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Infinite families of congruences modulo 4 for 16-regular partitions

by

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Abstract

Let $b_t(n)$ denote the number of t-regular partitions of n. In recent years, some parities for $b_t(n)$ have been proved for small t. In particular, some infinite families of congruences modulo 2 for $b_{16}(n)$ were established by Cui and Gu. Motivated by their work, we prove several infinite families of congruences modulo 4 for $b_{16}(n)$.

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1 Introduction

This paper is concerned with congruences modulo 4 for 16-regular partition. Recall that a partition of n is a non-increasing sequence of positive integers, called parts, whose sum is n. If $t \geq 2$ is an integer, then a partition is called an t-regular partition if there is no part divisible by t. Let $b_t(n)$ denote the number of t-regular partitions of n. As usual, set $b_t(0) = 1$. The generating function of $b_t(n)$ is

$$\sum_{n=0}^{\infty} b_t(n)q^n = \frac{f_t}{f_1}, \tag{1.1}$$

where here and throughout this paper,

$$f_k := \prod_{n=1}^{\infty}(1 - q^{nk}).$$

In recent years, a number of congruences for t-regular partitions have been proved. Merca [13] proved some congruences modulo 4 for $b_2(n)$. Yao [22] proved some infinite families of congruences modulo 2 for $b_3(n)$. Andrews, Hirschhorn and Sellers [1], Ballantine and Merca [2], and Xia [15] proved some congruences modulo 3 and powers of 2 for $b_4(n)$. Baruah and Das [3] proved some parity results for $b_7(n)$ and $b_{23}(n)$. Cui and Gu [6], Keith [8], Lin and Wang [11], and Xia and Yao [17, 18, 20] deduced several infinite families of congruences modulo powers of 2 and 3 for $b_9(n)$. Xia [16] proved some congruences modulo t for $b_t(n)$ by employing some theta functions, where $t \in \{13, 17, 19\}$. Recently, Keith and Zanello [9] studied the parity of the coefficients of certain eta-quotients and investigated the parity of $b_t(n)$ with $t \leq 28$. Cui and Gu [5] discovered infinite families of congruences modulo 2 for some t-regular partition functions, where $t \in \{2, 4, 5, 8, 13, 16\}$. In particular, they proved the following infinite families of congruences modulo 2 for $b_{16}(n)$:

$$b_{16}\left(p^{2k}n + \frac{(8i + 5p)p^{2k-1} - 5}{8}\right) \equiv 0 \pmod 2,$$

where p is a prime with $p \equiv 3 \pmod 4$, $k \geq 1$ and $1 \leq i \leq p-1$. For more details on congruence properties for partition functions, see for example [12, 19, 21].

In this paper, motivated by Cui and Gu's work, we prove some infinite families of congruences modulo 4 for $b_{16}(n)$. The main results of this paper can be stated as follows.

Theorem 1. *For $n \geq 0$ and $\alpha \geq 0$,*

$$b_{16}\left(3^{4\alpha}n + \frac{5(3^{4\alpha}-1)}{8}\right) \equiv (-1)^{\alpha} b_{16}(n) \pmod 4, \tag{1.2}$$

$$b_{16}\left(3^{4\alpha+4}n + \frac{189 \times 3^{4\alpha}-5}{8}\right) \equiv 0 \pmod 4, \tag{1.3}$$

$$b_{16}\left(3^{4\alpha+4}n + \frac{621 \times 3^{4\alpha}-5}{8}\right) \equiv 0 \pmod 4. \tag{1.4}$$

Based on (1.2) and the facts that $b_{16}(1) = 1$, $b_{16}(2) = 2$, $b_{16}(3) = 3$ and $b_{16}(11) = 56$, we can obtain the following corollary:

Corollary 1. *For $\alpha \geq 0$ and $0 \leq i \leq 3$,*

$$b_{16}\left(3^{4\alpha}r_i + \frac{5(3^{4\alpha}-1)}{8}\right) \equiv i \pmod 4,$$

where $r_0 = 11$, $r_1 = 1$, $r_2 = 2$ and $r_3 = 3$.

