrie $\mathrm{diag}(d_1, \ldots, d_r, 0, \ldots, 0) = \sum\limits_{k=1}^{m} M_k^{-1}$ cu $M_i \neq M_j$ pentru $i \neq j$, atunci vom avea $A = \sum\limits_{k=1}^{k} (VM_kU)^{-1}$ şi am terminat.

Rămâne deci să arătăm că o matrice de forma $\mathrm{diag}(d_1, \ldots, d_r, 0, \ldots, 0)$ se poate scrie ca o sumă ca cea din enunţ. Folosind reprezentarea numerelor raţionale ca fracţii egiptene scriem $d_r = \frac{1}{t_1} + \cdots + \frac{1}{t_m}$ cu $t_i \neq t_j$ pentru orice $i \neq j$. Cum $d_k | d_r$ pentru orice $1 \leq k \leq r$, avem $d_r = d_k s_k$ cu $s_k \in \mathbb{N}^*$ (evident $s_r = 1$) şi de aici rezultă că $d_k = \frac{1}{s_k} d_r$, aşadar $d_k = \frac{1}{s_k t_1} + \cdots + \frac{1}{s_k t_m}$ şi $s_k t_i \neq s_k t_j$ pentru orice $i \neq j$.

Dacă $r = n$, atunci scriem

$$\mathrm{diag}(d_1, \ldots, d_n) = \sum_{k=1}^{m} \mathrm{diag}(s_1 t_i, \ldots, s_n t_i)^{-1}.$$

Pentru $r < n$ considerăm trei cazuri:

Cazul I: m este număr par. Scriem $m = 2p$ şi atunci vom avea

$$\mathrm{diag}(d_1, \ldots, d_r, 0, \ldots, 0) =$$

$$= \sum_{i=1}^{p} [\mathrm{diag}(s_1 t_{2i-1}, \ldots, t_{2i-1}, 1, \ldots, 1)^{-1} + \mathrm{diag}(s_1 t_{2i}, \ldots, t_{2i}, -1, \ldots, -1)^{-1}].$$

Cazul II: m este număr impar, $m \geq 3$. Scriem $m = 2p + 1$ cu $p \geq 1$ şi apoi

$$\mathrm{diag}(d_1, \ldots, d_r, 0, \ldots, 0) =$$

$$= \sum_{i=1}^{p-1} [\mathrm{diag}(s_1 t_{2i-1}, \ldots, s_r t_{2i-1}, 1, \ldots, 1)^{-1} + \mathrm{diag}(s_1 t_{2i}, \ldots, s_r t_{2i}, -1, \ldots, -1)^{-1}] +$$

$$+ \mathrm{diag}(s_1 t_{2p-1}, \ldots, s_r t_{2p-1}, 1, \ldots, 1)^{-1} + \mathrm{diag}(s_1 t_{2p}, \ldots, s_r t_{2p}, -2, \ldots, -2)^{-1} +$$

$$+ \mathrm{diag}(s_1 t_{2p+1}, \ldots, s_r t_{2p+1}, -2, \ldots, -2)^{-1}.$$

Cazul III: $m = 1$. Atunci $d_1 = \cdots = d_r = 1$ şi scriem

$$\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0) = \mathrm{diag}(2, \ldots, 2, 1, \ldots, 1)^{-1} + \mathrm{diag}(2, \ldots, 2, -1, \ldots, -1)^{-1}.$$

Ideile principale din această soluţie s-au regăsit în lucrările lui Flavian Georgescu şi Ohad Livne. □

Soluţia a 2-a. Arătăm că orice matrice $A \in \mathcal{M}_n(\mathbb{Z})$, $n \geq 2$, se scrie sub forma $A = \sum\limits_{k=1}^{m} X_k$ cu $X_k \in \mathcal{M}_n(\mathbb{Z})$, $\det X_k = \pm 1$ pentru orice $1 \leq k \leq m$ şi $X_i \neq X_j$ pentru orice $i \neq j$. Pe scurt, orice matrice de ordin $n \geq 2$ cu elemente întregi se scrie ca o sumă de matrice cu elemente întregi, inversabile şi distincte două câte două. Dacă ţinem cont de faptul că pentru astfel de matrice $X_k^{-1} \in \mathcal{M}_n(\mathbb{Z})$, atunci $A = \sum\limits_{k=1}^{m} N_k^{-1}$, unde $N_k = X_k^{-1}$ şi rezolvarea se încheie.

Să demonstrăm acum afirmaţia de mai sus. Pentru început vom nota cu E_{ij} matricea $n \times n$ care are 1 pe poziţia (i,j) şi 0 în rest. Scriem $A = (a_{ij})_{i,j=1,\ldots,n}$ şi atunci vom avea $A = \sum a_{ij}E_{ij}$ (se consideră că în sumă sunt scrişi doar termenii pentru care $a_{ij} \neq 0$). Mai mult, putem scrie

$$A = \sum_{i<j}(a_{ij}E_{ij} + I_n) + \sum_{i>j}(a_{ij}E_{ij} + I_n) + mI_n + \sum a_{ii}E_{ii}.$$

Matricele $a_{ij}E_{ij} + I_n$, $i \neq j$, sunt inversabile şi distincte două câte două. Observăm că $mI_n + \sum a_{ii}E_{ii} = \sum\limits_{i=1}^{n}(m + a_{ii})E_{ii}$ şi rămâne să arătăm că o matrice nenulă de forma aE_{ii} se scrie ca o sumă de matrice cu elemente întregi, inversabile, distincte două câte două şi diferite de cele de forma $a_{ij}E_{ij} + I_n$, $i \neq j$. Dacă $a = 2k$, $k \geq 1$, atunci scriem $2kE_{ii} = \sum\limits_{j=1}^{k}(2E_{ii} - I_n + \alpha_j E_{1n}) + \sum\limits_{j=1}^{k}(I_n - \alpha_j E_{1n})$ cu $\alpha_j \in \mathbb{Z}$ aleşi corespunzător. Analog dacă $k \leq -1$. Dacă $a = 2k+1$, atunci $(2k+1)E_{ii} = 2kE_{ii} + E_{ii}$ şi rămâne să scriem matricea E_{ii} ca o sumă de matrice cu elemente întregi, inversabile, distincte două câte două şi diferite de cele folosite anterior. Notăm cu J_n matricea $n \times n$ care are 1 pe diagonala secundară şi 0 în rest. Avem scrierea $E_{ii} = (2E_{ii} - J_n) + (-E_{ii} + J_n)$ care este convenabilă cu excepţia cazului $n = 2p+1$ şi $i = p+1$, caz în care vom scrie $E_{p+1,p+1} = I_n - \sum\limits_{i \neq p+1} E_{ii}$ şi apoi folosim scrierea anterioară pentru matricele E_{ii}, $i \neq p+1$. (*Această soluţie, cu mici diferenţe de detaliu, a fost găsită în concurs de către Daniel Drimbe.*) □

Soluţia a 3-a. O matrice $M \in \mathcal{M}_n(\mathbb{Z})$ se numeşte „bună“ dacă se scrie sub forma $M = \sum\limits_{k=1}^{m} N_k^{-1}$, unde $m \in \mathbb{N}^*$ iar N_k sunt matrice $n \times n$ cu elemente întregi şi $N_i \neq N_j$ pentru orice $i \neq j$.

Să observăm că dacă $M_1, \ldots, M_r \in \mathcal{M}_n(\mathbb{Z})$ sunt matrice bune, atunci oricare ar fi $q_1, \ldots, q_r \in \mathbb{Q}$ rezultă că matricea $q_1M_1 + \cdots + q_rM_r$ este bună. În acest scop scriem matricele $M_1, \ldots, M_r$ ca sume de inverse de matrice cu elemente întregi diferite două câte două şi obţinem în final o scriere de forma

$$q_1M_1 + \cdots + q_rM_r = \sum q_i' N_i^{-1}$$

cu $N_i \neq qN_j$ oricare ar fi $q \in \mathbb{Q}$ şi $i \neq j$. Reprezentăm acum numerele raţionale q_i' ca fracţii egiptene şi obţinem astfel scrierea dorită.

Matricea I_n şi matricele $I_n - \frac{1}{2}E_{ii}$, $1 \leq i \leq n$, sunt în mod evident matrice bune. Matricele E_{ij}, $i \neq j$, sunt bune pentru că putem scrie, de exemplu,

$$E_{ij} = (I_n + E_{ij}) + (-I_n).$$

Fie A, B ca în enunţ. Notăm $AB^{-1} = (c_{ij})_{i,j=1,\ldots,n}$ cu $c_{ij} \in \mathbb{Q}$, scriem

$$AB^{-1} = \sum_{i \neq j} c_{ij}E_{ij} - \sum_i 2c_{ii}(I_n - \frac{1}{2}E_{ii}) + \sum_i 2c_{ii}I_n$$

şi acum problema este rezolvată. (*Această soluţie a fost dată în concurs de către Andrei Deneanu.*) □

A refinement of the Finsler-Hadwiger reverse inequality

Cezar Lupu[1)], Constantin Mateescu[2)],
Vlad Matei[3)], Mihai Opincariu[4)]

Abstract. In this short note we shall give a refinement of the celebrated Finsler-Hadwiger reverse inequality which states that in any triangle ABC with sides of lenghts a, b, c the following inequality is valid

$$a^2 + b^2 + c^2 \leq 4S\sqrt{3} + k[(a-b)^2 + (b-c)^2 + (c-a)^2],$$

where S denotes the area of the triangle ABC for $k = 3$. We refine the above inequality by proving that the inequality remains true for $k = 2$ and we also show that this new improvement fails in the case when the triangle ABC is not acute angled.

Keywords: Geometric inequality, Finsler-Hadwiger inequality, Popoviciu's inequality.

MSC : 51Mxx, 51Nxx, 51Axx, 26D15

1. Introduction & Main results

Many of the most important results in theory of geometric inequalities were discovered by the beginning of the 20-th century. The first important inequality was discovered in 1919 and it is due to *Weitzenbock* (see [2]), namely

Theorem 1.1. *In any triangle ABC with sides of lengths a, b, c, the following inequality holds*

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3},$$

where S denotes the area of the triangle ABC.

The above theorem also appeared in International Mathematical Olympiad in 1961 and many proofs of it can be found in [1]. A refinement of Theorem 1.1 is the *Finsler-Hadwiger* inequality

Theorem 1.2. *In any triangle ABC with sides of lengths a, b, c, the following inequality holds*

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3} + (a-b)^2 + (b-c)^2 + (c-a)^2,$$

where S denotes the area of the triangle ABC.

Many proofs of Theorem 1.2 can be found in [1] as well as other new proofs in [8], [5] or [6]. The reverse of Theorem 1.2 to which we give a simple proof in what follows, states that.

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Theorem 1.3. *In any triangle ABC with sides of lengths a, b, c, the following inequality holds*

$$a^2 + b^2 + c^2 \leq 4S\sqrt{3} + 3[(a-b)^2 + (b-c)^2 + (c-a)^2],$$

where S denotes the area of the triangle ABC.

