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Sections of $K3$ surfaces with Picard number two and Mercat's conjecture

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Dedicated to the memory of Lucian Bădescu

Abstract

In [6, Theorem 1.1], the authors present counterexamples to Mercat's conjecture by restricting to a hyperplane section C some suitable rank-two vector bundles on a $K3$ surface whose Picard group is generated by C and another very ample divisor. We prove that the same bundles produce other counterexamples by restriction to hypersurface sections $C_n \in |nC|$ for all $n \geq 2$. In the process, we compute the Clifford indices of the corresponding hypersurface sections C_n, noting their non-generic nature for $n \geq 2$ (refer to Theorem 1). A key ingredient to prove the (semi)stability of the restricted bundles, Theorem 2, is Green's Explicit H^0 Lemma (see [10, Corollary (4.e.4)]). In what concerns the (semi)stability, although general restriction theorems such as [9, Theorem 1.2] or [7, Theorem 1.1] are applicable for sufficiently large, explicit values of n, our approach works for all $n \geq 2$. It is also worth noting that our proof deviates slightly from the one presented in [6, Proposition 3.2]. Employing the same strategy leads to an enhancement of the main result of [21]; refer to Theorem 3 for counterexamples to the conjecture on curves in $|nC|$, where C now acts as a generator of the Picard group.

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1 Introduction

Mercat's conjecture aims to establish a connection between higher-rank Brill–Noether theory and classical Brill–Noether theory concerning curves. Let C be a smooth curve of genus $g \geq 3$, and consider $\mathcal{E}$ a semistable rank vector bundle on C satisfying $h^1(C, \mathcal{E}) \geq h^0(C, \mathcal{E}) \geq 2r$. The *Clifford index of* $\mathcal{E}$ is defined as

$$\gamma(\mathcal{E}) := \mu(\mathcal{E}) - \frac{2h^0(C, \mathcal{E})}{\mathrm{rk}(\mathcal{E})} + 2 \geq 0$$

and $\mathrm{Cliff}_r(C)$, the *rth Clifford index of* C is the minimum of the Clifford indices of bundles of rank r that can contribute, i.e.

$$\mathrm{Cliff}_r(C) := \min\{\gamma(\mathcal{E}) \ : \ \mathcal{E} \in \mathcal{U}_C(r, d), d \leq r(g-1), h^0(C, \mathcal{E}) \geq 2r\}.$$

In this context, Mercat [18] conjectured that for any $r \geq 1$, we have $\mathrm{Cliff}_r(C) = \mathrm{Cliff}(C)$. It is worth noting that the inequality $\mathrm{Cliff}_r(C) \leq \mathrm{Cliff}(C)$ is readily obtained taking direct sums of line bundles $A^{\oplus r}$. Originally, the conjecture was formulated as an explicit upper bound in terms of $\mathrm{Cliff}(C)$ for the number of sections of semistable bundles, [18, p. 786]. Precisely, the conjectured bound is given by:

$$h^0(C,\mathcal{E}) \leq \frac{d}{2} - r\left(\frac{\mathrm{Cliff}(C)}{2} - 1\right)$$

for all $\mathcal{E} \in \mathcal{U}_C(r,d)$, with $d \leq r(g-1)$ and $h^0(C,\mathcal{E}) \geq 2r$. In rank two, this inequality simplifies to $h^0(C,\mathcal{E}) \leq \frac{d}{2} - \mathrm{Cliff}(C) + 2$ for all $\mathcal{E} \in \mathcal{U}_C(2,d)$, with $d \leq 2g-2$ and $\mathcal{E}$ having at least 4 independent sections. Note that the number of independent sections is always bounded, [19, Proposition 3, Proposition 4], [18, Theorem 2.1] etc, but the known general bounds are weaker than those predicted by the conjecture.