Theorem 2. *Let $p \geq 5$ be a prime with $p \equiv 7, 11, 13, 17, 19, 23 \pmod{24}$. For $k \geq 0$, if $p \nmid n$, then*

$$b_{16}\left(27p^{2k+1}n + \frac{45p^{2k+2}-5}{8}\right) \equiv b_{16}\left(3p^{2k+1}n + \frac{5(p^{2k+2}-1)}{8}\right) \pmod 4. \tag{1.5}$$

Theorem 3. *Let $p \geq 5$ be a prime with $p \equiv 5 \pmod 6$. For $k \geq 0$, if $p \nmid n$, then*

$$b_{16}\left(27p^{2k+1}n + \frac{117p^{2k+2}-5}{8}\right) \equiv b_{16}\left(3p^{2k+1}n + \frac{13p^{2k+2}-5}{8}\right) \pmod 4. \tag{1.6}$$

Theorem 4. *We have*

$$\lim_{n\to\infty} \frac{\#\{m | b_{16}(27m+5) \equiv b_{16}(3m) \pmod 4,\ m \leq n\}}{n}$$
$$= \lim_{n\to\infty} \frac{\#\{m | b_{16}(27m+14) \equiv b_{16}(3m+1) \pmod 4,\ m \leq n\}}{n} = 1. \tag{1.7}$$

2 Proof of Theorem 1

Setting $t = 16$ in (1.1), we get

$$\sum_{n=0}^{\infty} b_{16}(n)q^n = \frac{f_{16}}{f_1}. \tag{2.1}$$

Andrews, Hirschhorn and Sellers [1] discovered the following 3-dissection formula for $\frac{f_4}{f_1}$:

$$\frac{f_4}{f_1} = \frac{f_{12}f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}. \tag{2.2}$$

Substituting (2.2) into (2.1) yields

$$\begin{aligned}\sum_{n=0}^{\infty} b_{16}(n)q^n =& \frac{f_4}{f_1}\cdot\frac{f_{16}}{f_4} = \left(\frac{f_{12}f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}\right)\\ &\times\left(\frac{f_{48}f_{72}^4}{f_{12}^3 f_{144}^2} + q^4\frac{f_{24}^2 f_{36}^3 f_{144}}{f_{12}^4 f_{72}^2} + 2q^8\frac{f_{24} f_{72} f_{144}}{f_{12}^3}\right)\\ \equiv& \frac{f_{18}^4 f_{48} f_{72}^4}{f_3^3 f_{12}^2 f_{36}^2 f_{144}^2} + q\frac{f_6^2 f_9^3 f_{36} f_{48} f_{72}^4}{f_3^4 f_{12}^3 f_{18}^2 f_{144}^2} + 2q^2\frac{f_6 f_{18} f_{36} f_{48} f_{72}^4}{f_3^3 f_{12}^3 f_{144}^2}\\ &+ q^4\frac{f_{18}^4 f_{24}^2 f_{36} f_{144}}{f_3^3 f_{12}^3 f_{72}^2} + q^5\frac{f_6^2 f_9^3 f_{24}^2 f_{36}^4 f_{144}}{f_3^4 f_{12}^4 f_{18}^2 f_{72}^2} + 2q^6\frac{f_6 f_{18} f_{24}^2 f_{36}^4 f_{144}}{f_3^3 f_{12}^4 f_{72}^2}\\ &+ 2q^8\frac{f_{18}^4 f_{24} f_{72} f_{144}}{f_3^3 f_{12}^2 f_{36}^2} + 2q^9\frac{f_6^2 f_9^3 f_{24} f_{36} f_{72} f_{144}}{f_3^4 f_{12}^3 f_{18}^2} \pmod 4. \end{aligned}\tag{2.3}$$

Picking out those terms in which the power of q is congruent to 2 modulo 3 in (2.3), then dividing them by q^2 and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} b_{16}(3n+2)q^n \equiv 2\frac{f_2 f_6 f_{12} f_{16} f_{24}^4}{f_1^3 f_4^3 f_{48}^2} + q\frac{f_2^2 f_3^3 f_8^2 f_{12}^4 f_{48}}{f_1^4 f_4^4 f_6^2 f_{24}^2} + 2q^2\frac{f_6^4 f_8 f_{24} f_{48}}{f_1^3 f_4^2 f_{12}^2} \pmod 4. \tag{2.4}$$