Proof. Let us denote by $a = y + z, b = z + x$ and $c = x + y$ where $x, y, z > 0$. The area is given by the formula $S = \sqrt{xyz(x+y+z)}$. Our inequality becomes

$$(x+y)^2 + (y+z)^2 + (z+x)^2 \leq 4\sqrt{3xyz(x+y+z)} + 3\sum_{cyc}(x-y)^2$$

which is equivalent after a few calculations with

$$2\sum_{cyc} xy - \sum_{cyc} x^2 \leq \sqrt{3xyz(x+y+z)}.$$

But, from *Schur*'s inequality (see [7]), i.e.

$$\left(\sum_{cyc} x^3\right) + 3xyz \geq \sum_{cyc} xy(x+y)$$

one can easily deduce that

$$2\sum_{cyc} xy - \sum_{cyc} x^2 \leq \frac{9xyz}{x+y+z}.$$

Now, we are only left to prove that $(x+y+z)^3 \geq 27xyz$, which follows immediately from AM-GM inequality. $\square$

We shall prove that in the case of an acute-angled triangle Theorem 1.3 can be improved in the following sense:

Theorem 1.4. *In any acute-angled triangle ABC with sides of lengths a, b, c, the following inequality holds*:

$$a^2 + b^2 + c^2 \leq 4S\sqrt{3} + 2[(a-b)^2 + (b-c)^2 + (c-a)^2],$$

where S denotes the area of the triangle ABC.

In order to prove our main result we shall use the following lemmas due to *Popoviciu* (see [4]) and *Walker* (see [9]), i.e.

Lemma 1.5. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex function. For any $x, y, z \in I$ the following inequality holds*:

$$\frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x+y+z}{3}\right) \geq$$

$$\geq \frac{2}{3}\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right].$$

Lemma 1.6. *In any acute triangle ABC with sides of lengths a, b, c and with circumradius R and inradius r the following inequality holds*:
$$a^2 + b^2 + c^2 \geq 4(R + r)^2.$$

First proof of Theorem 1.4. By expanding, the given inequality is equivalent to
$$a^2 + b^2 + c^2 \leq 4S\sqrt{3} + 2[2(a^2 + b^2 + c^2) - 2(ab + bc + ca)],$$
which is equivalent to
$$3(a^2 + b^2 + c^2) + 4S\sqrt{3} \geq 4(ab + bc + ca),$$
or
$$4(ab + bc + ca) - 2(a^2 + b^2 + c^2) \leq a^2 + b^2 + c^2 + 4S\sqrt{3}. \qquad (*)$$
On the other hand, by the cosine law, we have
$$a^2 = b^2 + c^2 - 2bc\cos A = (b - c)^2 + 2bc(1 - \cos A) = (b - c)^2 + 4S\frac{1 - \cos A}{\sin A} =$$
$$= (b - c)^2 + 4S\tan\frac{A}{2},$$
where we used the well-known formulae $1 - \cos\alpha = 2\sin^2\frac{\alpha}{2}$ and $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$. Thus, we obtain
$$a^2 + b^2 + c^2 = (a - b)^2 + (b - c)^2 + (c - a)^2 + 4S\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right),$$
which is equivalent to
$$2(ab + bc + ca) - (a^2 + b^2 + c^2) = 4S\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right). \qquad (**)$$
Now, by $(*)$ and $(**)$, we have
$$8S\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right) \leq a^2 + b^2 + c^2 + 4S\sqrt{3},$$
which is equivalent to
$$2\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right) \leq \frac{a^2 + b^2 + c^2}{4S} + \sqrt{3}.$$
Again, by the cosine and sine law, we have
$$\cot A + \cot B + \cot C = \frac{R(b^2 + c^2 - a^2) + R(c^2 + a^2 - b^2) + R(a^2 + b^2 - c^2)}{abc},$$
where R denotes the circumradius of the triangle ABC. Since $R = \frac{abc}{4S}$, we get
$$\cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4S}.$$

Now, our inequality finally reduces to

$$\cot A + \cot B + \cot C + \sqrt{3} \geq 2\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right).$$

Let us consider the function $f(x) = \cot x$ where $f : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$. A simple calculation of the second derivative shows that f is convex. By applying lemma 1.5 for $f(x) = \cot x$, we obtain

$$\frac{1}{3}(\cot A + \cot B + \cot C) + \cot\left(\frac{A+B+C}{3}\right) \geq \frac{2}{3}\sum_{cyc}\cot\left(\frac{A+B}{2}\right),$$

which is equivalent to

$$\cot A + \cot B + \cot C + \sqrt{3} \geq 2\left(\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}\right),$$

exactly what we wanted to prove.

Second proof of Theorem 1.4. The conclusion rewrites as

$$4(a^2+b^2+c^2) - 4(ab+bc+ca) \geq a^2+b^2+c^2 - 4S\sqrt{3},$$

which is equivalent to

$$3(a^2+b^2+c^2) + 4S\sqrt{3} \geq 4(ab+bc+ca).$$

Since $ab+bc+ca = s^2+4Rr+r^2$ and $a^2+b^2+c^2 = 2(s^2-4Rr-r^2)$, our inequality is equivalent to

$$s^2 + 2S\sqrt{3} \geq 20Rr + 5r^2,$$

where s is the semiperimeter of the triangle ABC. By *Gerretsen*'s inequality, $s^2 \geq 16Rr - 5r^2$, we have $s^2 + 2S\sqrt{3} \geq 16Rr - 5r^2 + 2S\sqrt{3}$. Now, we are left to prove

$$16Rr - 5r^2 + 2S\sqrt{3} \geq 20Rr + 5r^2$$

which is succesively equivalent to

$$S\sqrt{3} \geq 2Rr + 5r^2,$$

$$s\sqrt{3} \geq 2R + 5r$$

and by squaring we have

$$3s^2 \geq 4R^2 + 20Rr + 25r^2.$$

Applying the Lemma 1.6 written in the form $s^2 \geq 2R^2 + 8Rr + 3r^2$, we deduce $3s^2 \geq 6R^2 + 24Rr + 9r^2$. In this moment, it is enoungh to prove that

$$6R^2 + 24Rr + 9r^2 \geq 4R^2 + 20Rr + 25r^2$$

which is nothing else that $(R-2r)(R+4r) \geq 0$. This is evident by *Euler*'s inequality. $\square$

Remark. The inequality does not hold in any obtuze triangle due to the following counterexample:

Let there be an isosceles triangle ABC with $AB = AC = x$ and $BC = 1$. We consider it to be obtuse in A so $AB^2 + AC^2 < BC^2$ and thus $x < \frac{1}{\sqrt{2}}$. Also to exist such a triangle x must fulfill $x > \frac{1}{2}$.

For the inequality to hold we must have that $\forall\, x \in \left(\frac{1}{2}; \frac{1}{\sqrt{2}}\right)$, $2x^2 + 1 \leq$ $\leq 4S\sqrt{3} + 4(x-1)^2$. In our case it is easy to see that $4S\sqrt{3} = \sqrt{3(4x^2-1)}$ and thus the inequality is equivalent to $-3x^2 + 8x - 3 \leq \sqrt{3\,(4x^2-1)}$. It is obvious that for $x = 0,55$ the inequality fails and this is a counterexample.

Thus, we think that the constant $k = 2$ is not optimal. Concerning this matter, we formulate the following:

Conjecture. *Find the optimal constant k such that in any acute-angled triangle ABC with sides of lengths a, b, c and area S, the following inequality is valid:*

$$a^2 + b^2 + c^2 \leq 4S\sqrt{3} + k[(a-b)^2 + (b-c)^2 + (c-a)^2].$$

In our research we have found that the constant $k = \dfrac{2-\sqrt{3}}{3-2\sqrt{2}}$ is optimal and it is attained for a right angled isosceles triangle, but we do not know if the reversed *Finsler-Hadwiger* inequality holds true in this case.

References

[1] A. Engel, *Problem Solving Strategies*, Springer Verlag, 1998.

[2] R. Weitzenbock, *Uber eine Ungleichung in der Dreiecksgeometrie*, Math. Z., **5**(1919), 137–146.

[3] P. Finsler, H. Hadwiger, *Einige Relationen im Dreieck*, Comm. Mat. Helv., **10**(1937), 316–326.

[4] S. Savchev, T. Andreescu, *Mathematical Miniatures*, Mathematical Association of America, 2002.

[5] C. Alsina, R. Nelsen, *Geometric proofs of the Weitzenbock and Finsler-Hadwiger inequality*, Math. Mag., **81**(2008), 216–219.

[6] D. Marinescu, M. Monea, M. Opincariu, M. Stroe, *A sequence of triangles and geometric inequalities*, Forum Geom., **9**(2009), 291–295.

[7] G. N. Watson, *Schur's inequality*, The Math. Gaz., **39**(1955), 208–209.

[8] C. Lupu, *An elementary proof of the Hadwiger-Finsler inequality*, Arhimede, **3**(2003), 18–19.

[9] C. Lupu, *Asupra Inegalităţii lui Gerretsen*, Revista de Matematică din Timişoara, **11** (2006), 3–10.

A note on two convergence tests

JÓZSEF SÁNDOR[1)]

Abstract. In this note we present a relation between *Cauchy*'s second test and *Raabe-Duhamel-Bolyai*'s test, by improving a result by *M. S. Jovanović* [2].

Keywords: series, convergence

MSC : 40A05

1. Introduction

In the theory of series with positive terms there are used two convergence tests that depend on the following limits:

$$\lim_{n\to\infty} \frac{\log\left(\dfrac{1}{a_n}\right)}{\log n} = \lambda_1 \quad (\textit{Cauchy}\text{'s second test}), \tag{1}$$

and

$$\lim_{n\to\infty} \left(1 - \frac{a_{n+1}}{a_n}\right) = \lambda_2 \quad (\textit{Raabe-Duhamel-Bolyai}\text{'s test}), \tag{2}$$

where a_n is the nth term in the series, and log denotes natural logarithm. For the association of (2) with the name of *Farkas Bolyai* (the father of *János Bolyai*), see the historical discoveries by *B. Szénássy* [7] (see also the recent paper by the author [6]).

It is a known result (see [4]) that if λ_2 exists, finite or infinite, then λ_1 exists and $\lambda_1 = \lambda_2$. The two tests are then equivalent, the first (1) being more general than the second (2).

In 2005 *M. S. Jovanović* [2] proved the following result:

Theorem 1. *Let (a_n) be a sequence in $\mathbb{R}$ such that $a_n > 0$ for all $n \in \mathbb{N}$. Then*

$$\liminf R_n \le \liminf C_n \le \limsup C_n \le \limsup R_n, \tag{3}$$

where

$$R_n := n \cdot \left(1 - \frac{a_{n+1}}{a_n}\right) \quad \text{and} \quad C_n := \frac{\log\left(\dfrac{1}{a_n}\right)}{\log n}.$$

The aim of this note is to provide a new proof of a stronger version of Theorem 1.