While the conjecture has been confirmed in various cases, e.g. in rank two, it holds for arbitrary k-gonal curves of genus $g > 2(k-1)(k-2)$, for general curves [2], for general k-gonal curves of genus $g > 4k-4$, for plane curves [12], [15] etc, it fails for large values k of the gonality. Specifically, several counterexamples have been provided by curves on $K3$ surfaces, as seen, for instance in [5], [13], [14], [6], [21] (see also [1], [7] for higher ranks). A current challenge is to discover additional examples of pairs (g,k) where Mercat's conjecture fails in rank two or to determine whether the existing list of counterexamples is exhaustive. In view of [15, Section 4], the problem needs to be addressed for curves of Clifford dimension one.

In this short Note, we present a new infinite set of counter-examples for the conjecture. Our methodology also revolves around the utilization of curves on $K3$ surfaces, specifically drawing upon the counterexamples identified in [6], i.e. curves on $K3$ surfaces of Picard number two. A distinctive aspect of our investigation is the transition from hyperplane sections to hypersurface sections, aligning with the exploration of $K3$ surfaces with Picard number one, as discussed in [21]. The two main technical difficulties that we have to overcome are: the computation of Clifford indices, and the semistability of the restricted bundles. These issues are addressed in Theorem 1 and Theorem 2, respectively. For the computation of the Clifford indices, we use the Main Theorem of [11], and the verification of semistability relies on Green's explicit H^0 Lemma, see [10, Corollary (4.e.4)]. Employing the same strategy leads to an enhancement of the main result of [21]; see Theorem 3 for counterexamples to the conjecture on curves in $|nC|$, where C now acts as a generator of the Picard group.

2 Basic properties of Lazarsfeld–Mukai bundles

We follow closely the presentation of [16]. Let S be a $K3$ surface, C be a smooth connected curve of genus g in S, and A be a base-point-free complete g^r_d on C. Denote by M_A the kernel of the evaluation map

$$\mathrm{ev}_A : H^0(A) \otimes \mathcal{O}_C \to A.$$

The map ev_A induces a surjective morphism $H^0(A) \otimes \mathcal{O}_S \to A$ of sheaves on S whose kernel $\mathcal{F}_{C,A}$ is a vector bundle of rank $(r+1)$. Its dual $\mathcal{E}_{C,A} = \mathcal{F}^\vee_{C,A}$ is called a *Lazarsfeld–Mukai bundle*. The defining sequences of $\mathcal{F}_{C,A}$ and $\mathcal{E}_{C,A}$ are

$$0 \to \mathcal{F}_{C,A} \to H^0(A) \otimes \mathcal{O}_S \to A \to 0. \tag{2.1}$$

and, respectively

$$0 \to H^0(A)^\vee \otimes \mathcal{O}_S \to \mathcal{E}_{C,A} \to K_C(-A) \to 0. \tag{2.2}$$

The bundles $\mathcal{E}_{C,A}$ and $\mathcal{F}_{C,A}$ have the following properties:

1. $\det(\mathcal{E}_{C,A}) = \mathcal{O}_S(C)$,
2. $c_2(\mathcal{E}_{C,A}) = d$,
3. $h^0(S, \mathcal{F}_{C,A}) = h^1(S, \mathcal{F}_{C,A}) = 0$,
4. $\chi(S, \mathcal{F}_{C,A}) = h^2(S, \mathcal{F}_{C,A}) = 2(r+1) + g - d - 1$,
5. $h^0(S, E_{C,A}) = r + 1 + h^0(C, K_C(-A))$,
6. $\mathcal{E}_{C,A}$ is generated off the base locus of $|K_C(-A)|$ inside C.

Restricting the sequence (2.1) to the curve C, we obtain a short exact sequence:

$$0 \to K_C^\vee(A) \to \mathcal{F}_{C,A}|_C \to M_A \to 0 \tag{2.3}$$

which implies, twisting by $K_C(-A)$ and using the adjunction formula,

$$0 \to \mathcal{O}_C \to \mathcal{F}_{C,A} \otimes K_C(-A) \to M_A \otimes K_C(-A) \to 0. \tag{2.4}$$

Note that $H^0(M_A \otimes K_C(-A)) = \ker(\mu_{0,A})$, where $\mu_{0,A} : H^0(A) \otimes H^0(K_C(-A)) \to H^0(K_C)$ is the Petri map.

