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On the totally positive grassmannian

by
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Abstract

We give an alternative proof for the equivalence of two definitions of the totally positive grassmannian.

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0.1. Let V be an $\mathbf{R}$-vector space of dimension $N \geq 2$ with a fixed basis $e_1, e_2, \ldots, e_N$. Let $\mathcal{B}$ be the manifold whose points are the complete flags $F_1 \subset F_2 \subset \ldots \subset F_{N-1}$ in V. Here F_i is a subspace of dimension i of V. In [4] I defined the totally positive part $\mathcal{B}_{>0}$ of $\mathcal{B}$, a certain open subset of $\mathcal{B}$ homeomorphic to $\mathbf{R}_{>0}^{N(N-1)/2}$. We fix $k \in [1, N-1]$. Let Gr^k be the manifold whose points are the subspaces of V of dimension k. In [5] I defined the totally positive part $Gr^k_{>0}$ of Gr^k as the image of $\mathcal{B}_{>0}$ under the map $\mathcal{B} \to Gr^k$ which takes $F_1 \subset F_2 \subset \ldots \subset F_{N-1}$ to F_k. This is an open subset of Gr^k homeomorphic to $\mathbf{R}_{>0}^{k(N-k)}$. In [5, 3.4] it was shown that $Gr^k_{>0}$ can be described in terms of inequalities involving elements in the canonical basis of the irreducible representation of $SL(V)$ corresponding to a multiple $c\varpi_k$ of the k-th fundamental weight ϖ_k where c is any integer $\geq N-1$. (The result in [5, 3.4] applies to any real partial flag manifold.) In a note added in the proof of [5] I stated (quoting Rietsch [7]) that c can be taken to be any number ≥ 1 (including 1); a similar statement was made for any partial flag manifold. However the proof in [7] contained an error, see Geiss, Leclerc, Schröer [1]. (I thank B. Leclerc for providing this reference.) In 2009, Rietsch (unpublished, but mentioned in [2]) has shown that c above can indeed be taken to be 1. Proofs of Rietsch's result appeared in [8] and [3]. But for a general partial flag manifold it is not known to what extent the result in [5, 3.4] can be improved. In this paper we present a method which could possibly yield an improvement of the result of [5, 3.4]. In the case of the Grassmannian this recovers the result of Rietsch, see Theorem 0.5; but one can hope that our method applies also in other cases. This method is based on the observation of [5, §2] that the positive part of a partial flag manifold is a single connected component of an explicit algebraic open subset of that partial flag manifold. For a further study of $Gr^k_{>0}$, see [6]. I thank P. Galashin and L. Williams for comments on an earlier version of this paper.

0.2. *Notation.* For two integers $a \leq b$ we set $[a, b] = \{z \in \mathbf{Z}; a \leq z \leq b\}$. For a finite set I let $|I|$ be the cardinal of I. For any $I \in [1, N]$ let V_I be the subspace of V with basis $\{e_i; i \in I\}$. For any $I \subset [1, N]$ with $|I| = k$, we set $e_I = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \in \Lambda^k V$ where I consists of the numbers $i_1 < i_2 < \ldots < i_k$ and $\Lambda^k V$ is the k-th exterior power of V. Let $\Lambda^k V_{>0}$ (resp. $\Lambda^k V_{\geq 0}$) be the set of nonzero vectors in $\Lambda^k V$ whose coordinates with respect to the basis $\{e_I; |I| = k\}$ are all in $\mathbf{R}_{>0}$ (resp. $\mathbf{R}_{\geq 0}$). Let $P\Lambda^k V_{>0}$ (resp. $P\Lambda^k V_{\geq 0}$) be the set of lines in $\Lambda^k V$ which are spanned by vectors in $\Lambda^k V_{>0}$ (resp. $\Lambda^k V_{\geq 0}$). Define a linear map $A : V \to V$ by $A(e_1) = e_2$, $A(e_i) = e_{i+1} + e_{i-1}$ if $1 < i < N$, $A(e_N) = e_{N-1}$. For any $r \in \mathbf{R}_{\geq 0}$ let $g_r = \exp(rA) \in GL(V)$. If $r > 0$, the matrix of g_r is totally positive.

