Bull. Math. Soc. Sci. Math. Roumanie
Tome 66 (114), No. 4, 2023, 409–431

Equidistribution and inequalities for partitions into powers

by
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Abstract

If $p_k(a,m;n)$ denotes the number of partitions of n into kth powers with a number of parts that is congruent to a modulo m, recent work of the author (2020) showed that $p_2(0,2;n) \sim p_2(1,2;n)$ and that the sign of the difference $p_2(0,2;n) - p_2(1,2;n)$ alternates with the parity of n as $n \to \infty$. The aim of this paper is to study this problem in its full generality. By an analytic argument using the circle method and an upper bound on exponential Gauss sums related to center density estimates arising from the sphere packing problem, we prove that the same results hold for any $k \geq 2$. In addition, by a purely combinatorial argument, we show that the sign of the difference $p_k(0,2;n) - p_k(1,2;n)$ alternates with the parity of n for a larger class of partitions.

Key Words: Asymptotics, circle method, equidistribution, Gauss sums, partition inequalities, power partitions, saddle-point method, sphere packing problem.
2020 Mathematics Subject Classification: Primary 11P82; Secondary 11P55, 11P83.

1 Introduction and statement of results

1.1 Motivation

A *partition* of a positive integer n is a non-increasing sequence (often written as a sum) of positive integers, called *parts*, that add up to n. By $p(n)$ we denote the number of partitions of n, and by convention we let $p(0) = 1$. For example, $p(4) = 5$ as the partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$, this being the case of *unrestricted* partitions. One can also consider, however, *restricted* partitions with various conditions imposed on their parts. Generally, these are partitions with all parts being in a set S satisfying certain properties. If $S \subseteq \mathbb{N}$ is any (finite or infinite) set, we denote by $p_S(n)$ the number of partitions of n into parts that all belong to the set S. If $S = \mathbb{N}$, then $p_S(n) = p(n)$.

In the particular case when $S = \{n^k : n \in \mathbb{N}\}$ is the set of perfect kth powers, with $k \in \mathbb{N}$, we will use the shorthand notation $p_k(n)$ instead. It is with this class of restricted partitions that the current paper will mostly be concerned. Additionally, we let $p_S(m;n)$ denote the number of partitions of n with exactly m parts, all from S, and $p_S(a,m;n)$ that of partitions of n with a number of parts, all from S, which is congruent to a modulo m. The quantities $p_k(m;n)$ and $p_k(a,m;n)$ are defined in a similar fashion for the special case when $S = \{n^k : n \in \mathbb{N}\}$ as explained above. Answering a conjecture formulated by Bringmann and Mahlburg [7], the author proved the following [8].

Theorem 1 (Ciolan [8]). *As $n \to \infty$, we have*

$$p_2(0,2;n) \sim p_2(1,2;n) \sim \frac{p_2(n)}{2}$$

and

$$\begin{cases} p_2(0,2;n) > p_2(1,2;n) & \text{if } n \text{ is even,} \\ p_2(0,2;n) < p_2(1,2;n) & \text{if } n \text{ is odd.} \end{cases} \tag{1.1}$$

The statement of Theorem 1, which is about partitions into squares, raises the natural question whether the same type of result holds for partitions into higher powers, or, more generally, into parts that are certain polynomial functions. Also, one might wonder whether similar results hold for moduli $m > 2$.

1.2 Historical background

The earliest result of which the author is aware in the literature goes back to 1876 and is due to Glaisher [15], who proved that $p_1(0,2;n) - p_1(1,2;n) = (-1)^n p_{\mathrm{o}}(n)$, where $p_{\mathrm{o}}(n)$ counts partitions of n into odd parts without repetitions. To compute asymptotics for $p(n)$, Hardy and Ramanujan [16] designed the famous *circle method*, a breakthrough of their times, while Wright [26] improved on their method and computed asymptotics for $p_k(n)$. More recently, Vaughan [25] gave a simpler asymptotic formula for $k = 2$. His approach was extended to any $k \geq 2$ by Gafni [13], who also found asymptotics for the number of partitions into kth prime powers [14], whereas Berndt, Malik and Zaharescu [6] further generalized the results from [13] to partitions into kth powers in a residue class.