3 Clifford indices of hypersurface sections of a $K3$ surface with Picard number two

Given integers $p \geq 3$ and $a \geq 2p+3$, let S be a $K3$ surface whose Picard group is generated by two very ample smooth divisors, $\mathrm{Pic}(S) = \langle C, D\rangle$, where $C^2 = 4a$, $D \cdot C = 2a + 2p + 1$, $D^2 = 4p + 2$. The existence of such surfaces is established through the surjectivity of the period map, as noted in [6]. We focus on the embedding $S \subset \mathbb{P}^{2a+1}$ defined by the complete linear system $|C|$. It is worth noting that in [6], the authors consider the surface S as being embedded via the other linear system $|D|$, denoted by $|H|$ in that context.

For the convenience of the reader, we highlight the following simple fact that was implicitly used in [6].

Lemma 1. *Put $E = C - D$. Then $E^2 = 0$, $h^0(S, \mathcal{O}_S(E)) = 2$ and $h^1(S, \mathcal{O}_S(E)) = 0$.*

Proof. The numerical data makes it evident that $E^2 = 0$. Notably, as $(-E) \cdot D = -2a + 2p + 1 < 0$ and D is ample, it implies that $-E$ cannot be effective, and it cannot be zero either. By the Riemann-Roch Theorem, we derive $h^0(S, \mathcal{O}_S(E)) \geq 2$.

Suppose the linear system $|E|$ is base-point-free; in this case, according to [20, Proposition 2.6], it follows that E is a multiple of a smooth elliptic curve. Since E is a generator of the Picard group, it must be a smooth elliptic curve. Consequently, we have $h^0(S, \mathcal{O}_S(E)) = 2$ and $h^1(S, \mathcal{O}_S(E)) = 0$.

Now, assume the linear system $|E|$ has base points. According to [20, Proposition 2.6] the linear system $|E|$ has a fixed component Δ. Write $E = \Delta + E'$ where E' is an effective divisor with $h^0(S, \mathcal{O}_S(E)) = h^0(S, \mathcal{O}_S(E'))$. From [4, Proposition 2.2], we deduce that $(E')^2 \geq 0$ and $\Delta^2 \geq 0$. Since $|E'|$ is the moving part of the linear system $|E|$, we must also have $E' \cdot \Delta > 0$. On the other hand $E^2 = 0$, which is a contradiction. □

We aim to prove that the Clifford index of any curve in the linear system $|nC|$, for $n \geq 2$, is computed by $\mathcal{O}(E)$.

Theorem 1. *For any $n \geq 2$, and any smooth curve $C_n \in |nC|$, we have* $\mathrm{Cliff}(C_n) = n(2a - 2p - 1) - 2$.

Proof. We remark that the genus of C_n is $g(C_n) = 2an^2 + 1$, and the bundle $\mathcal{O}_{C_n}(E)$ contributes to the Clifford index of C_n, with its Clifford index strictly smaller than the generic Clifford index $(an^2 - 1)$. Applying the Main Theorem of [11], and [17, Lemma 2.2] the Clifford index of C_n is computed by the restriction of a line bundle $\mathcal{O}_S(F) \in \mathrm{Pic}(S)$ by the formula

$$\mathrm{Cliff}(C_n) = \mathrm{Cliff}(\mathcal{O}_{C_n}(F)) = F \cdot C_n - F^2 - 2.$$

To simplify calculations, we work with the basis $\{C, E\}$ of $\mathrm{Pic}(S)$ instead of the original $\{C, D\}$, considering $E^2 = 0$. Note that $C \cdot E = 2a - 2p - 1 > 0$. Therefore, expressing $F = sC + tE$ with $s, t \in \mathbb{Z}$, we compute:

$$f(s,t) := \mathrm{Cliff}(\mathcal{O}_{C_n}(sC + tE)) = (n - 2s)(2a - 2p - 1)t - 4as^2 + 4ans - 2. \tag{3.1}$$

The condition $f(s,t) \geq 0$ must be satisfied due to the definition of the Clifford index.