0.3. As in [5], let $Gr^k_{\geq 0}$ be the closure of $Gr^k_{>0}$ in Gr^k. We define Gr'^k to be the set of all $E \in Gr^k$ such that $E \cap V_I = 0$ for any $I \subset [1, N]$ such that $|I| = N - k$ (an open subset of Gr^k). Let $Gr'^k_{>0}$ (resp. $Gr'^k_{\geq 0}$) be the set of all $E \in Gr^k$ such that the line $\Lambda^k E$ in $\Lambda^k V$ is in $P\Lambda^k(V)_{>0}$ (resp. $P\Lambda^k(V)_{\geq 0}$). According to [5, 3.2], we have

(a) $Gr^k_{>0} \subset Gr'^k_{>0}$, $Gr^k_{\geq 0} \subset Gr'^k_{\geq 0}$.

We show:

(b) $Gr'^k_{>0} \subset Gr'^k$.

Assume that $E \in Gr'^k_{>0}$, $E \notin Gr'^k$. We can find $I \subset [1, N]$ such that $|I| = N - k$, $E \cap V_I \neq 0$. Let $e'_1 \in E \cap V_I - \{0\}$. We can find a basis $e'_1, e'_2, \dots, e'_k$ of E containing e'_1. Since $e'_1 \in V_I$, $\epsilon := e'_1 \wedge e'_2 \wedge \dots \wedge e'_k$ is a linear combination of elements of the form $e_{I'}$ with $I' \subset [1, N], |I'| = k$, $I' \cap I \neq \emptyset$. In particular, $e_{[1,N]-I}$ appears with coefficient 0 when ϵ is expressed as a linear combination of $e_{I''}, |I''| = k$. Thus, for any $c \in \mathbf{R} - \{0\}$ we have $c\epsilon \notin \Lambda^k V_{>0}$ so that $E \notin Gr'^k$. This proves (b).

0.4. Let $E \in Gr^k$. We say that E is *generic* if

(i) $E \cap V_{[1,N-k]} = 0$,

(ii) $E \cap V_{[k+1,N]} = 0$,

and if, setting

$E_i = E \cap V_{[1,N-k+i]}$ if $i \in [1, k-1]$, $E_k = E$, $E_i = E \oplus V_{[1,i-k]}$ if $i \in [k+1, N-1]$,

$E'_i = E \cap V_{[k-i+1,N]}$ if $i \in [1, k-1]$, $E'_k = E$, $E'_i = E \oplus V_{N-i+k+1,N}$ if $i \in [k+1, N-1]$,

so that $E_1 \subset E_2 \subset \dots \subset E_{N-1}$, $\dim E_i = i$, $E'_1 \subset E'_2 \subset \dots \subset E'_{N-1}$, $\dim E'_i = i$, we have:

(iii) $E'_i \cap E_{k-i} = 0$ if $i \in [1, k-1]$,

(iv) $E'_{k+i} \cap E_{N-i} = E$ if $i \in [1, N-k-1]$.

Let Gr^{k*} be the set of all $E \in Gr_k$ which are generic. (An open subset of Gr_k.) According to [5]:

(a) $Gr^k_{>0}$ *is a connected component of* Gr^{k*}.

We show:

(b) $Gr'^k \subset Gr^{k*}$.

Let $E \in Gr'^k$. Then E clearly satisfies conditions (i),(ii). For $i \in [1, k-1]$ we have

$$\begin{aligned} E'_i \cap E_{k-i} &= (E \cap V_{[k-i+1,N]}) \cap (E \cap V_{[1,N-i]}) \\ &= E \cap V_{[k-i+1,N] \cap [1,N-i]} \\ &= E \cap V_{[k-i+1,N-i]} \\ &= 0 \end{aligned}$$

since $|[k-i+1, N-i]| = N-k$. Thus, (iii) holds. For $i \in [1, N-k-1]$ and for

$$x \in E'_{k+i} \cap E_{N-i} = (E \oplus V_{[N-i+1,N]}) \cap (E \oplus V_{[1,N-k-i]})$$