Roth and Szekeres [20] computed asymptotics for $p_U(n)$ in the case when $U = \{u_n\}_{n \geq 1}$ is a sequence of positive integers which is increasing for $n \geq n_0$ and which satisfies a few growth properties, see conditions (I)–(II) from [20, p. 241], under the restriction that no repeated parts are allowed. Liardet and Thomas [17] computed $p_U(n)$ while removing this condition and allowing repetitions. An example of such a set $U = \{u_n\}_{n \geq 1}$ is given by $u_n = f(n)$, where f is a polynomial such that $f(\mathbb{N}) \subseteq \mathbb{N}$ and with the property that for every prime p there exists an integer n such that $p \nmid f(n)$. Certainly, $f(n) = n^k$ is such a polynomial, for any $k \in \mathbb{N}$. Most recently, Zhou [27] proved that if $U = \{f(n)\}_{n \geq 1}$, with $f : \mathbb{N} \to \mathbb{N}$ a polynomial function satisfying similar conditions to those from [20], then

$$p_U(a,m;n) \sim \frac{p_U(n)}{m}$$

holds uniformly as $n \to \infty$ for all $m = o\Big(n^{\frac{1}{2+2\deg f}} (\log n)^{-\frac{1}{2}}\Big)$, for any $0 \leq a \leq m-1$. However, the methods from [27] do not help in proving the inequalities (1.1).

In a modest attempt to paint a full picture of this otherwise very rich and active area, we mention also the work of Dunn and Robles [12], who, using polylogarithms and the Matsumoto-Weng zeta function (see [19]), studied partitions into polynomial parts and established asymptotics for $p_{\mathcal{A}_f}$, where $f \in \mathbb{Z}[x]$ is any polynomial such that $f(\mathbb{N}) \subseteq \mathbb{N}$ and $\mathcal{A}_f = \{f(n) : n \in \mathbb{N}\}$.

1.3 Statement of results

The purpose of this paper is two-fold. First, we prove that Theorem 1 extends, indeed, to partitions into perfect kth powers, for any $k \geq 2$. Second, we investigate whether inequalities of the form (1.3) hold for more general types of partitions. For this purpose, we would like to give both an analytic and a combinatorial proof to Theorem 2, since we believe that the two approaches, independent of one another, are instructive in their own right.

Theorem 2. *For any $k \geq 2$ we have, as $n \to \infty$,*

$$p_k(0,2;n) \sim p_k(1,2;n) \sim \frac{p_k(n)}{2} \tag{1.2}$$

and

$$\begin{cases} p_k(0,2;n) > p_k(1,2;n) & \text{if } n \text{ is even,} \\ p_k(0,2;n) < p_k(1,2;n) & \text{if } n \text{ is odd.} \end{cases} \tag{1.3}$$

The analytic approach relies on Wright's modular transformations for partitions into kth powers [26], on a modification of Meinardus's Theorem [18] on asymptotics of infinite product generating functions which combines the circle and the saddle-point method, and on estimates of exponential Gauss sums; in particular, we invoke a bound that was established by Banks and Shparlinski [5] with the somewhat unexpected and surprising help of the effective lower estimates on center density found by Cohn and Elkies [10] in their work on the sphere packing problem.

We find the connection between our partition question and the sphere packing problem to be rather interesting, and it is also this precise step that allows for a generalization to $k \geq 2$ of the argument given in [8] for dealing with the case $k = 2$. While the equidistribution statement follows as a particular case of Corollary 1.2 from [27], our argument, which is independent from that in [27], proves *both* the equidistribution and the inequalities.

Whereas the combinatorial approach simplifies our work greatly in establishing the inequalities (1.3), it is not of much help in proving equidistribution results, at least not in a more general framework. Nevertheless, it allows us to prove the following.