Following the proof of [4, Theorem 3] and the proof of [6, Proposition 3.3], we observe that F is subject to the following restrictions:

(i) $F^2 \geq 0$,

(ii) $F \cdot D > 2$,

(iii) $F \cdot C_n \leq g(C_n) - 1$,

Taking into account that $g(C_n) = 2an^2 + 1$, these constraints are translated into the following conditions:

(i) $s(2as + (2a - 2p - 1)t) \geq 0$,

(ii) $4as + (2a - 2p - 1)(t - s) > 2$,

(iii) $4as + (2a - 2p - 1)t \leq 2an$.

The objective is to prove that the minimum of f, when s and t are integers satisfying conditions (i), (ii), and (iii) is attained at $(0, 1)$. Since $E \cdot C_n - E^2 - 2 = n(2a - 2p - 1) - 2$ that would conclude the proof of the theorem.

We note that $s \geq 0$. Indeed, if $s < 0$, then (i) implies that $2as + (2a - 2p - 1)t \leq 0$ and hence we obtain from (ii) that $(2p + 1)s > 0$ which is a contradiction with the assumption $s < 0$. Consequently, condition (i) is reformulated as:

(i) $2as + (2a - 2p - 1)t \geq 0$.

Additionally, we have $s \leq n$, due to (i) and (iii) leading to $2an - 2as \geq 0$.

If $s = 0$, then $f(0,t) = n(2a - 2p - 1)t - 2$ and the minimal positive value is $f(0,1) = n(2a - 2p - 1) - 2$ which we wanted to prove.

We analyze next the case $s \geq 1$. The inequalities (i) and (iii) give the following bounds for t:

$$t_{min} := -\frac{2as}{2a - 2p - 1} \leq t \leq t_{max} := \frac{2a(n - 2s)}{2a - 2p - 1}.$$

If n is even and $s = \frac{n}{2}$, we observe that $f\left(\frac{n}{2}, t\right) = an^2 - 2 > n(2a - 2p - 1) - 2$ for $n \geq 2$.

If $n > 2s$, since the coefficient of t in the expression of f is positive and $s \geq 1$, it follows that

$$f(s,t) \geq f(s, t_{min}) = 2ans - 2 > n(2a - 2p - 1) - 2.$$

If $n < 2s$, since the coefficient of t in the expression of f is negative, it holds that

$$f(s,t) \geq f(s, t_{max}) = 4as^2 - 4ans + 2an^2 - 2.$$

On the interval $\left[\frac{n}{2}, n\right]$ the degree-two function $g(s) := f(s, t_{max})$ is increasing and hence

$$f(s,t) \geq f(s, t_{max}) \geq f\left(\frac{n}{2}, t_{max}\right) = an^2 - 2 > n(2a - 2p - 1) - 2$$

for $n \geq 2$. This completes the proof. □

Remark 1. *For any integer $n \geq 2$, any K3 surface S, and any very ample line bundle $\mathcal{O}_S(C)$, consider a smooth curve C_n in the linear system $|nC|$. In this context, the Clifford index of C_n is smaller than the generic value $\left[\frac{g(C_n)-1}{2}\right]$. Specifically, the restriction of the bundle $\mathcal{O}_S(C)$ to C_n contributes to the Clifford index, and upon direct computation, its Clifford index is found to be smaller than the generic value. If $\mathcal{O}_S(C)$ generates the Picard group of S, then* $\operatorname{Cliff}(C_n)$ *is computed by the restriction of $\mathcal{O}_S(C)$. In contrast to the very generic case, the explicit situation presented here yields* $\operatorname{Cliff}(\mathcal{O}_{C_n}(C)) = 4(n-1)a - 2 > n(2a - 2p - 2) - 2 = \operatorname{Cliff}(C_n)$.