we have $x = a + b = c + d$ with

$$a \in E, b \in V_{[N-i+1,N]}, c \in E, d \in V_{[1,N-k-i]}.$$

We have

$$b - c \in V_{[N-i+1,N]} + V_{[1,N-k-i]} = V_{[N-i+1,N]\cup[1,N-k-i]}.$$

Also, $b-c \in E$ and $E \cap V_{[N-i+1,N]\cup[1,N-k-i]} = 0$ since $|[N-i+1,N]\cup[1,N-k-i]| = N-k$. Thus $b = c$ so that $b = c = 0$ since $V_{[N-i+1,N]} \cap V_{[1,N-k-i]} = 0$. We see that $x = a \in E$. Thus, $E'_{k+i} \cap E_{N-i} \subset E$. The reverse inclusion is obvious. We see that (iv) holds. This proves (b).

Theorem 0.5 (Rietsch) *We have $Gr'^k_{>0} = Gr^k_{>0}$.*

The proof is similar to that of [4, 8.17]. The inclusion $Gr^k_{>0} \subset Gr'^k_{>0}$ follows from 0.3(a). We show the reverse inclusion. Let $E \in Gr'^k_{>0}$. Since g_r is totally positive for $r > 0$ and is the identity for $r = 0$, for any $r \in \mathbf{R}_{\geq 0}$ we have $g_r(E) \in Gr'^k_{>0}$. Applying [4, 5.2] to $g = g_1 : \Lambda^k V \to \Lambda^k V$ and noting that $g_1^n = g_n$ for all integers $n \geq 1$, we see that the sequence $g_n \Lambda^k E$ $(n = 1, 2, \ldots)$ converges in $P\Lambda^k V_{>0}$ to the Perron line $L_{g_1} \in P\Lambda^k V_{>0}$ (notation of [4, 5.2(a)]). From [4, 8.9(a)] we see that there exists $E_1 \in Gr^k_{>0}$ such that $g_1(E_1) = E_1$. By 0.3(a) we have $E_1 \in Gr'^k_{>0}$. Since L_{g_1} is the unique g_1-stable line in $P\Lambda^k V_{>0}$, we must have $L_{g_1} = \Lambda^k E_1$. Since $g_n \Lambda^k E$ converges to $\Lambda^k E_1$ as $n \to \infty$, it follows that $g_n E$ converges to E_1 as $n \to \infty$ (in Gr^k). Since $E_1 \in Gr^k_{>0}$ and $Gr^k_{>0}$ is open in Gr^k, it follows that $g_{n_0} E \in Gr^k_{>0}$ for some integer $n_0 \geq 1$. Since the map $r \mapsto g_n E$ from $\mathbf{R}_{\geq 0}$ to $Gr'^k_{>0}$ is continuous, its image is contained in a single connected component of Gr^{k*} (recall that $Gr'^k_{>0} \subset Gr'^k \subset Gr^{k*}$, see 0.3(b),0.4(b)). In particular the image of 0 and that of n_0 (namely E and $g_{n_0} E$) belong to the same connected component of Gr^{k*}. Since $g_{n_0} E \in Gr^k_{>0}$ and $Gr^k_{>0}$ is a connected component of Gr^{k*} (see 0.4(a)), it follows that $E \in Gr^k_{>0}$. The theorem is proved.

0.6. The following result is a consequence of the theorem in 0.5.

(a) $Gr'^k_{\geq 0} = Gr^k_{\geq 0}$.

The inclusion $Gr^k_{\geq 0} \subset Gr'^k_{\geq 0}$ follows from 0.3(a). We show the reverse inclusion. Let $E \in Gr'^k_{\geq 0}$. Since g_r is totally positive for $r > 0$, for such r we have $g_r \Lambda^k E \in P\Lambda^k V_{>0}$ that is, $\Lambda^k(g_r E) \in P\Lambda^k V_{>0}$. Using 0.5 we see that $g_r E \in Gr^k_{>0}$ for $r > 0$. Taking the limit as $r \to 0$ we see that E is in the closure of $Gr^k_{>0}$, that is, $E \in Gr^k_{\geq 0}$. This proves (a).

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