2. Main result

Theorem 2. *Let (a_n), (R_n), (C_n) be defined as in Theorem 1. Then one has*

$$\begin{gathered}\liminf R_n \le \liminf D_n \le \liminf C_n \le \limsup C_n \le \\ \le \limsup D_n \le \limsup R_n,\end{gathered} \tag{4}$$

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where

$$D_n := n \log \frac{a_n}{a_{n+1}}.$$

Proof. We shall use the extended version of the *Stolz-Cesàro* theorem (see e.g. [1], [6]) as follows: Let (x_n), (y_n) be two sequences of real numbers, with $y_n \nearrow +\infty$ as $n \to \infty$. Then

$$\liminf \frac{x_{n+1} - x_n}{y_{n+1} - y_n} \leq \liminf \frac{x_n}{y_n} \leq \limsup \frac{x_n}{y_n} \leq \limsup \frac{x_{n+1} - x_n}{y_{n+1} - y_n}. \tag{5}$$

Put in (5) $y_n = \log n$ and $x_n = \log \dfrac{1}{a_n}$. Then we get

$$\liminf \left(-n \log \frac{a_{n+1}}{a_n} \right) \leq \liminf \frac{\log \frac{1}{a_n}}{\log n} \leq$$
$$\leq \limsup \frac{\log \frac{1}{a_n}}{\log n} \leq \limsup \left(-n \log \frac{a_{n+1}}{a_n} \right), \tag{6}$$

where we have used that $n \log \left(1 + \dfrac{1}{n} \right) \to 1$ as $n \to \infty$.

As $-n \log \dfrac{a_{n+1}}{a_n} = D_n$; it will be sufficient to prove that

$$l = \liminf R_n \leq \liminf D_n \text{ and } \limsup D_n \leq \limsup R_n. \tag{7}$$

We will prove only the first part of (7), the second one being completely analogous.

(i) Assume first that l is finite. Then for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$n \left(1 - \frac{a_{n+1}}{a_n} \right) > l - \varepsilon = l_1 \quad \text{for} \quad n \geq n_0.$$

This may be written also as

$$\frac{a_n}{a_{n+1}} > \frac{n}{n - l_1},$$

so

$$n \log \frac{a_n}{a_{n+1}} > n \log \frac{n}{n - l_1} = \log \left(\frac{n}{n - l_1} \right)^n. \tag{8}$$

As

$$\lim_{n \to \infty} \left(\frac{n}{n - l_1} \right)^n = \lim_{n \to \infty} \left[\left(1 + \frac{l_1}{n - l_1} \right)^{\frac{n - l_1}{l_1}} \right]^{\frac{n l_1}{n - l_1}} = \mathrm{e}^{l_1},$$

we get from (8) that

$$\liminf D_n \geq \log \mathrm{e}^{l_1} = l_1 = l - \varepsilon,$$

which implies $\liminf D_n \geq l = \liminf R_n$; i.e. the left side of (7).

(ii) When $l = +\infty$, clearly $R_n \to \infty$ as $n \to \infty$. Thus for any $M > 0$

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) > M \quad \text{for all} \quad n \geq k\ (= k(M)),$$

implying as above that

$$\liminf D_n > M,$$

so, M being arbitrary, $\lim D_n = +\infty$.

(iii) When $l = -\infty$, there is nothing to prove.

Remark. We note that the proof of the Theorem 1, presented in [2] is much more complicated.

References

[1] K. Knopp, *Theory and application of infinite series*, London and Glasgow, 1954.

[2] M. S. Jovanović, *On two convergence tests*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. **16**(2005), 139–141.

[3] T. J. I'A.Bromwich, *An introduction to the theory of infinite series,* AMS Chelsea Publishing, Third ed., 1991.

[4] J. Michelow, *A note on two convergence tests*, Amer. Math. Monthly, **67**(1960), 581–583.

[5] J. Sándor, *On sequences, series and applications in prime number theory* (Romanian), Gazeta Mat. A. **6**(1985), No. 1–2, 38–48.

[6] J. Sándor and R. Oláh-Gál, *On the work of Farkas Bolyai in the theory of infinite series* (Hungarian), to appear Matematikai Lapok, Budapest.

[7] B. Szénássy, *The history of Hungarian mathematics up to the beginning of the XXth century* (Hungarian), Third ed., Polygon, Szeged, 2008.

PROBLEMS

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PROPOSED PROBLEMS

295. *Proposed by Vlad Matei, student University of Bucharest, Bucharest, Romania.* Determine all nonconstant polynomials $P \in \mathbb{Z}[X]$ such that $P(p)$ is square-free for all prime numbers p.

296. *Proposed by Alin Gălăţan, student University of Bucharest, Bucharest, Romania.* Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n > 0$ and $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ be real numbers. Show that

$$\left(\sum_{1\le i,j\le n} x_i x_j \min(a_i, a_j)\right)\left(\sum_{1\le i,j\le n} y_i y_j \min(b_i, b_j)\right) \ge \left(\sum_{1\le i,j\le n} x_i y_j \min(a_i, b_j)\right).$$

297. *Proposed by Marius Cavachi, Ovidius University of Constanţa, Constanţa, Romania.* Let S^2 be the bidimensional sphere and $\alpha > 0$. Show that for any positive integer n and $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$ and $C_1, C_2, \ldots, C_n$ arbitrary points on S^2, there exists $P_n \in S^2$ such that

$$\sum_{i=1}^{n} P_n A_i^{\alpha} = \sum_{i=1}^{n} P_n B_i^{\alpha} = \sum_{i=1}^{n} P_n C_i^{\alpha}$$

if and only if $\alpha = 2$.

298. *Proposed by Octavian Ganea, student École Polytechnique Federale de Lausanne, Lausanne, Switzerland* and *Cristian Tălău, student Polytechnic University of Bucharest, Bucharest, Romania.* Let t be an odd number. Find all monic polynomials $P \in \mathbb{Z}[X]$ such that for all integers n there exists an integer m for which $P(m) + P(n) = t$.

299. *Proposed by Gabriel Dospinescu, École Normale Supérieure de Paris, Paris, France* and *Fedja Nazarov, University of Wisconsin, Madison,* *U.S.A.* Find all functions $f : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$ satisfying the following properties:

i) $a - b$ divides $f(a) - f(b)$ for all $a, b \in \{1, 2, \ldots\}$;

ii) if a, b are relatively prime, so are $f(a)$ and $f(b)$.

300. *Proposed by Alin Gălățan, student University of Bucharest, Bucharest* and *Cezar Lupu, student University of Bucharest, Bucharest, Romania.* Consider the sequence $(a_n)_{n\geq 1}$ defined by $a_1 = 2$ and $a_{n+1} = 2a_n + \sqrt{3(a_n^2 - 1)}$, $n \geq 1$. Show that the terms of a_n are positive integers and for any odd prime number p divides $a_p - 2$.

301. *Proposed by Victor Vuletescu, University of Bucharest, Bucharest, Romania.* Determine the greatest prime number $p = p(n)$ such that there exists a matrix $X \in \mathrm{SL}(n, \mathbb{Z})$, $X \neq I_n$ with $X^p = I_n$.

302. *Proposed by Remus Nicoară, University of Tennessee, Knoxville, Tennessee, U.S.A.* Let $n \geq 2$ and denote by $\mathcal{D} \subset M_n(\mathbb{C})$ and $\mathcal{C} \subset M_n(\mathbb{C})$ the diagonal and circulant set of matrices.

Consider $\mathcal{V} = [D, C] = \mathrm{span}\{dc - cd; d \in \mathcal{D}, c \in \mathcal{C}\}$. Prove that $\mathcal{C}, \mathcal{D}$ and $\mathcal{V}$ span $M_n(\mathbb{C})$ if and only if n is prime.

303. *Proposed by Cezar Lupu, student University of Bucharest, Bucharest* and *Tudorel Lupu, Decebal High School, Constanța, Romania.* Prove that

$$\sum_{n=0}^{\infty} \frac{1}{\Gamma\left(n + \dfrac{3}{2}\right)} = \frac{2\mathrm{e}}{\sqrt{\pi}} \int_0^1 \mathrm{e}^{-x^2} \mathrm{d}x,$$

and

$$\sum_{n=0}^{\infty} \frac{\Gamma\left(-n + \dfrac{1}{2}\right)\Gamma\left(n + \dfrac{1}{2}\right)}{\Gamma\left(n + \dfrac{3}{2}\right)} = \frac{\sqrt{\pi}}{2\mathrm{e}} \int_0^1 \mathrm{e}^{x^2} \mathrm{d}x,$$

where $\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1} \mathrm{d}t$ is the *Euler*'s Gamma function.

304. *Proposed by Andrei Ciupan, student Harvard University, Boston, U.S.A.* Let $f, g : [0, 1] \to \mathbb{R}$ be two functions such that f is continuous and g is increasing and differentiable, with $g(0) \geq 0$. Prove that if, for any $t \in [0, 1]$,

$$\int_t^1 f(x)\mathrm{d}x \geq \int_t^1 g(x)\mathrm{d}x,$$

then

$$\int_0^1 f^2(x)\mathrm{d}x \geq \int_0^1 g^2(x)\mathrm{d}x.$$

305. *Proposed by Marius Cavachi, Ovidius University of Constanţa, Constanţa, Romania.* Let K be a field and $f \in K[X]$ with $\deg(f) = n \geq 1$ having distinct roots $x_1, x_2, \ldots, x_n$. For $p \in \{1, 2, \ldots, n\}$ let $S_1, S_2, \ldots, S_p$ be the symmetric fundamental polynomials in $x_1, x_2, \ldots, x_p$. Show that

$$[K(S_1, S_2, \ldots, S_p) : K] \leq \binom{n}{p}.$$

306. *Proposed by Radu Gologan, Institute of Mathematics Simion Stoilow of the Romanian Academy, Bucharest, Bucharest, Romania.* Prove that for any set of positive numbers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ such that $x_1 + x_2 + \cdots + x_n = 1$, denoting by $\mathcal{A}$ the set of pairs (i,j) such that $y_1 + \cdots + y_j \leq x_1 + \cdots + x_i$, one has

$$\sum_{(i,j)\in\mathcal{A}} \frac{x_{i+1}y_j}{1 + (x_1 + \cdots + x_{i+1})(y_1 + \cdots + y_j)} \leq \frac{\pi^2}{24}.$$

Prove that the constant $\frac{\pi^2}{24}$ is the best satisfyng the property.

307. *Proposed by Benjamin Bogoşel, student West University of Timisoara, Timişoara, Romania.* Let $f : \mathbb{R} \to \mathbb{R}$ twice differentiable with f'' continuous and $\lim\limits_{x\to\pm\infty} f(x) = \infty$ such that $f(x) > f''(x)$ for all $x \in \mathbb{R}$. Show that $f(x) > 0$ for any real number x.