4 New counterexamples to Mercat's conjecture

We adopt the notation from in the previous sections. Consider a g^1_{p+2} denoted as A on D, and let $\mathcal{E} = \mathcal{E}_{C,A}$ be the associated Lazarsfeld–Mukai bundle. As affirmed by [6, Theorem 1.1], it follows that $\operatorname{Cliff}(C) = a$, and additionally, $\gamma(\mathcal{E}|_C) < \operatorname{Cliff}(C)$. Notably, $\mathcal{E}|_C$ is semistable ([6, Proposition 3.2]), consequently providing a counterexample to Mercat's conjecture.

In the subsequent discussion, we establish the following result.

Theorem 2. *Notation as above. Assume $a \geq 3p + 2$. For any $n \geq 2$, the bundle $\mathcal{E}|_{C_n}$ is stable and it is a counter-example to Mercat's conjecture.*

Proof. We prove first the semistability of $\mathcal{E}|_{C_n}$. Suppose, for a contradiction, that $\mathcal{E}|_{C_n}$ is not stable and consider

$$0 \to \mathcal{O}_{C_n}(B) \to \mathcal{E}|_{C_n} \to \mathcal{O}_{C_n}(D-B) \to 0$$

a destabilizing sequence. In particular,

$$\deg(B) \geq \mu(\mathcal{E}|_{C_n}) = \frac{n(2a+2p+1)}{2}. \tag{4.1}$$

Since $\mathcal{E}$ is globally generated, it follows that $\mathcal{O}_{C_n}(D-B)$, along with any other quotient of $\mathcal{E}$, is also globally generated.

If $\mathcal{O}_{C_n}(D-B) \neq \mathcal{O}_{C_n}$, then $h^0(C_n, \mathcal{O}_{C_n}(D-B)) \geq 2$. Furthermore, since

$$h^0(C_n, \mathcal{O}_{C_n}(C_n - D + B)) \geq h^0(C_n, \mathcal{O}_{C_n}(C_n - D)) \geq h^0(S, \mathcal{O}_S(C_n - D)) \geq 2,$$

it follows that $\mathcal{O}_{C_n}(D-B)$ contributes to the Clifford index of C_n. Using the inequality (4.1) we evaluate

$$\mathrm{Cliff}(\mathcal{O}_{C_n}(D-B)) \leq n(2a+2p+1) - \deg(B) - 2 \leq \frac{n(2a+2p+1)}{2} - 2$$

and the latter value is smaller than $n(2a-2p-1)-2 = \mathrm{Cliff}(C_n)$, by the assumption $a \geq 3p+2$, leading to a contradiction.

In conclusion, we have $\mathcal{O}_{C_n}(D-B) = \mathcal{O}_{C_n}$. This implies the existence of a short exact sequence

$$0 \to \mathcal{O}_{C_n}(D) \to \mathcal{E}|_{C_n} \to \mathcal{O}_{C_n} \to 0$$

and, as a consequence, we have:

$$h^0(C_n, \mathcal{E}|_{C_n}) \geq h^0(C_n, \mathcal{O}_{C_n}(D)). \tag{4.2}$$

Since $h^0(S, \mathcal{O}_S(D - nC)) = 0$ for all $n \geq 1$, it follows that

$$h^0(C_n, \mathcal{O}_{C_n}(D)) \geq h^0(S, \mathcal{O}_S(D)) = 2p+1.$$

Moreover, the two dimensions are equal, as shown below.

Claim 1. $h^1(S, \mathcal{O}_S(D - nC)) = 0$ for all $n \geq 1$.

We proceed by induction on n. For the base case $n = 1$, we apply Lemma 1. For the induction step, for $n \geq 2$, consider the long cohomology sequence of the short exact sequence

$$0 \to \mathcal{O}_S((n-1)C - D) \to \mathcal{O}_S(nC - D) \to \mathcal{O}_C(nC - D) \to 0$$

and observe that $h^1(C, K_C^{\otimes n}(-D))) = 0$ by degree reasons.