By the binomial theorem, it is easy to prove that for all positive integers m, k and any prime p,

$$f_k^{p^m} \equiv f_{pk}^{p^{m-1}} \pmod{p^m}. \tag{2.5}$$

In view of (2.4) and (2.5),

$$\sum_{n=0}^{\infty} b_{16}(3n+2)q^n \equiv 2\frac{f_2^2}{f_1}\cdot\frac{f_{12}^2}{f_6} + q\frac{f_{48}}{f_3} + 2q^2\cdot\frac{f_1}{f_2^2}\cdot f_{24}^3 \pmod 4. \tag{2.6}$$

The following 3-dissection formula for $\frac{f_2^2}{f_1}$ follows from Berndt's book [4, Corollary (ii), p. 49]:

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{2.7}$$

Hirschhorn and Sellers [7] proved the following 3-dissection formula for $\frac{f_1}{f_2^2}$:

$$\frac{f_1}{f_2^2} = \frac{f_3^2 f_9^3}{f_6^6} - q\frac{f_3^3 f_{18}^3}{f_6^7} + q^2\frac{f_3^4 f_{18}^6}{f_6^8 f_9^3}. \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.6) yields

$$\sum_{n=0}^{\infty} b_{16}(3n+2)q^n \equiv 2\frac{f_{12}^2}{f_6}\left(\frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}\right) + q\frac{f_{48}}{f_3} + 2q^2 f_{24}^3\left(\frac{f_3^2 f_9^3}{f_6^6} - q\frac{f_3^3 f_{18}^3}{f_6^7} + q^2\frac{f_3^4 f_{18}^6}{f_6^8 f_9^3}\right)$$

$$\equiv 2\frac{f_9^2 f_{12}^2}{f_3 f_{18}} + 2q\frac{f_{12}^2 f_{18}^2}{f_6 f_9} + q\frac{f_{48}}{f_3} + 2q^2\frac{f_3^2 f_9^3 f_{24}^3}{f_6^6}$$

$$+ 2q^3\frac{f_3^3 f_{18}^3 f_{24}^3}{f_6^7} + 2q^4\frac{f_3^4 f_{18}^6 f_{24}^3}{f_6^8 f_9^3} \pmod 4. \tag{2.9}$$

Extracting those terms in which the power of q is congruent to 1 modulo 3 in (2.9), then dividing them by q and replacing q^3 by q, we arrive at

$$\sum_{n=0}^{\infty} b_{16}(9n+5)q^n \equiv 2\frac{f_4^2 f_6^2}{f_2 f_3} + \frac{f_{16}}{f_1} + 2q\frac{f_1^4 f_6^6 f_8^3}{f_2^8 f_3^3}$$

$$\equiv 2\frac{f_4^2}{f_2}\cdot\frac{f_6^2}{f_3} + \frac{f_4}{f_1}\cdot\frac{f_{16}}{f_4} + 2q\frac{f_6^5}{f_3}\cdot\frac{f_8^2}{f_4} \pmod 4. \qquad \text{(by (2.5))} \tag{2.10}$$

Substituting (2.2) and (2.7) into (2.10) yields

$$\sum_{n=0}^{\infty} b_{16}(9n+5)q^n \equiv 2\frac{f_6^2}{f_3}\left(\frac{f_{12} f_{18}^2}{f_6 f_{36}} + q^2\frac{f_{36}^2}{f_{18}}\right) + \left(\frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}\right)$$

$$\times\left(\frac{f_{48} f_{72}^4}{f_{12}^3 f_{144}^2} + q^4\frac{f_{24}^2 f_{36}^3 f_{144}}{f_{12}^4 f_{72}^2} + 2q^8\frac{f_{24} f_{72} f_{144}}{f_{12}^3}\right) + 2q\frac{f_6^5}{f_3}\left(\frac{f_{24} f_{36}^2}{f_{12} f_{72}} + q^4\frac{f_{72}^2}{f_{36}}\right)$$