308. *Proposed by Flavian Georgescu, student University of Bucharest, Bucharest, Romania.* Let $M_n(\mathbb{Q})$ be the ring of square matrices of size n and $X \in M_n(\mathbb{Q})$. Define the adjugate (classical adjoint) of X by $\operatorname{adj}(X)$ as: the (i,j)-minor M_{ij} of X is the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting row i and column j of X, and the i,j cofactor of X as $C_{ij} = (-1)^{i+j}M_{ij}$. The adjugate of X is the transpose of the "cofactor matrix" C_{ij} of X. Consider $A, B \in M_n(\mathbb{Q})$ such that

$$(\operatorname{adj}(A))^3 - (\operatorname{adj}(B))^3 = 2((\operatorname{adj}(A)) - (\operatorname{adj}(B))) \;\neq O_n.$$

Show that

$$\operatorname{rank}(AB) \in \{\operatorname{rank}(A), \operatorname{rank}(B)\}.$$

SOLUTIONS

275. *Proposed by Jose Luis Diaz-Barrero, Barcelona, Spain*, and *Pantelimon George Popescu, Bucharest, Romania.* Let $A(z) = z^n + \sum\limits_{k=0}^{n-1} a_k z^k$ be a polynomial with complex coefficients of degree $n \geq 2$ that has n distinct zeros $z_1, z_2, \ldots, z_n$ and let α be a nonzero complex number such that no ratio of two zeros of $A(z)$ is equal to α. Prove that

$$\sum_{k=1}^{n} \left(\frac{1}{A(\alpha z_k) A'(z_k)} + \frac{1}{\alpha A\left(\frac{z_k}{\alpha}\right) A'(z_k)} \right) = 0.$$

Solution by the authors. We will evaluate the integral

$$I = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{1}{A(z)A(\alpha z)} \mathrm{d}z$$

over the interior and exterior domains limited by γ, a circle centered at the origin and radius $\min\limits_{1\leq k\leq n} \{|z_k|\} > r > \max\limits_{1\leq k\leq n} \left\{ \left| \frac{z_k}{\alpha} \right| \right\}$. Integrating in the region outside of γ contour we have

$$I_1 = \frac{1}{2\pi \mathrm{i}} \oint\limits_\gamma \frac{1}{A(z)A(\alpha z)} \mathrm{d}z = \sum_{k=1}^{n} \mathrm{Res} \left\{ \frac{1}{A(z)A(\alpha z)}, z = z_k \right\} =$$

$$= \sum_{k=1}^{n} \frac{1}{A(\alpha z_k)} \prod_{j=1, j\neq k}^{n} \frac{1}{z_k - z_j}.$$

Integrating in the region inside of γ contour we obtain

$$I_2 = \frac{1}{2\pi \mathrm{i}} \oint\limits_\gamma \frac{1}{A(z)A(\alpha z)} \mathrm{d}z = \sum_{k=1}^{n} Res \left\{ \frac{1}{A(z)A(\alpha z)}, z = \frac{z_k}{\alpha} \right\} =$$

$$= \sum_{k=1}^{n} \frac{1}{\alpha A\left(\frac{z_k}{\alpha}\right)} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{z_k - z_j}.$$

According to a well-known result on contour integrals, we have $I_1 + I_2 = 0$ and we are done.

Solution by Marian Tetiva, Bârlad, Romania. We begin with a small remark. We assume that the roots of the polynomial must be nonzero, although the nature of our polynomial is not important as we shall use the classic decompositon $A(z) = (z - z_1)(z - z_2) \ldots (z - z_n)$. Next, observe that

$$A(\alpha z_k) = (\alpha - 1) z_k A_k(z_k),$$

where $A_k(z) = \frac{A(z)}{z - z_k}$. On the other hand, we have

$$\alpha A\left(\frac{z_k}{\alpha}\right) = -(\alpha - 1) z_k A_k\left(\frac{z_k}{\alpha}\right)$$

and thus our identity can be rewritten as

$$\sum_{k=1}^{n} \frac{1}{z_k A'(z_k)} \left(\frac{1}{A_k(\alpha z_k)} - \frac{1}{A_k\left(\frac{z_k}{\alpha}\right)} \right) = 0.$$

Let $B(z) = (z - w_1)(z - w_2)\dots(z - w_m)$. It is well-known that

$$\frac{1}{B(z)} = \sum_{j=1}^{m} \frac{1}{B'(w_j)(z - w_j)},$$

where $B'(w_j) = B_j(w_j) = \prod\limits_{i \neq j} (w_j - w_i)$. Using this type of formula for A_k-s, we find

$$\frac{1}{A_k(\alpha z_k)} = \sum_{\substack{j=1 \\ j \neq k}} \frac{1}{A'(z_j)(\alpha z_k - z_j)}.$$

Similarly we have

$$\frac{1}{A_k\left(\frac{z_k}{\alpha}\right)} = \sum_{j \neq k} \frac{1}{A_k'(z_j)\left(\frac{z_k}{\alpha} - z_j\right)} = -\sum_{j \neq k} \frac{\alpha}{A_k'(z_j)(\alpha z_j - z_k)},$$

and thus the identity that we have to prove becomes

$$\sum_{k=1}^{n} \frac{1}{z_k A'(z_k)} \left(\sum_{j \neq k} \frac{1}{A_k'(z_j)(\alpha z_k - z_j)} + \sum_{j \neq k} \frac{\alpha}{A_k'(z_j)(\alpha z_j - z_k)} \right) = 0.$$

The coefficient of $\frac{1}{\alpha z_k - z_j}$ (for indices $j, k \in \{1, 2, \dots, n\}, j \neq k$) is

$$\frac{1}{z_k A'(z_k) A_k'(z_j)} + \frac{\alpha}{z_j A'(z_j) A_j'(z_k)}.$$

On the other hand, one has $A'(z_k) = (z_k - z_j)A_j'(z_k)$ and $A'(z_j) = (z_j - -z_k)A_k'(z_j)$. Thus, the coefficient of $\frac{1}{\alpha z_k - z_j}$) is given by

$$\frac{1}{(z_j - z_k)A_j'(z_k)A_k'(z_j)} \left(-\frac{1}{z_k} + \frac{\alpha}{z_j} \right) = \frac{\alpha z_k - z_j}{z_j z_k (z_j - z_k) A_j'(z_k) A_k'(z_j)}.$$

In other words, this fraction gives the term $\frac{1}{z_j z_k (z_j - z_k) A_j'(z_k) A_k'(z_j)}$. Now, it clear that this term will be cancelled by $\frac{1}{z_j z_k (z_k - z_j) A_j'(z_k) A_k'(z_j)}$ and our sum can be divided into $\frac{n(n-1)}{2}$ sums of opposite terms of the type described above.

In conclusion, we finally obtain that our sum is equal to zero.

Remark. One can prove in the same manner the following identity

$$\sum_{k=1}^{n} z_k \left(\frac{1}{A(\alpha z_k) A'(z_k)} + \frac{1}{\alpha^2 A\left(\frac{z_k}{\alpha}\right) A'(z_k)} \right) = 0.$$

This is problem 11190 from *no.* 10/2005, *American Mathematical Monthly* proposed by the same authors.

276. *Proposed by Marian Tetiva, Bârlad, Romania.* Let d be a positive square-free integer and (x_n, y_n) be the sequence of solutions of *Pell*'s equation $x^2 - dy^2 = 1$. Show that for any positive integer N, there are many infinitely pairs of positive integers distinct from m and n such that $\gcd(x_m, x_n) > N$.

Solution by the author. It is well-known that x_n and y_n are defined by

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d}\,)^n,$$

for all $n \in \mathbb{N}^*$, where (x_1, y_1) is the smallest solution. The identity

$$(x_1 + y_1\sqrt{d})^{n+k} = \left(x_1 + y_1\sqrt{d}\right)^k \left(x_1 + y_1\sqrt{d}\right)^n$$

can be rewritten in the form

$$x_{n+k} + y_{n+k}\sqrt{d} = \left(x_k + y_k\sqrt{d}\right)\left(x_n + y_n\sqrt{d}\right),$$

or equivalently $x_{n+k} = x_k x_n + dy_k y_n$ and $y_{n+k} = y_k x_n + dx_k y_n$, for all positive integers k and n. One can easily derive that

$$y_n = \frac{x_{n+k} - x_k x_n}{dy_k}$$

and replacing into the second equality we find $x_{n+2k} - 2x_k x_{n+k} + x_n = 0$, for all $n, k \in \mathbb{N}^*$. In particular, we have $x_{3k} - 2x_k x_{2k} + x_k = 0$ and this shows that x_{3k} divides x_k for any k positive integer.

Thus $(x_{3k}, x_k) = x_k > N$ for k big enough, because $x_k \to \infty$ for $k \to \infty$ and the solution ends here. One can observe that the sequence $(y_n)_n$ has the same property.

277. *Proposed by Robert Szász, Târgu Mureş, Romania.* If $u \in C^1([0, 1]$; with $u(0) = 0$ and $u'(0) = 1$, then

$$\int_0^1 \mathrm{e}^{u(x)}\mathrm{d}x + \int_0^1 (u'(x))^2\mathrm{d}x \geq 4.$$

Remark. Unfortunately, the problem is obviously wrong! Consider the function $u : [0, 1] \to \mathbb{R}$, $u(x) = x$. Clearly, u is C^1 on $[0, 1]$ and $u(0) = 0$ and $u'(0) = 1$. On the other hand,

$$\int_0^1 \exp(u(x))\mathrm{d}x + \int_0^1 (u'(x))^2\mathrm{d}x = e - 1 + 1 = e < 4.$$

The editorial board apologises for this error.

278. *Proposed by Marius Cavachi, Ovidius University of Constanţa, Romania.* Prove that there does not exist a rational function R with real coefficients such that

$$R(n) = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2},$$

for an infinitely of natural numbers n.

Solution by Marian Tetiva, Bârlad, Romania. We shall justify this affirmation by contradiction. Indeed we assume that such a function R exists. Denote $R(n) = \frac{P(n)}{Q(n)}$ for infinitely many n.

It is clear that $\deg(P)$ must be equal to $\deg R$, if not since the equality is valid for infinitely many n, by passing to the limit when $n \to \infty$, we have:

$$\lim_{n\to\infty} R(n) = \lim_{n\to\infty} \left(1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}\right) = \frac{\pi^2}{6}.$$

Therefore

$$R(n) = \frac{a_k n^k + \ldots + a_0}{b_k n^k + \ldots + b_0},$$

where $a_k, \ldots, a_0$, $b_k, \ldots, b_0$ are real numbers with $a_k \neq 0$, $b_k \neq 0$ and $\frac{a_k}{b_k} = \frac{\pi^2}{6}$.

Then we have

$$n\left(R(n) - \frac{\pi^2}{6}\right) = n\left(1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2} - \frac{\pi^2}{6}\right)$$

for an infinity of natural numbers n. By passing to the limit again

$$\lim_{n\to\infty} n\left(R(n) - \frac{\pi^2}{6}\right) = \lim_{n\to\infty} n\left(1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2} - \frac{\pi^2}{6}\right) = -1.$$

Further more we obtain that $R(n)$ and $1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}$ have the same asymptotic expansion

$$l_0 + \frac{l_1}{n} + \frac{l_2}{n^2} + \ldots,$$

where the iterated limits $l_0, l_1, \ldots$, for the sequence $1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}$ are well-known; we have $l_0 = \frac{\pi^2}{6}$, $l_1 = -1$, then $l_2 = \frac{1}{2}$, $l_3 = -\frac{1}{6}$, and generally written $l_n = -B_{n-1}$ for all $n \geq 1$. The expansion

$$1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2} = \frac{\pi^2}{6} - \sum_{j=0}^{\infty} \frac{B_j}{n^{j+1}}$$

can be found in the paper „Pi, Euler Numbers, and Asymptotic Expansions“ by *J. M. Borwein*, *P. B. Borwein*, *K. Dilcher* in The American Mathematical Monthly, vol. 96, No. 8 (Oct. 1989), pg. 681-687. As usual, denote B_n the n^{th} *Bernoulli* number ($B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ etc.; we recall that $B_{2l+1} = 0$ for $l \geq 1$).