Claim 2. $h^1(S, \mathcal{E}(-nC)) = 0$.

To establish Claim 2, we begin with the defining exact sequence (2.1) of the dual of the Lazarsfeld-Mukai bundle:

$$0 \to \mathcal{E}^\vee \to H^0(A) \otimes \mathcal{O}_S \to A \to 0$$

(where A was a g^1_{p+2} on D) twist it with $\mathcal{O}_S(nC)$ and take the long cohomology sequence:

$$0 \to H^0(\mathcal{E}^\vee(nC)) \to H^0(A) \otimes H^0(\mathcal{O}_S(nC)) \to H^0(D, A(nC)) \to H^1(\mathcal{E}^\vee(nC)) \to 0.$$

Claim 1 implies that the restriction map $H^0(S, \mathcal{O}_S(nC)) \to H^0(D, \mathcal{O}_D(nC))$ is surjective. Hence, Claim 2 would follow from the surjectivity of the multiplication map

$$H^0(D, A) \otimes H^0(D, \mathcal{O}_D(nC)) \to H^0(D, A(nC))$$

To this end, we apply [10, Corollary (4.e.4)]; the hypothesis

$$\deg(A) + \deg(\mathcal{O}_D(C_n)) \geq 4g(D) + 2$$

is verified for $n \geq 2$, as the genus of D is $g(D) = 2p + 2$, and $\deg(A) + \deg(\mathcal{O}_D(C_n)) = (p+2) + n(2a + 2p + 1) \geq 13p + 16$. Claim 2 is proved.

We consider the short exact sequence

$$0 \to \mathcal{E}(-nC) \to \mathcal{E} \to \mathcal{E}|_{C_n} \to 0.$$

Since $h^0(S, \mathcal{E}(-nC)) = h^1(S, \mathcal{E}(-nC)) = 0$, (the vanishing of h^0 follows from the defining sequence (2.2)), we infer that $h^0(S, \mathcal{E}) = p + 3$. This leads to a contradiction with (4.2).

Finally, we note that $\mathcal{E}|_{C_n}$ contributes to $\mathrm{Cliff}(C_n)$. We compute $\gamma(\mathcal{E}|_{C_n}) = \mu(\mathcal{E}|_{C_n}) - h^0(C_n, \mathcal{E}|_{C_n}) + 2$. We have proved that $H^1(S, \mathcal{E}(-nC)) = 0$, and hence $h^0(C_n, \mathcal{E}|_{C_n}) = h^0(S, \mathcal{E}) = p + 3$, implying

$$\gamma(\mathcal{E}|_{C_n}) = \frac{n(2a + 2p + 1)}{2} - p - 1 < n(2a - 2p - 1) - 2$$

by the assumption $a \geq 3p + 2$, which concludes the proof. ∎

Remark 2. *As mentioned in the preamble of [6, Section 4], Mercat's conjecture holds for any curve of genus g and gonality k if $g > 2(k-1)(k-2)$. In our case, note that, as S contains no (-2)–curve, the results of [3] imply that $\mathrm{gon}(C_n) = n(2a - 2p - 1)$. Since $g(C_n) = 2an^2 + 1$, the above inequality fails, even though both expressions are quadratic in n. The gonality of C_n is small compared to the genus, and yet not sufficiently small to satisfy the conditions for Mercat's conjecture.*

The same strategy yields the following improvement of the main result of [21].