$$\equiv 2\frac{f_6 f_{12} f_{18}^2}{f_3 f_{36}} + \frac{f_{18}^4 f_{48} f_{72}^4}{f_3^3 f_{12}^2 f_{36}^2 f_{144}^2} + q\frac{f_6^2 f_9^3 f_{36} f_{48} f_{72}^4}{f_3^4 f_{12}^3 f_{18}^2 f_{144}^2} + 2q\frac{f_6^5 f_{24} f_{36}^2}{f_3 f_{12} f_{72}}$$

$$+ 2q^2\frac{f_6 f_{18} f_{36} f_{48} f_{72}^4}{f_3^3 f_{12}^3 f_{144}^2} + 2q^2\frac{f_6^2 f_{36}^2}{f_3 f_{18}} + q^4\frac{f_{18}^4 f_{36} f_{24}^2 f_{144}}{f_3^3 f_{12}^3 f_{72}^2}$$

$$+ q^5\frac{f_6^2 f_9^3 f_{24}^2 f_{36}^4 f_{144}}{f_3^4 f_{12}^4 f_{18}^2 f_{72}^2} + 2q^5\frac{f_6^5 f_{72}^2}{f_3 f_{36}} + 2q^6\frac{f_6 f_{18} f_{24}^2 f_{36}^4 f_{144}}{f_3^3 f_{12}^4 f_{72}^2}$$

$$+ 2q^8\frac{f_{18}^4 f_{24} f_{72} f_{144}}{f_3^3 f_{12}^2 f_{36}^2} + 2q^9\frac{f_6^2 f_9^3 f_{24} f_{36} f_{72} f_{144}}{f_3^4 f_{12}^3 f_{18}^2} \pmod 4. \tag{2.11}$$

Extracting the terms of the form q^{3n+2} in (2.11), then dividing them by q^2 and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} b_{16}(27n+23)q^n \equiv 2\frac{f_2 f_6 f_{12} f_{16} f_{24}^4}{f_1^3 f_4^3 f_{48}^2} + 2\frac{f_2^2 f_{12}^2}{f_1 f_6} + q\frac{f_2^2 f_3^3 f_8^2 f_{12}^4 f_{48}}{f_1^4 f_4^4 f_6^2 f_{24}^2}$$

$$+ 2q\frac{f_2^5 f_{24}^2}{f_1 f_{12}} + 2q^2\frac{f_6^4 f_8 f_{24} f_{48}}{f_1^3 f_4^2 f_{12}^2} \pmod 4. \tag{2.12}$$

Thanks to (2.5),

$$\frac{f_2f_6f_{12}f_{16}f_{24}^4}{f_1^3f_4^3f_{48}^2} \equiv \frac{f_2^2f_{12}^2}{f_1f_6} \pmod 2, \tag{2.13}$$

$$\frac{f_2^2f_3^3f_8^2f_{12}^4f_{48}}{f_1^4f_4^4f_6^2f_{24}^2} \equiv \frac{f_{48}}{f_3} \pmod 4, \tag{2.14}$$

$$\frac{f_2^5f_{24}^2}{f_1f_{12}} \equiv \frac{f_2^2}{f_1}\frac{f_4^2}{f_2}\frac{f_{24}^2}{f_{12}} \pmod 2, \tag{2.15}$$

$$\frac{f_6^4f_8f_{24}f_{48}}{f_1^3f_4^2f_{12}^2} \equiv \frac{f_1f_{24}^3}{f_2^2} \pmod 2. \tag{2.16}$$

Combining (2.12)–(2.16) yields

$$\sum_{n=0}^{\infty} b_{16}(27n+23)q^n \equiv q\frac{f_{48}}{f_3} + 2q\cdot\frac{f_2^2}{f_1}\cdot\frac{f_4^2}{f_2}\cdot\frac{f_{24}^2}{f_{12}} + 2q^2\frac{f_1}{f_2^2}\cdot f_{24}^3 \pmod 4. \tag{2.17}$$