On the other hand , the asymptotic expansion of sequence $R(n)$ can be obtained from the *MacLaurian* expansion of function $R\left(\frac{1}{x}\right)$. So, if we have

$$R\left(\frac{1}{x}\right) = \frac{a_k + \ldots + a_0 x^k}{b_k + \ldots + b_0 x^k} = l_0 + l_1 x + l_2 x^2 + \ldots$$

(in a neighbourhood of the origin), then $l_0, l_1, \ldots$ are exactly the iterated limits mention above. Therefore one can verify that $l_0 = \lim\limits_{n\to\infty} R(n)$, $l_1 = \lim\limits_{n\to\infty} n\left(R(n) - l_0\right)$, $l_2 = \lim\limits_{n\to\infty} n\left(n\left(R(n) - l_0\right) - l_1\right)$ and so on.

The above equality could be rewritten as a formal series equality

$$a_k + \ldots + a_0 x^k = \left(b_k + \ldots + b_0 x^k\right)\left(l_0 + l_1 x + l_2 x^2 + \ldots\right).$$

Further on, if we identify the coefficients of x^j from the two members, we obtain

$$a_{k-j} = \sum_{s+t=j} b_{k-s} l_t,$$

for all $j \in \mathbb{N}$ (the sum takes into account every pair (s,t) which are natural numbers and their sum is j_i by convention $a_i = 0$ and $b_i = 0$ for $i < 0$). One can also remark that we find the identit $a_k = b_k l_0$, even though it isn't essential for our solution.

For $j \geq k+1$ this equalities prove that the sequence $(l_n)_{n\geq 1}$ verifies a linear recurrence of order k (where coefficients are exactly those of the R denominator). Taking into account the general term formula for a solution of such a recurrence we deduce that there is $a > 0$ such that $\lim\limits_{n\to\infty} \frac{l_n}{a^n} = 0$.

Finally we recall the celebrated *Euler*'s formula for the calculus of the zeta function values of argument a natural even number:

$$\zeta(2n) = \frac{(2\pi)^{2n} |B_{2n}|}{2(2n)!} \Leftrightarrow |B_{2n}| = \frac{2(2n)!\zeta(2n)}{(2\pi)^{2n}}.$$

It is obvious that $\zeta(2n) > 1$ and therefore we obtain

$$|B_{2n}| > \frac{2(2n)!}{(2\pi)^{2n}},\ \forall\, n \geq 1.$$

This means under the assumed hypothesis that

$$|l_{2n+1}| > \frac{2(2n)!}{(2\pi)^{2n}},$$

for all $n \geq 1$ and would evidently contradict what we previously obtained regarding the sequence $(l_n)_{n\geq 1}$, specifically

$$\lim_{n\to\infty} \left| \frac{l_{2n+1}}{a^{2n+1}} \right| = 0$$

(because $\lim\limits_{n\to\infty} \frac{n!}{q^n} = \infty$, for all $q > 0$). This contradiction proves that the initial hypothesis is false and therefore there does not exist rational functions which represent the sequence with the general term $1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}$ for infinitely many n.

Remark. One could use the same method in order to prove that the above result is also valid in the following form: there does not exists rational functions R such that $R(n) = 1 + \frac{1}{2^s} + \ldots + \frac{1}{n^s}$ for infinitely many natural numbers n (where $s > 1$ is a fixed real number).

279. *Proposed by Dumitru Popa, Ovidius University of Constanţa, Romania.* Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous and strictly increasing function, twice fifferentiable in 0 with $f(0) = f'(0) = 0$, $f''(0) > 0$, $\lim\limits_{x\to\infty} f(x) = \infty$ and $a > 0$.

Show that for all $n \in \mathbb{N}$ the equation

$$\sum_{k=1}^{n} f\left(\frac{kx}{n^2}\right) = a$$

has an unique solution x_n in the interval $(0,\infty)$, and:

$$\lim_{n\to\infty}\frac{x_n}{\sqrt{n}}=\sqrt{\frac{6a}{f''(0)}}.$$

Solution by the author. For all $n\in\mathbb{N}$, let there be $\varphi_n:[0,\infty)\to\mathbb{R}$, $\varphi_n(x)=$ $=\sum\limits_{k=1}^{n} f\left(\frac{kx}{n^2}\right)-a$. Then φ_n is continuously strictly increasing. Moreover, from $f(0)=0$, we can deduce that $\varphi_n(0)=-a<0$ and $\lim\limits_{x\to\infty}\varphi_n(x)=\infty$ because $\lim\limits_{x\to\infty} f(x)=\infty$. Therefore the equation $\varphi_n(x)=0$ has an unique solution in the interval $(0,\infty)$. And as every root verifies the equation, we have

$$\varphi_n(x_n)=0,\quad\text{or}\quad \sum_{k=1}^{n} f\left(\frac{kx_n}{n^2}\right)=a,\quad \forall\, n\in\mathbb{N}. \tag{1}$$

In robat follows we shall use a method similar to the one used in the solution for exercise 4.11 from [1].

As $f(0)=f'(0)=0$ this implies $\lim\limits_{\substack{x\to 0\\ x>0}}\frac{f(x)}{x^2}=\frac{1}{2}f''(0)$.

Because $f''(0)>0$, for $\varepsilon=\frac{f''(0)}{4}>0$, there exists $\delta>0$ such that $\forall\, 0<x<\delta$ we have $\left|\frac{f(x)}{x^2}-\frac{f''(0)}{2}\right|<\frac{f'(0)}{4}$ i.e.

$$\exists\,\delta>0\quad\text{such that}\ \ \forall\, 0<x<\delta\quad\text{ we have }\ f(x)>\frac{f''(0)}{4}\cdot x^2. \tag{2}$$

Let there be $M=\sqrt{\frac{12a}{f''(0)}}>0$. As $\frac{M}{\sqrt{n}}\to 0$, for $\delta>0$, we have $n_0\in\mathbb{N}$ such that $\frac{M}{\sqrt{n}}<\delta$, $\forall\, n\geq n_0$. Let there be $n\geq n_0$. For all $1\leq k\leq n$ we have $\frac{Mk\sqrt{n}}{n^2}\leq\frac{M}{\sqrt{n}}<\delta$. Therefore from (2) we obtain

$$f\left(\frac{Mk\sqrt{n}}{n^2}\right)>\frac{f''(0)}{4}\cdot\frac{M^2k^2n}{n^4}.$$

Summing from $k=1$ to $k=n$ we obtain

$$\varphi_n(M\sqrt{n})=\sum_{k=1}^{n} f\left(\frac{Mk\sqrt{n}}{n^2}\right)-a>\frac{f''(0)}{4}\cdot\frac{M^2n(n+1)(2n+1)}{6n^3}-a.$$

From the choice of M we have

$$\varphi_n\left(M\sqrt{n}\right)>\frac{an(n+1)(2n+1)}{2n^3}-a=\frac{(3n+1)a}{2n^2}>0.$$

Therefore $0=\varphi_n(x_n)<\varphi_n\left(M\sqrt{n}\right)$, $\forall\, n\geq n_0$ and because φ_n is strictly increasing we obtain $x_n<M\sqrt{n}$, $\forall\, n\geq n_0$, i.e. the sequence

$$\left(\frac{x_n}{\sqrt{n}}\right)_{n\in\mathbb{N}}\qquad\text{is bounded from above.} \tag{3}$$

Let $g : [0, \infty) \to \mathbb{R}$, $g(x) = \begin{cases} \frac{f(x)}{x}, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0. \end{cases}$ We have $f(x) = xg(x)$ and the equation (1) becomes

$$\frac{x_n}{n} \sum_{k=1}^{n} \frac{k}{n} g\left(\frac{kx_n}{n^2}\right) = a. \tag{4}$$

Because

$$\frac{kx_n}{n^2} = \frac{x_n}{\sqrt{n}} \cdot \frac{k}{n} \cdot \frac{1}{\sqrt{n}} \leq \frac{x_n}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}}, \quad \forall\, 1 \leq k \leq n, \tag{5}$$

using (3), (5) and the sandwich theorem we obtain

$$\max_{1 \leq k \leq n} \frac{kx_n}{n^2} \to 0. \tag{6}$$

As $g'(0) = \frac{1}{2} f''(0) > 0$, using a celebrated result (see example [1], exercice 3.20), from (6) we obtain

$$\sum_{k=1}^{n} \frac{k}{n} g\left(\frac{kx_n}{n^2}\right) \sim g'(0) \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{kx_n}{n^2}$$

from which we deduce

$$\sum_{k=1}^{n} \frac{k}{n} g\left(\frac{kx_n}{n^2}\right) \sim x_n \cdot \frac{1}{3} g'(0). \tag{7}$$

From (4) and also taking into account (7) we have $\frac{x_n^2}{n} \cdot \frac{1}{3} g'(0) \to a$ i.e. our conclusion.

We denoted by $a_n \sim b_n$ if and only if $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$.

Remark. In the above solution we have denoted by [1].

One can generalize the problem in the following way : $f \in C^{\infty}$ and p is a natural number with the property $f(0) = f'(0) = f''(0) = \ldots = f^{(p-1)}(0) = 0$ and $f^{(p)}(0) > 0$.

Solution by Ilie Bulacu, Erhardt+Leiner Romania P.T.S., Bucharest, Romania. We will prove the generalization of the given problem which is:

Let $f : [0, \infty) \to \mathbb{R}$ be a continuous and strictly decreasing function, $m-$times differentiable in 0 with $f(0) = \ldots = f^{(m-1)}(0) = 0$, $f^{(m)}(0) > 0$, $\lim\limits_{x\to\infty} f(x) = \infty$ and $a > 0$.

Prove that for all $n \in \mathbb{N}$ the equation

$$\sum_{k=1}^{n} f\left(\frac{kx}{n^2}\right) = a$$

has a unique solution x_n in the interval $[0, \infty)$ and

$$\lim_{n\to\infty} \frac{x_n}{\sqrt[m]{n^{m-1}}} = \sqrt[m]{\frac{(m+1)!a}{f^{(m)}(0)}}.$$

Now we prove the started generalization.

Let $\varphi : [0,\infty) \to \mathbb{R}$, $\varphi(x) = \sum\limits_{k=1}^{n} f\left(\frac{kx}{n^2}\right) - a$. Because f is a continuously strictly increasing function it implies that function φ is also continuously strictly increasing. In addition, as $f(0) = 0$ we have $\varphi(0) = -a < 0$ and from $\lim\limits_{x\to\infty} f(x) = \infty$ we can deduce $\lim\limits_{x\to\infty} \varphi(x) = \infty$. Therefore equation $\varphi(0) = 0$ has a unique solution x_n in the interval $(0,\infty)$.

According to *MacLaurian*'s formula, because f is $m-$times differentiable in 0 we have the equality

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(m-1)}(0)}{(m-1)!}x^{m-1} + \frac{f^{(m)}(0)}{m!}x^m + o\left(x^m\right),$$

for $x \to 0$, where o is the *Landau* symbol.

Taking into account be equalities from the hypothesis, it immediately follows that

$$\lim_{\substack{x\to 0\\ x>0}} \frac{f(x)}{x^m} = \frac{f^{(m)}(0)}{m!},$$

or equivalently

$$\forall\,\varepsilon > 0,\ \exists\,\delta_\varepsilon > 0 \text{ such that } \left|\frac{f(x)}{x^m} - \frac{f^{(m)}(0)}{m!}\right| < \varepsilon,\ \forall\,x,\ 0 < x < \delta_\varepsilon,$$

or

$$\forall\,\varepsilon > 0,\ \exists\,\delta_\varepsilon > 0 \ \text{ such that } \ \alpha x^m < f(x) < \beta x^m,\ \forall\,x,\ 0 < x < \delta_\varepsilon, \tag{1}$$

where $\alpha = \frac{f^{(m)}(0)}{m!} - \varepsilon$ and $\beta = \frac{f^{(m)}(0)}{m!} + \varepsilon$.