Theorem 3. *Let α be any positive integer and let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function such that*

a) $f(1) = 1$;

b) $f(n)$ *is odd if n is odd*;

c) $f(2n) = 2\alpha f(n)$ *for every $n \in \mathbb{N}$.*

If $S = \{f(n)\}_{n\geq 1}$, then

$$\begin{cases} p_S(0,2;n) \geq p_S(1,2;n) & \text{if } n \text{ is even,} \\ p_S(0,2;n) \leq p_S(1,2;n) & \text{if } n \text{ is odd,} \end{cases} \tag{1.4}$$

and the inequalities are strict for any large enough n.

As it is easy to see that the power functions $f_k(n) = n^k$ satisfy the conditions of Theorem 3 (with $\alpha = 2^{k-1}$), the inequalities (1.3) follow as an immediate consequence of the above result.

1.4 Notation

Before proceeding further, let us introduce some notation used in the sequel. By $\zeta_n = e^{\frac{2\pi i}{n}}$ we will denote the standard primitive nth root of unity. Whenever required to take logarithms or to extract roots of complex numbers, we will use principal branches, and the principal branch of the complex logarithm will be denoted by Log. The "little" and "big oh" notation o and O are used throughout with their standard meaning, and we will also make use of the Vinogradov symbol $\ll$, writing $f \ll g$ to denote the fact that $f(x) = O(g(x))$ as $x \to \infty$. For reasons of space, we will sometimes use $\exp(z)$ instead of e^z.

1.5 Outline

The paper is structured as follows. In Section 2 we use generating functions to give a reformulation of our problem. In Section 3 we discuss the combinatorial approach and we give the proof of Theorem 3. In Section 4 we present the strategy of the analytic proof and we discuss the similarities and differences with the proof of the same result from [8] in the case $k = 2$. This will also be done, throughout the paper, in the form of commentaries placed at the end of the relevant sections. We consider this to be for the benefit of the reader interested in comparing the present paper with [8]. In Sections 5 and 6 we prove two estimates which, combined, will provide the analytic proof of Theorem 2, given in Section 7. We conclude this paper by proposing some open problems and future research directions in Section 8.

2 A reformulation

2.1 Generating functions

It is well-known (see, for example, [1, Ch. 1]) that

$$\prod_{n=1}^{\infty}(1-q^{n^k})^{-1} = \sum_{n=0}^{\infty} p_k(n)q^n, \tag{2.1}$$

where, as usual, for $\tau \in \mathbb{H}$ (the upper half-plane) we set $q = e^{2\pi i\tau}$. Letting

$$H_k(q) = \sum_{n=0}^{\infty} p_k(n)q^n, \quad H_k(w;q) = \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} p_k(\ell;n)w^{\ell}q^n, \quad H_{k,a,m}(q) = \sum_{n=0}^{\infty} p_k(a,m;n)q^n,$$

it is not difficult to see, from the orthogonality relations satisfied by the roots of unity, that

$$H_{k,a,m}(q) = \frac{1}{m}H_k(q) + \frac{1}{m}\sum_{j=1}^{m-1}\zeta_m^{-aj}H_k(\zeta_m^j;q). \tag{2.2}$$

Using, in turn, (2.2) and eq. (2.1.1) from [1, p. 16], we obtain

$$H_{k,0,2}(q) - H_{k,1,2}(q) = H_k(-1;q) = \prod_{n=1}^{\infty}\frac{1}{1+q^{n^k}},$$

and, on substituting $q \mapsto -q$, we have

$$H_k(-1;-q) = \prod_{n=1}^{\infty} \frac{1}{1+(-q)^{n^k}} = \prod_{n=1}^{\infty} \frac{1}{(1+q^{2^k n^k})(1-q^{(2n+1)^k})}$$
$$= \prod_{n=1}^{\infty} \frac{(1-q^{2^k n^k})^2}{(1-q^{2^{k+1} n^k})(1-q^{n^k})}. \tag{2.3}$$