Substituting (2.7) and (2.8) into (2.17), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} b_{16}(27n+23)q^n &\equiv q\frac{f_{48}}{f_3} + 2q\frac{f_{24}^2}{f_{12}}\left(\frac{f_6f_9^2}{f_3f_{18}} + q\frac{f_{18}^2}{f_9}\right)\left(\frac{f_{12}f_{18}^2}{f_6f_{36}} + q^2\frac{f_{36}^2}{f_{18}}\right)\\ &\quad + 2q^2f_{24}^3\left(\frac{f_3^2f_9^3}{f_6^6} - q\frac{f_3^3f_{18}^3}{f_6^7} + q^2\frac{f_3^4f_{18}^6}{f_6^8f_9^3}\right)\\ &\equiv q\frac{f_{48}}{f_3} + 2q\frac{f_9^2f_{18}f_{24}^2}{f_3f_{36}} + 2q^2\frac{f_{18}^4f_{24}^2}{f_6f_9f_{36}} + 2q^2\frac{f_3^2f_9^3f_{24}^3}{f_6^6} + 2q^3\frac{f_6f_9^2f_{24}^2f_{36}^2}{f_3f_{12}f_{18}^2}\\ &\quad + 2q^3\frac{f_3^3f_{18}^3f_{24}^3}{f_6^7} + 2q^4\frac{f_{18}f_{24}^2f_{36}^2}{f_9f_{12}} + 2q^4\frac{f_3^4f_{18}^6f_{24}^3}{f_6^8f_9^3} \pmod 4. \end{aligned}\tag{2.18}$$

By (2.5),

$$\frac{f_9^2f_{18}f_{24}^2}{f_3f_{36}} \equiv \frac{f_{48}}{f_3} \pmod 2, \tag{2.19}$$

$$\frac{f_{18}^4f_{24}^2}{f_6f_9f_{36}} \equiv \frac{f_3^2f_9^3f_{24}^3}{f_6^6} \pmod 2, \tag{2.20}$$

$$\frac{f_6f_9^2f_{24}^2f_{36}^2}{f_3f_{12}f_{18}^2} \equiv \frac{f_3^3f_{18}^3f_{24}^3}{f_6^7} \pmod 2, \tag{2.21}$$

$$\frac{f_{18}f_{24}^2f_{36}^2}{f_9f_{12}} \equiv \frac{f_3^4f_{18}^6f_{24}^3}{f_6^8f_9^3} \pmod 2. \tag{2.22}$$

Based on (2.18)–(2.22), we deduce that

$$\sum_{n=0}^{\infty} b_{16}(81n+50)q^n \equiv 3\frac{f_{16}}{f_1} \pmod 4 \tag{2.23}$$

and for $n \geq 0$,

$$b_{16}(81n+23) \equiv b_{16}(81n+77) \equiv 0 \pmod{4}. \tag{2.24}$$

Combining (2.1) and (2.23), we see that for $n \geq 0$,

$$b_{16}(81n+50) \equiv -b_{16}(n) \pmod{4}. \tag{2.25}$$

By (2.25) and mathematical induction, we get (1.2). Replacing n by $81n+23$ and $81n+77$ in (1.2), respectively, and using (2.24), we obtain (1.3) and (1.4). This completes the proof of Theorem 1. ∎

3 Proofs of Theorems 2–4

We first prove Theorem 2. Define

$$a(n) := b_{16}(9n+5) - b_{16}(n). \tag{3.1}$$

In view of (2.1) and (2.10),

$$\sum_{n=0}^{\infty} a(n)q^n \equiv 2\frac{f_4^2}{f_2} \cdot \frac{f_6^2}{f_3} + 2q\frac{f_6^5}{f_3} \cdot \frac{f_8^2}{f_4} \pmod{4}. \tag{3.2}$$

Substituting (2.7) into (3.2), we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} a(n)q^n \equiv& 2\frac{f_6^2}{f_3}\left(\frac{f_{12}f_{18}^2}{f_6 f_{36}} + q^2\frac{f_{36}^2}{f_{18}}\right) + 2q\frac{f_6^5}{f_3}\left(\frac{f_{24}f_{36}^2}{f_{12}f_{72}} + q^4\frac{f_{72}^2}{f_{36}}\right)\\ \equiv& 2f_3f_{12} + 2qf_3 \cdot \frac{f_{24}^2}{f_{12}} + 2q^2\frac{f_6^2 f_{36}^2}{f_3 f_{18}} + 2q^5\frac{f_6^5 f_{72}^2}{f_3 f_{36}} \pmod{4}, \quad \text{(by (2.5))}\end{aligned}$$

which yields

$$\sum_{n=0}^{\infty} a(3n)q^n \equiv 2f_1f_4 \pmod{4} \tag{3.3}$$

and

$$\sum_{n=0}^{\infty} a(3n+1)q^n \equiv 2f_1\frac{f_8^2}{f_4} \pmod{4}. \tag{3.4}$$