Next we will prove the following equivalence:

$$\text{the sequence } \left(\frac{x_n}{\sqrt[m]{x^{m-1}}}\right)_{n\in\mathbb{N}} \text{ is bounded if and only if } f^{(m)}(0) > 0.$$

Suppose that the sequence $\left(\frac{x_n}{\sqrt[m]{n^{m-1}}}\right)_{n\in\mathbb{N}}$ is bounded, meaning that $\exists\,M > 0$ such that $0 < \frac{x_n}{\sqrt[m]{n^{m-1}}} \le M$, $\forall\,n$, $n \in \mathbb{N}$. As f is an increasing function, we deduce that

$$a = \sum_{k=1}^{n} f\left(\frac{kx_n}{n^2}\right) \le \sum_{k=1}^{n} f\left(\frac{x_n}{n}\right) = \sum_{k=1}^{n} f\left(\frac{1}{\sqrt[m]{n}} \cdot \frac{x_n}{\sqrt[m]{n^{m-1}}}\right) \le$$

$$\le \sum_{k=1}^{n} f\left(\frac{M}{\sqrt[m]{n}}\right) = nf\left(\frac{M}{\sqrt[m]{n}}\right) = \frac{f\left(\frac{M}{\sqrt[m]{n}}\right)}{\frac{M^m}{n}} \cdot M^m,\ \forall\,n,\ n \in \mathbb{N}.$$

For $n \to \infty$ we obtain $a \le \frac{f^{(m)}(0)}{m!} \cdot M^m$ and as $a > 0$, $M > 0$, it implies that $f^{(m)}(0) > 0$.

Conversely, let us assume that $f^{(m)}(0) > 0$. Let there be $M = \sqrt[m]{\frac{(m+1)\cdot(m+1)!\cdot a}{f^{(m)}(0)}}$. Because $\lim\limits_{\substack{x\to 0\\ x>0}} \frac{f(x)}{x^m} = \frac{f^{(m)}(0)}{m!}$, for $\varepsilon = \frac{mf^{(m)}(0)}{(m+1)!}$, $\exists\,\delta > 0$ such that

$$\left|\frac{f(x)}{x^m} - \frac{f^{(m)}(0)}{m!}\right| < \frac{mf^{(m)}(0)}{(m+1)!}, \quad \forall\,x,\ 0 < x < \delta,$$

meaning that $\exists\,\delta > 0$ such that

$$f(x) > \frac{f^{(m)}(0)}{(m+1)!} \cdot x^m, \quad \forall\, x,\ 0 < x < \delta. \tag{2}$$

Because $\frac{M}{\sqrt[m]{n}} \to 0$, for the above $\delta > 0$, $\exists\, n_0$, $n_0 \in \mathbb{N}$, such that $\frac{M}{\sqrt[m]{n}} < \delta$, $\forall\, n$, $n \in \mathbb{N}$, $n \geq n_0$.

Let there be $n \in \mathbb{N}$, $n \geq n_0$. Evidently we have

$$0 < \frac{kx_n}{n^2} < \frac{x_n}{n} = \frac{1}{\sqrt[m]{n}} \cdot \frac{x_n}{\sqrt[m]{n^{m-1}}} < \frac{M}{\sqrt[m]{n}} < \delta,\ \forall\, k,\ 1 \leq k \leq n$$

and also considering (2) we obtain

$$f\left(\frac{k}{n\sqrt[m]{n}} \cdot M\right) > \frac{f^{(m)}(0)}{(m+1)!} \cdot \frac{k^m}{n^{m+1}} \cdot M^m,\ \forall\, k,\ 1 \leq k \leq n. \tag{3}$$

Let $\Psi : [0,\infty) \to \mathbb{R}$, $\Psi(x) = \sum\limits_{k=1}^{n} f\left(\frac{kx}{n\sqrt[m]{n}}\right) - a$. Evidently $\Psi\left(\frac{x}{\sqrt[m]{n^{m-1}}}\right) = \varphi(x)$ and Ψ has the same properties as φ, which are: Ψ is continuous and strictly increasing, $\Psi(0) = -a < 0$, $\lim\limits_{x\to\infty} \Psi(x) = \infty$ and $\Psi\left(\frac{x_n}{\sqrt[m]{n^{m-1}}}\right) = \varphi(x_n) = 0$.

Summing inequalities (3) from $k = 1$ to $k = n$ and taking into consideration the choice of M we obtain:

$$\Psi(M) = \sum_{k=1}^{n} f\left(\frac{k}{n\sqrt[m]{n}} \cdot M\right) - a > \sum_{k=1}^{n} \frac{f^{(m)}(0)}{(m+1)!} \cdot \frac{k^m}{n^{m+1}} \cdot M^m - a >$$

$$> \frac{f^{(m)}(0)}{(m+1)!} \cdot \frac{M^m}{n^{m+1}} \cdot \left(\sum_{k=1}^{n} k^m\right) - a = (m+1)a \cdot \frac{\sum\limits_{k=1}^{n} k^m}{n^{m+1}} - a > (m+1)a \cdot \frac{1}{m+1} - a = a - a = 0,$$

$\forall\, n$, $n \geq n_0$.

On other hand, we have

$$\Psi\left(\frac{x_n}{\sqrt[m]{n^{m-1}}}\right) = \sum_{k=1}^{n} f\left(\frac{k}{n\sqrt[m]{n}} \cdot \frac{x_n}{\sqrt[m]{n^{m-1}}}\right) - a = \sum_{k=1}^{n} f\left(\frac{kx_n}{n^2}\right) - a = 0,\ \forall\, n,\ n > n_0.$$

Therefore, $0 = \Psi\left(\frac{x_n}{\sqrt[m]{n^{m-1}}}\right) < \Psi(M)$, and as Ψ is strictly increasing we obtain

$$0 < \frac{x_n}{\sqrt[m]{n^{m-1}}} < M,\ \forall\, n,\ n \geq n_0,$$

meaning that the sequence $\left(\frac{x_n}{\sqrt[m]{n^{m-1}}}\right)_{n\in\mathbb{N}}$, is bounded.

In conclusion, the sequence $\left(\frac{x_n}{\sqrt[m]{n^{m-1}}}\right)_{n\in\mathbb{N}}$ is bounded if and only if

$$f^{(m)}(0) > 0.$$

As a result, the sequence $\left(\frac{x_n}{\sqrt[m]{n^{m-1}}}\right)_{n\in\mathbb{N}}$ is bounde because we have $f^{(m)}(0) > 0$ from the hypothesis.

Therefore $\exists\, M > 0$ such that $0 < \frac{x_n}{\sqrt[m]{n^{m-1}}} < M$, $\forall\, n$, $n \in \mathbb{N}$.

Conversely, as $\frac{M}{\sqrt[m]{n}} \to 0$, for $\delta_\varepsilon > 0$, $\exists\, n_\varepsilon$, $n_\varepsilon \in \mathbb{N}$, such that $\frac{M}{\sqrt[m]{n}} < \delta_\varepsilon$, $\forall\, n \in \mathbb{N}$, $n \geq n_\varepsilon$.

Let $n \in \mathbb{N}$, $n \geq n_\varepsilon$. Evidently we have

$$0 < \frac{kx_n}{n^2} < \frac{x_n}{n} = \frac{1}{\sqrt[m]{n}} \cdot \frac{x_n}{\sqrt[m]{n^{m-1}}} < \frac{M}{\sqrt[m]{n}} < \delta_\varepsilon,\ \forall\, k,\ 1 \leq k \leq n$$

and taking into accont (1) we obtain

$$\frac{\alpha k^m x_n^m}{n^{2m}} < f\left(\frac{kx_n}{n^2}\right) < \frac{\beta k^m x_n^m}{n^{2m}},\ \forall\, k,\ 1 \leq k \leq n$$

Summing further on from $k = 1$ to $k = n$ we succesively obtain

$$\frac{\alpha\left(\sum\limits_{k=1}^{n} k^m\right)}{n^{2m}} < \sum_{k=1}^{n} f\left(\frac{kx_m}{n^2}\right) < \frac{\beta\left(\sum\limits_{k=1}^{n} k^m\right) x_n^m}{n^{2m}}, \quad \forall\, n \geq n_\varepsilon \Leftrightarrow$$

$$\Leftrightarrow \frac{\alpha\left(\sum\limits_{k=1}^{n} k^m\right)}{n^{2m}} < a < \frac{\beta\left(\sum\limits_{k=1}^{n} k^m\right) x_n^m}{n^{2m}},\ \forall\, n \geq n_\varepsilon \Leftrightarrow$$

$$\Leftrightarrow \alpha < \frac{an^{2m}}{\left(\sum\limits_{k=1}^{n} k^m\right) x_n^m} < \beta,\ \forall\, n \geq n_\varepsilon \Leftrightarrow$$

$$\Leftrightarrow \frac{f^{(m)}(0)}{m!} - \varepsilon < \frac{an^{2m}}{\left(\sum\limits_{k=1}^{n} k^m\right) x_m^n} < \frac{f^{(m)}(0)}{m!} + \varepsilon,\ \forall\, n \geq n_\varepsilon \Leftrightarrow$$

$$\Leftrightarrow -\varepsilon < \frac{a_n^{2m}}{\left(\sum\limits_{k=1}^{n} k^m\right) x_n^m} - \frac{f^{(m)}(0)}{m!} < +\varepsilon, \quad \forall\, n \geq n_\varepsilon \Leftrightarrow$$

$$\Leftrightarrow \left| \frac{\alpha n^{2m}}{\left(\sum\limits_{k=1}^{n} k^m\right) x_n^m} - \frac{f^{(m)}(0)}{m!} \right| < +\varepsilon,\ \forall\, n \geq n_\varepsilon \Leftrightarrow \left| a\frac{n^{m+1}}{\sum\limits_{k=1}^{n} k^m} \cdot \frac{n^{m-1}}{x_n^m} - \frac{f^{(m)}(0)}{m!} \right| < \varepsilon,$$

$$\forall\, n \geq n_\varepsilon \Leftrightarrow \left| a\frac{n^{m+1}}{\sum\limits_{k=1}^{n} k^m} \cdot \frac{n^{m-1}}{x_n^m} - \frac{f^{(m)}(0)}{m!} \right| < \varepsilon,\ \forall\, n \geq n_\varepsilon \Leftrightarrow$$

$$\Leftrightarrow \left| a\frac{n^{m+1}}{\sum\limits_{k=1}^{n} k^m} \cdot \left(\frac{\sqrt[m]{n^{m-1}}}{x_n}\right)^m - \frac{f^{(m)}(0)}{m!} \right| < \varepsilon,\ \forall\, n \geq n_\varepsilon.$$

Passing to the limit for $n \to \infty$ and $\varepsilon \to 0$ and considering the known limit

$$\lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} k^m}{n^{m+1}} = \frac{1}{m+1},$$

we obtain

$$a(m+1)\cdot \lim_{n\to\infty}\left(\frac{\sqrt[m]{n^{m-1}}}{x_n}\right)^m = \frac{f^{(m)}(0)}{m!} \Leftrightarrow \lim_{n\to\infty}\left(\frac{x_n}{\sqrt[m]{n^{m-1}}}\right)^m = \frac{(m+1)!a}{f^{(m)}(0)} \Leftrightarrow$$

$$\Leftrightarrow \lim_{n\to\infty} \frac{x_m}{\sqrt[m]{n^{m-1}}} = \sqrt[m]{\frac{(m+1)!a}{f^{(m)}(0)}}.$$

Particular case

1. For $m = 1$ we have $\lim\limits_{n\to\infty} x_n = \dfrac{2a}{f'(0)}$, which is problem 4.11, pag. 129, 178-181, from [1].