Theorem 3. *Let S be a $K3$ surface with $\mathrm{Pic}(S) = \langle C \rangle$, where C is a smooth curve of genus $g \geq 2$. Denote by $k = \left[\frac{g+3}{2}\right]$ and by $\mathcal{E}$ the Lazarsfeld-Mukai bundle associated to a g^1_k, A on C. Let $n \geq 2$ and $C_n \in |nC|$ be a smooth curve. If either $n \geq 3$, or $n = 2$ and $g \geq 9$, then $\mathcal{E}|_{C_n}$ is semistable with $\gamma(\mathcal{E}|_{C_n}) < \mathrm{Cliff}(C_n)$ and thus it is a counterexample to Mercat's conjecture. Furthermore, if $n \geq 3$, then $\mathcal{E}|_{C_n}$ is stable.*

Proof. We proceed along the lines of the proof of Theorem 2. Note that $g(C_n) = n^2(g-1)+1$ and $\mathrm{Cliff}(C_n) = 2(n-1)(g-1) - 2$.

We first establish that $\mathcal{E}|_{C_n}$ is semistable, and it is stable if $n \geq 3$. Suppose $\mathcal{E}|_{C_n}$ is unstable and consider

$$0 \to \mathcal{O}_{C_n}(B) \to \mathcal{E}|_{C_n} \to \mathcal{O}_{C_n}(C-B) \to 0$$

a destabilizing sequence. If $\mathcal{O}_{C_n}(C-B) \neq \mathcal{O}_{C_n}$, then it contributes to the Clifford index of C_n, and

$$\mathrm{Cliff}(\mathcal{O}_{C_n}(C-B)) < \mu(\mathcal{E}|_{C_n}) - 2 = n(g-1) - 2 \leq \mathrm{Cliff}(C_n)$$

which leads to a contradiction. Note that, if $n \geq 3$, we have the stronger inequality $\mu(\mathcal{E}|_{C_n}) - 2 < \mathrm{Cliff}(C_n)$.

Therefore, the destabilizing sequence is, in fact,

$$0 \to \mathcal{O}_{C_n}(C) \to \mathcal{E}|_{C_n} \to \mathcal{O}_{C_n} \to 0$$

and it follows that $h^0(C_n, \mathcal{E}|_{C_n}) \geq h^0(C_n, \mathcal{O}_{C_n}(C)) = h^0(S, \mathcal{O}_S(C)) = g+1$.

We prove that the restriction map $H^0(S,\mathcal{E}) \to H^0(C_n, \mathcal{E}|_{C_n})$ is an isomorphism. Since $h^0(S,\mathcal{E}) = g-k+3$, this will be in contradiction with the inequality above. The vanishing of $H^0(S, \mathcal{E}(-nC))$ follows immediately from the sequence (2.2), twisted with $\mathcal{O}_S(-nC)$. The surjectivity of the restriction map reduces to $H^1(S, \mathcal{E}(-nC)) = 0$ which is equivalent to the vanishing of $H^1(S, \mathcal{E}^\vee(nC))$. Consider the defining sequence

$$0 \to \mathcal{E}^\vee \to H^0(A) \otimes \mathcal{O}_S \to A \to 0,$$

twist it by $\mathcal{O}_S(nC)$ and take the long cohomology sequence. This reduces the problem to proving the surjectivity of the multiplication map

$$H^0(C,A) \otimes H^0(S, \mathcal{O}_S(nC)) \to H^0(C, A(nC)).$$

Since the restriction map $H^0(S, \mathcal{O}_S(nC)) \to H^0(C, \mathcal{O}_C(nC))$ is surjective, it suffices to prove that the multiplication map

$$H^0(C,A) \otimes H^0(C, \mathcal{O}_C(nC)) \to H^0(C, A(nC))$$

is surjective. To verify this, we apply once again Green's explicit H^0 Lemma, [10, Corollary (4.e.4)], as in the proof of Theorem 2. We observe that the condition

$$\deg(A) + \deg(\mathcal{O}_C(C_n)) \geq 4g+2$$

is verified for any $n \geq 2$, by the hypothesis.

To finish the proof, we compute $\gamma(\mathcal{E}|_{C_n}) = \mu(\mathcal{E}|_{C_n}) - h^0(\mathcal{E}|_{C_n}) + 2 = (n-1)(g-1)+k-2 < 2(n-1)(g-1) - 2$. □

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