The following identity is commonly known as Euler's pentagonal number theorem and is worth highlighting here:

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \tag{3.5}$$

In view of (3.3) and (3.5),

$$\sum_{n=0}^{\infty} a(3n)q^n \equiv 2 \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m+1)/2} \sum_{k=-\infty}^{\infty} (-1)^k q^{2k(3k+1)} \pmod 4,$$

which yields

$$\begin{aligned} a(3n) &\equiv 2 \sum_{\substack{m(3m+1)/2+2k(3k+1)=n,\\ (m,k)\in\mathbb{Z}\times\mathbb{Z}}} (-1)^{m+k} \\ &\equiv 2 \sum_{\substack{(6m+1)^2+(12k+2)^2=24n+5,\\ (m,k)\in\mathbb{Z}\times\mathbb{Z}}} 1 \pmod 4. \end{aligned} \tag{3.6}$$

From (3.6), we know if $24n+5$ is not of the form x^2+y^2, then $a(3n) \equiv 0 \pmod 4$. Note that if N is of the form x^2+y^2, then $\nu_p(N)$ is even since p is a prime with $p \equiv 3 \pmod 4$ and $\left(\frac{-1}{p}\right) = -1$. Here $\nu_p(N)$ denotes the highest power of p dividing N and $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. It is easy to check that if $p \nmid n$, then

$$\nu_p\left(24\left(p^{2k+1}n + \frac{5(p^{2k+2}-1)}{24}\right) + 5\right) = \nu_p\left(24p^{2k+1}n + 5p^{2k+2}\right) = 2k+1$$

is odd. Therefore, $24\left(p^{2k+1}n + \frac{5(p^{2k+2}-1)}{24}\right) + 5$ is not of the form x^2+y^2 and

$$a\left(3p^{2k+1}n + \frac{5(p^{2k+2}-1)}{8}\right) \equiv 0 \pmod 4, \qquad p \equiv 3 \pmod 4. \tag{3.7}$$

The following identity is a classical identity of theta functions due to Gauss:

$$\frac{f_2^2}{f_1} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{3.8}$$

In view of (2.5) and (3.3),

$$\sum_{n=0}^{\infty} a(3n)q^n \equiv 2f_2 \cdot \frac{f_2^2}{f_1} \pmod 4. \tag{3.9}$$

By (3.5) and (3.8), we can rewrite (3.9) as

$$\sum_{n=0}^{\infty} a(3n)q^n \equiv 2 \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m+1)} \sum_{k=0}^{\infty} q^{k(k+1)/2} \pmod 4,$$

which yields

$$a(3n) \equiv 2 \sum_{\substack{m(3m+1)+k(k+1)/2=n,\\ (m,k)\in\mathbb{Z}\times\mathbb{N}}} (-1)^m$$

$$\equiv 2 \sum_{\substack{2(6m+1)^2+3(2k+1)^2=24n+5,\\ (m,k)\in\mathbb{Z}\times\mathbb{N}}} 1 \pmod 4.$$

Therefore, if $24n+5$ is not of the form $2x^2+3y^2$, then $a(3n)\equiv 0 \pmod 4$. Note that if N is of the form $2x^2+3y^2$, then $\nu_p(N)$ is even since p is a prime with $p\equiv 13,17,19,23 \pmod{24}$ and $\left(\frac{-6}{p}\right)=-1$. It is easy to verify if $p\nmid n$, then

$$\nu_p\left(24\left(p^{2k+1}n+\frac{5(p^{2k+2}-1)}{24}\right)+5\right)=\nu_p\left(24p^{2k+1}n+5p^{2k+2}\right)=2k+1$$

is odd. Therefore, $24\left(p^{2k+1}n+\frac{5(p^{2k+2}-1)}{24}\right)+5$ is not of the form $2x^2+3y^2$ and

$$a\left(3p^{2k+1}n+\frac{5(p^{2k+2}-1)}{8}\right)\equiv 0 \pmod 4, \qquad p\equiv 13,17,19,23 \pmod{24}. \tag{3.10}$$

Replacing n by $3p^{2k+1}n+\frac{5(p^{2k+2}-1)}{8}$ ($p\equiv 7,11,13,17,19,23 \pmod{24}$) in (3.1) and using (3.7) and (3.10), we arrive at (1.5). This completes the proof of Theorem 2.