2. For $m = 2$ we have $\lim\limits_{n\to\infty} \dfrac{x_n}{\sqrt{n}} = \sqrt{\dfrac{6}{f''(0)}}$, which is problem **279** from G.M.-A no. 1/2009.

REFERENCES

[1] D. Popa, *Exerciţii de analiză matematică*, Biblioteca S.S.M.R., Editura Mira, Bucureşti, 2007.

A partial solution has received from *Marius Olteanu*, S. C. Hidroconstrucţia S. A., Râmnicu Vâlcea, Romania.

280. *Proposed by Marian Tetiva, Bârlad, Romania.* Let $n \geq 2$ be a natural number. Find the greatest positive number C such that the inequality:

$$2^{n-1}\left(x^n + y^n\right) - (x+y)^n \geq C\left[(x+3y)^n + (3x+y)^n - 2^{n+1}(x+y)^n\right]$$

is true for all $x, y \geq 0$.

Solution by the author. As both the left member and the espression written between paranthesis are non negative, there follows that for $C \leq 0$ the inequality is trivial and here C must be nonnegative.

Let us first remark that if the is true for all $x, y \geq 0$, then it is also true for all $x \geq 0$ and $y = 1$. Therefore we must have

$$f(x) = 2^{n-1}\left(x^n + 1\right) - (x+1)^n - C\left[(x+3)^n + (3x+1)^n - 2^{n+1}(x+1)^n\right] \geq 0,\ \forall\, x \geq 0.$$

For the polynomial function f one could immediately verify that the first two differentials are

$$f'(x) = n\cdot 2^{n-1}\cdot x^{n-1} - n(x+1)^{n-1} -$$
$$- C\left[n(x+3)^{n-1} + 3n(3x+1)^{n-1} - n\cdot 2^{n+1}(x+1)^{n-1}\right]$$

and

$$f''(x) = n(n-1)\cdot 2^{n-1}\cdot x^{n-2} - n(n-1)(x+1)^{n-2} -$$
$$- C\left[n(n-1)(x+3)^{n-2} + 9n(n-1)(3x+1)^{n-2} - n(n-1)\cdot 2^{n+1}(x+1)^{n-2}\right],$$

respectively.

So, we have $f(1) = f'(1) = 0$ and according to the *Taylor* formula

$$f(x) = (x-1)^2 \left[\frac{f''(1)}{2!} + \frac{f^{(3)}(1)}{3!}(x-1) + \ldots + \frac{f^{(n)}(1)}{n!}(x-1)^{n-2} \right];$$

the requirement that $f(x) \geq 0$ for all $x \geq 0$ leads to the conclusion that the expression between the brackets is also ≥ 0, for all $x \geq 0$, $x \neq 1$. In addition, owing to continuity issues, this inequality is also valid for $x = 1$. In other words we obtain $f''(1) \geq 0$, or

$$n(n-1) \cdot 2^{n-1} - n(n-1) \cdot 2^{n-2} \geq$$
$$\geq C \left[n(n-1) \cdot 4^{n-2} + 9n(n-1) \cdot 4^{n-2} - n(n-1) \cdot 2^{n+1} \cdot 2^{n-2} \right]$$

from which we obtain

$$C \leq \frac{1}{2^{n-1}}.$$

We will now prove that the inequality from the hypothesis is valid for all $x, y \geq 0$ if we consider $C = \frac{1}{2^{n-1}}$ which means that this is the maximum value we are searching for. A few simple transformations which lead to inequality

$$2^{2n-2}\left(x^n + y^n\right) + 3 \cdot 2^{n-1}(x+y)^n \geq (x+3y)^n + (3x+y)^n,$$

which we will prove next.

If we have $x = 0$ or $y = 0$, the inequality can be deduced from

$$2^{2n-2} + 3 \cdot 2^{n-1} \geq 3^n + 1$$

this is valid as an equality for $n = 1, 2, 3$ and is true for $n = 4$. If it takes place for a natural number $n \geq 2$, then we also have:

$$3 \cdot 2^{2n-2} + 9 \cdot 2^{n-1} \geq 3^{n+1} + 3,$$

and, because

$$2^{n-1} \geq 2 \Rightarrow \left(2^{n-1} - 1\right)\left(2^{n-1} - 2\right) \geq 0 \Rightarrow 2^{2n} + 3 \cdot 2^n + 2 \geq 3 \cdot 2^{2n-2} + 9 \cdot 2^{n-1},$$

the desired inequality can be obtained by induction.

Of course, the best complicated case have x or y nul (or both), so further on we will consider $x, y > 0$. Moreover, due to its simetry we can also assume that $x \geq y$ and divide the inequality by y^n in order to obtain an equivalent form

$$h(t) = 2^{n-2}\left(t^n + 1\right) + 3 \cdot 2^{n-1}(t+1)^n - (t+3)^n - (3t+1)^n \geq 0, \ \forall\, t > 0$$

(where we denoted by $t = \frac{x}{y}$, so the proof for $t \geq 1$ would be satisfying). This inequality can be obtained starting from *Taylor*'s formula as well. One can easily remark that the differentials of the polinomial function h are given by the formula

$$h^{(k)}(t) = n(n-1)\ldots(n-k+1)\left[2^{2n-2}t^{n-k} + 3 \cdot 2^{n-1}(t+1)^{n-k} - \right.$$
$$\left. -(t+3)^{n-k} - 3^k \cdot (3t+1)^{n-k}\right],$$

for every $k \geq 1$ and $t \in \mathbb{R}$ such that

$$h^{(k)}(1) = n(n-1)\ldots(n-k+1) \cdot 2^{2n-2k}\left(2^{2k-2} + 3 \cdot 2^{k-1} - 3^k - 1\right),$$

according to the previous inequality is

$$h^{(k)}(1) \geq 0,$$

for all $k \geq 1$. Therefore we have

$$h(t) = h(1) + \frac{h'(1)}{1!}(t-1) + \frac{h''(1)}{2!}(t-1)^2 + \ldots + \frac{h^{(n)}}{n!}(t-1)^n \geq 0,$$

for all $t \geq 1$.

In conclusion, the inequality

$$2^{n-1}(x^n + y^n) - (x+y)^n \geq \frac{1}{2^{n-1}}\left[(x+3y)^n + (3x+y)^n - 2^{n+1}(x+y)^n\right]$$

is „the best“ among all inequalities with the general form

$$2^{n-1}(x^n + y^n) - (x+y)^n \geq C\left[(x+3y)^n + (3x+y)^n - 2^{n-1}(x+y)^n\right]$$

because it implies all other inequalities which are true for all $x, y \geq 0$.

Also solved by *Nicuşor Minculete*, Dimitrie Cantemir, University, Braşov, Romania and by *Marius Olteanu*, S.C. Hidroconstrucţia S.A., Râmnicu Vâlcea, Romania.

282. *Proposed by Mihai Dicu, Fraţii Buzeşti National College, Craiova, Romania.* Denote

$$E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!},\ \forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}.$$

Prove that

a) $\displaystyle\sum_{n=0}^{\infty}(e^x - E_n(x)) = xe^x$; b) $\displaystyle\sum_{n=0}^{\infty} n\,(e^x - E_n(x)) = \frac{x^2}{2}e^x$.

Solution by the author. a) $\displaystyle\sum_{n=0}^{\infty}(e^x - E_n(x)) =$

$$= (e^x - 1) + \left(e^x - 1 - \frac{x}{1!}\right) + \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!}\right) + \ldots + \left(e^x - 1 - \frac{x}{1!} - \ldots - \frac{x^n}{n!}\right) =$$

$$= (n+1)e^x - (n+1) - n\frac{x}{1!} - (n+1)\frac{x^2}{2!} - (n+1)\frac{x^2}{2!} - \ldots - 2\frac{x^{n-1}}{(n-1)!} - \frac{x^n}{n!} =$$

$$= (n+1)\left(e - E_n(x)\right) + x\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!}\right) =$$

$$= (n+1)\left(e - E_n(x)\right) + xE_{n-1}(x)$$

We consider well-known the following fact

$$\sum_{n=1}^{\infty}(n+1)!\,(e^x - E_n(x)) = 0,$$

which together with previous proof implies

$$\sum_{n=0}^{\infty}(e^x - E_n(x)) = xe^x.$$

b) $\displaystyle\sum_{n=0}^{\infty} k\left(e^x - E_k(x)\right) =$

$$= \left(e^x - 1 - \frac{x}{1!}\right) + 2\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!}\right) + \ldots + n\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{1!} - \ldots - \frac{x^n}{n!}\right) =$$

$$= \frac{n(n+1)}{2}e^x - \left(E_1(x) + 2E_2(x) + \ldots + nE - n(x)\right) =$$

$$= \frac{n(n+1)}{2}e^x - \left[1 + \frac{x}{1!} + 2\left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right) + \ldots + n\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}\right)\right] =$$

$$= \frac{n(n+1)}{2} - \left[\frac{n(n+1)}{2} + \frac{x}{1!}\frac{n(n+1)}{2} + \frac{x^2}{2!}\left(\frac{n(n+1)}{2} - 2\right) + \right.$$

$$\left. + \frac{x^3}{3!}\left(\frac{n(n+1)}{2} - 1 - 2\right) + \ldots + \frac{x^n}{n!}\left(\frac{n(n+1)}{2} - 1 - 2 - \ldots - (n-1)\right)\right] =$$

$$= \frac{n(n+1)}{2}\left(e^x - E_n(x)\right) = x^2\left(\frac{1}{2!}\frac{1\cdot 2}{2} + \frac{x}{3!}\frac{2\cdot 3}{2} + \frac{x^2}{4!}\frac{2\cdot 4}{2} + \ldots + \frac{x^{n-2}}{n!}\frac{n(n+1)}{2}\right) =$$

$$= \frac{n(n+1)}{2}\left(e^x - E_n(x)\right) + \frac{x^2}{2}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-3}}{(n-3)!} + \frac{x^{n-2}}{(n-2)!}\right) =$$

$$= \frac{n(n+1)}{2}\left(e^x - E_n(x)\right) + \frac{x^2}{E_{\ n-2}}(x), \text{which converges to } \frac{x^2}{2}e^x.$$

Also solved by *Nicuşor Minculete*, Dimitrie Cantemir, University, Braşov, Romania and by *Marius Olteanu*, S.C. Hidroconstrucţia S.A., Râmnicu Vâlcea, Romania.