Now, we turn to prove Theorem 3.

It follows from (3.4), (3.5) and (3.8) that

$$\sum_{n=0}^{\infty} a(3n+1)q^n \equiv 2\sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m+1)/2}\sum_{k=0}^{\infty} q^{2k(k+1)} \pmod 4.$$

Thus,

$$\begin{aligned} a(3n+1) \equiv& 2\sum_{\substack{m(3m+1)/2+2k(k+1)=n,\\ (m,k)\in\mathbb{Z}\times\mathbb{N}}}(-1)^m \\ \equiv& 2\sum_{\substack{(6m+1)^2+3(4k+2)^2=24n+13,\\ (m,k)\in\mathbb{Z}\times\mathbb{N}}} 1 \pmod 4. \end{aligned} \tag{3.11}$$

Thus, if $24n+13$ is not of the form x^2+3y^2, then $a(3n+1)\equiv 0 \pmod 4$. Note that if N is of the form x^2+3y^2, then $\nu_p(N)$ is even since p is a prime with $p\equiv 5 \pmod 6$ and $\left(\frac{-3}{p}\right)=-1$. It is easy to check that if $p\nmid n$, then

$$\nu_p\left(24\left(p^{2k+1}n+\frac{13(p^{2k+2}-1)}{24}\right)+13\right)=\nu_p\left(24p^{2k+1}n+13p^{2k+2}\right)=2k+1$$

is odd. Therefore, $24\left(p^{2k+1}n+\frac{13(p^{2k+2}-1)}{24}\right)+13$ is not of the form x^2+3y^2 and

$$a\left(3p^{2k+1}n+\frac{13\times p^{2k+2}-5}{8}\right)\equiv 0 \pmod 4, \qquad p\equiv 5 \pmod 6. \tag{3.12}$$

Replacing n by $3p^{2k+1}n+\frac{13\times p^{2k+2}-5}{8}$ in (3.1) and utilizing (3.12), we get (1.6). The proof of Theorem 3 is complete.

Recall that an integral power series $\sum_{n\in\mathbb{Z}} a(n)q^n$ is called lacunary if

$$\lim_{X\to\infty} \frac{\#\{n|a(n)=0,\ n\le X\}}{X} = 1.$$

To prove Theorem 4, we require the following classical result due to Landau [10]; see also [14].

Theorem 5. *[10, 14] Let $r(n)$ and $s(n)$ be quadratic polynomials. Then*

$$\left(\sum_{n=-\infty}^{\infty} q^{r(n)}\right)\left(\sum_{n=-\infty}^{\infty} q^{s(n)}\right)$$

is lacunary.

It follows from (3.6), (3.11) and Theorem 5 that

$$\lim_{n\to\infty} \frac{\#\{m|a(3m)\equiv 0 \pmod 4,\ 0\le m\le n\}}{n}$$
$$= \lim_{n\to\infty} \frac{\#\{m|a(3m+1)\equiv 0 \pmod 4,\ 0\le m\le n\}}{n} = 1. \quad (3.13)$$

Theorem 4 follows from (3.1) and (3.13). This completes the proof. □

4 Concluding remarks

As seen in Introduction, congruences for $b_t(n)$ have received a lot of attention in recent years. In this study, we prove some infinite families of congruences modulo 4 on $b_{16}(n)$. A natural question is to extend the congruences in this paper to modulo 8, 16, etc. However, it will likely require a different approach since the methods used in this paper run into serious limitations beyond the modulus of 8. Furthermore, it would be interesting to determine the arithmetic density of the set of integers such that $b_t(n) \equiv 0 \pmod{2^k}$ for fixed positive integers t and k.

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