283. *Proposed by Dumitru M. Bătineţu-Giurgiu, Bucharest, Romania.* Let $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that there exists

$$\lim_{x\to\infty} \frac{(f(x))^{\frac{1}{x}}}{x} = a \in \mathbb{R}_+^*$$

and $s, t \in \mathbb{R}$ with $s + t = 1$. Show that

$$\lim_{x\to\infty}\left((x+1)^s\,(f(x+1))^{\frac{t}{x+1}} - x^s\,(f(x))^{\frac{t}{x}}\right) = b \in \mathbb{R},$$

if and only if there exists

$$\lim_{x\to\infty} \frac{f(x+1)}{xf(x)} = c \in \mathbb{R}_+^*,$$

we have $a^t\left(s + t\ln\dfrac{c}{a}\right) = b$.

Solution by the author. Denote $v(x) = (f(x))^{\frac{1}{x}}$, $\forall\, x \in \mathbb{R}_+^*$. We have to show that there exists $\lim\limits_{x\to\infty}\left((x+1)^s \cdot (v(x+1))^t - (x^s)\,(v(x))^t\right) = b \in \mathbb{R}$, if and only if there is $\lim\limits_{x\to\infty} \frac{f(x+1)}{xf(x)} = c \in \mathbb{R}_+^*$.

Observe that

$$(x+1)^s \cdot (v(x+1))^t - x^s\,(v(x))^t = x^s\,(v(x))^t \cdot \left(\left(\frac{x+1}{x}\right)^s \cdot \left(\frac{v(x+1)}{v(x)}\right)^t - 1\right) =$$

$$= x^s\,(v(x))^t \cdot (u(x)-1) = x^{s+t}\left(\frac{v(x)}{x}\right)^t \cdot (u(x)-1) =$$

$$= x \cdot \left(\frac{v(x)}{x}\right)^t \cdot \frac{u(x)-1}{\ln u(x)} \cdot \ln u(x) = \left(\frac{v(x)}{x}\right)^t \cdot \frac{u(x)-1}{\ln u(x)} \cdot \ln\,(u(x))^x\,, \quad \forall\, x \in \mathbb{R}_+^*, \tag{1}$$

where $u : \mathbb{R}_+^* \to \mathbb{R}_+^*$, $u(x) = \left(\frac{x+1}{x}\right)^s \left(\frac{v(x+1)}{v(x)}\right)^t$. It is evident that

$$\lim_{x\to\infty} u(x) = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^s \lim_{x\to\infty}\left(\frac{v(x+1)}{v(x)}\right)^t =$$

$$= \lim_{x\to\infty}\left(\frac{v(x+1)}{x+1} \cdot \frac{x}{v(x)} \cdot \frac{x+1}{x}\right)^t = \left(a \cdot \frac{1}{a} \cdot 1\right)^t = 1,$$

and thus $\lim\limits_{x\to\infty} \frac{u(x)-1}{\ln u(x)} = 1$. Here we tock into account that

$$\lim_{x\to\infty} \frac{v(x)}{x} = \lim_{x\to\infty} \frac{(f(x))^{1/2}}{x} = a \in \mathbb{R}_+^*.$$

We obtain

$$(u(x))^x = \left(\frac{x+1}{x}\right)^{sx}\left(\frac{v(x+1)}{v(x)}\right)^{tx} = \left(1+\frac{1}{x}\right)^{xs}\left(\frac{f(x+1)}{xf(x)} \cdot \frac{x}{v(x+1)}\right)^t =$$

$$= \left(1+\frac{1}{x}\right)^{xs}\left(\frac{f(x+1)}{xf(x)} \cdot \frac{x+1}{v(x+1)} \cdot \frac{x}{x+1}\right)^t ,;\ \forall\, x \in \mathbb{R}_+^*. \tag{2}$$

If there is $c = \lim\limits_{x\to\infty} \frac{f(x+1)}{x-f(x)} \in \mathbb{R}_+^*$, then (2) implies

$$\lim_{x\to\infty} (u(x))^x = \mathrm{e}^s\left(c \cdot \frac{1}{a} \cdot 1\right)^t = \mathrm{e}^1\left(\frac{c}{a}\right)^t$$

and from (1) we infer

$$\lim_{x\to\infty}\left((x+1)^s \cdot (v(x+1))^t - x^s\,(v(x))^t\right) =$$

$$= \lim_{x\to\infty}\left(\frac{v(x)}{x}\right)^t \cdot \lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)} \cdot \ln\left(\lim_{x\to\infty}(u(x))^x\right) = a^t \cdot 1 \cdot \ln\left(\mathrm{e}^s\left(\frac{c}{a}\right)^t\right) = a^t\left(s + t\ln\frac{a}{c}\right).$$

Conversely, if there exists $b = \lim\limits_{x\to\infty}\left((x+1)^s \cdot (v(x+1))^t - x^s\,(v(x))^t\right) \in \mathbb{R}$, from (1) one can deduce

$$b = a^t \cdot 1 \cdot \ln\left(\lim_{x\to\infty}(u(x))^x\right) \Leftrightarrow b \cdot a^{-t} = \ln\left(\lim_{x\to\infty}(u(x))^x\right) \Leftrightarrow \lim_{x\to\infty}(u(x))^x = \mathrm{e}^{b\cdot a^{-t}} \Leftrightarrow$$

$$\Leftrightarrow \mathrm{e}^{ba^{-t}} = \mathrm{e}^s \lim_{x\to\infty}\left(\frac{f(x+1)}{xf(x)}\right)^t \cdot \lim_{x\to\infty}\left(\frac{x+1}{v(x+1)}\right)^t \cdot \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^t \Leftrightarrow$$

$$\Leftrightarrow \mathrm{e}^{b\cdot a^{-t}-s} = a^{-t} \cdot \lim_{x\to\infty}\left(\frac{f(x+1)}{xf(x)}\right)^t \Leftrightarrow a^t \cdot \mathrm{e}^{b\cdot a^{-t}-s} = \lim_{x\to\infty}\left(\frac{f(x+1)}{xf(x)}\right)^t \Leftrightarrow$$

$$\Leftrightarrow \lim_{x\to\infty}\frac{f(x+1)}{xf(x)} = a \cdot \mathrm{e}^{\left(b\cdot a^{-t}-s\right)\frac{1}{t}} \in \mathbb{R}_+^* \Leftrightarrow c = a \cdot \mathrm{e}^{\frac{b\cdot a^{-t}-s}{t}} \in \mathbb{R}_+^*.$$

Thus, the problem is solved.

Also solved by *Nicuşor Minculete*, Dimitrie Cantemir, University, Braşov, Romania and by *Marius Olteanu*, S.C. Hidroconstrucţia S.A., Râmnicu Vâlcea, Romania.

284. *Proposed by Marius Olteanu, S.C. Hidroconstrucţia S.A., Râmnicu Vâlcea, Romania.* In the orthocentric tetrahedron $[ABCD]$ denote by r_A, r_B, r_C, r_D the inradius of BCD, ACD, ABD, ABC, and by r and R the inradius and circumradius of the inscribed and circumscribed spheres of the tetrahedron, respectively. Show that we have the following refinement of *Euler-Durrande* inequality ($R \geq 3r$):

$$R^2 \geq r^2 + r_A^2 + r_B^2 + r_C^2 + r_D^2 \geq 9r^2.$$

Solution by the author. Denote $a = BC$, $b = CA$, $c = AB$, $l = AD$, $m = BD$, $n = CD$, R_X is the circumradius of the triangle oposed to $X \in \{A, B, C, D\}$, h_X is the altitude of the tetrahedron which contains the vortex $X \in \{A, B, C, D\}$, O is the center of circumscribed spfere of the tetrahedron, I is the center of inscribed spherei of the tetrahedron and H is the orthocenterof the given tetrahedron.

By theorem 5, pag. 165 from [1] we have

$$HI^2 = R^2 + 3r^2 - \frac{1}{12}\left(a^2 + b^2 + c^2 + l^2 + m^2 + n^2\right);$$

and since $HI^2 \geq 0$ it follows that

$$R^2 + 3^2 \geq \frac{1}{12}\left(a^2 + b^2 + c^2 + l^2 + m^2 + n^2\right). \tag{1}$$

On te other hand, since $[ABCD]$ is orthocentric, we have

$$a^2 + l^2 = b^2 + m^2 = c^2 + n^2, \quad ([3]); \tag{2}$$

$$h_A^2 + 4R_A^2 = h_B^2 + 4R_B^2 = h_C + 4R_C^2 = h_D^2 + 4R_D^2 = a^2 + l^2, \quad ([3]). \tag{3}$$

From (1), (2) and (3) we infer

$$R^2 + 3r^2 \geq \frac{1}{12} \cdot 3\left(a^2 + l^2\right) = \frac{a^2 + l^2}{4} = \frac{4\left(a^2 + l^2\right)}{16} =$$

$$= \frac{\left(h_A^2 + h_B^2 + h_C^2 + h_D^2\right) + 4\left(R_A^2 + R_B^2 + R_C^2 + R_D^2\right)}{16} = \frac{1}{16}\sum h_A^2 + \frac{1}{4}\sum R_A^2. \tag{4}$$

But, by Euler's inequality,

$$\frac{1}{16} \cdot \sum h_A^2 \geq 4r^2, \quad ([3]), \tag{5}$$

$$R_A^2 \geq 4r_A^2 \tag{6}$$

and analogues, and we obtain from (6)

$$\frac{1}{4}\sum R_A^2 \geq \sum r_A^2. \tag{7}$$

Now, from (4), (5) and (7) it follows that

$$R^2 + 3r^2 \geq 4r_A^2 + r_B^2 + r_C^2 + r_D^2 \tag{8}$$

or equivalently

$$R^2 \geq r^2 + r_A^2 + r_B^2 + r_C^2 + r_D^2.$$

But, according to (8), pag. 99, from [2], we have

$$r_A^2 + r_B^2 + r_C^2 + r_D^2 \geq 8r^2;$$

or

$$r^2 + r_A^2 + r_B^2 + r_C^2 + r_D^2 \geq 9r^2. \tag{9}$$

Se observă imediat că din inegalităţile (8) şi (9) obţinem:

$$R^2 \geq r^2 + r_A^2 + r_B^2 + r_C^2 + r_D^2 \geq 9r^2,$$

cu egalitate dacă şi numai dacă $[ABCD$ este tetraedru regulat.

References

[1] L. Niculescu, A. Bumbăcea, A. Catană, P. Hoja, Gh. G. Niculescu, N. Oprea, C. Zara, *Metode de rezolvare a problemelor de geometrie*, Editura Universităţii din Bucureşti, 1998.

[2] M. Olteanu, *Noi rafinări ale inegalităţii Durrande în tetraedru*, G.M.-A, nr. 2/2008.

[3] M. Olteanu, *Inegalităţi în tetraedru*, Editura Universitară Conspress, Bucureşti, 2003.

Solution by Nicuşor Minculete, Dimitrie Cantemir University, Braşov, Romania. In [1] at page 99, we find the inequality

$$\frac{8}{9}R^2 \geq r_A^2 + r_B^2 + r_C^2 + r_D^2 \geq 8r^2,$$

which is valid in every tetrahedron. As a consequence, using the *Euler-Durrande* inequality we have

$$R^2 = \frac{8}{9}R^2 + \frac{R^2}{9} \geq \frac{8}{9}R^2 + r^2 \geq r^2 + r_A^2 + r_B^2 + r_C^2 + r_D^2 \geq 9r^2,$$

which is exactly the inequality given in the hypthesis.

References

[1] M. Olteanu, *Noi rafinări ale inegalităţii Durrande în tetraedru*, G.M.-A, nr. 2/2008.