On noting now that

$$H_k(-1;-q) = H_{k,0,2}(-q) - H_{k,1,2}(-q) = \sum_{n=0}^{\infty} a_k(n) q^n,$$

where

$$a_k(n) = \begin{cases} p_k(0,2;n) - p_k(1,2;n) & \text{if } n \text{ is even}, \\ p_k(1,2;n) - p_k(0,2;n) & \text{if } n \text{ is odd}, \end{cases}$$

we see that proving the inequalities from Theorem 2 is equivalent to showing that the coefficients $a_k(n)$ of the infinite product $H_k(-1;-q)$, expressed as a q-series, are positive as $n \to \infty$. For simplicity, we will denote $G_k(q) = H_k(-1;-q)$. In order to extract more information about the coefficients $a_k(n)$, one might try to compute them asymptotically. Indeed, this is the motivation of our analytic approach, and this is what we are going to do in Sections 4–7.

3 A combinatorial approach

The attentive reader (and especially the reader familiar with partition generating functions) might have noticed that all the previous identities hold not only for partitions into parts from the set $\{n^k : n \in \mathbb{N}\}$ but, more generally, for any set $S \subseteq \mathbb{N}$ of positive integers. Therefore, all steps in Section 2.1 can be redone with any set S instead of the set of kth powers. In this regard, identity (2.1) becomes

$$\prod_{n \in S} (1-q^n)^{-1} = \sum_{n=0}^{\infty} p_S(n) q^n,$$

and all the other identities following it, including the orthogonality relation (2.2), turn into their corresponding analogues. More precisely, for a given (infinite) set $S \subseteq \mathbb{N}$, if we let

$$a_S(n) = \begin{cases} p_S(0,2;n) - p_S(1,2;n) & \text{if } n \text{ is even}, \\ p_S(1,2;n) - p_S(0,2;n) & \text{if } n \text{ is odd}, \end{cases}$$

and

$$H_S(w;q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_S(m;n) w^m q^n,$$

then we obtain

$$H_S(-1;-q) = \prod_{n \in S} \frac{1}{1+(-q)^n} = \sum_{n=0}^{\infty} a_S(n) q^n.$$

In Section 2.1, in the particular case when $S = \{n^k : n \in \mathbb{N}\}$, we expressed the above product as shown in (2.3). However, we can rewrite this product in another way. Indeed, if $S = \{f(n) : n \in \mathbb{N}\}$ with $f : \mathbb{N} \to \mathbb{N}$ as in Theorem 3, we have

$$\begin{aligned} H_S(-1; -q) &= \prod_{\ell \in S} \frac{1 - (-q)^{\ell}}{1 - (-q)^{2\ell}} = \prod_{n \geq 1} \frac{1 - (-q)^{f(n)}}{1 - (-q)^{2f(n)}} \\ &= \prod_{n \geq 1} \frac{1 - (-q)^{f(2n)}}{1 - q^{2f(n)}} \prod_{n \geq 1} \left(1 - (-q)^{f(2n-1)}\right) \\ &= \prod_{n \geq 1} \frac{1 - q^{f(2n)}}{1 - q^{2f(n)}} \prod_{n \geq 1} \left(1 + q^{f(2n-1)}\right), \end{aligned} \tag{3.1}$$

where the last identity follows from the fact that f satisfies properties b) and c) of Theorem 3. In light of c), we further have

$$H_S(-1; -q) = \prod_{n \geq 1} \frac{1 - q^{2\alpha f(n)}}{1 - q^{2f(n)}} \prod_{n \geq 1} \left(1 + q^{f(2n-1)}\right). \tag{3.2}$$

Proof of Theorem 3. If $\alpha = 1$, the first product in the right-hand side of (3.2) vanishes and the only function satisfying the conditions is $f(n) = n$, which corresponds to unrestricted partitions. From (3.2) one recovers precisely Glaisher's identity mentioned in Section 1.2. Let us now assume $\alpha \geq 2$. Regarded as a series in q^2, the first product in (3.2) counts partitions into parts from S with no part appearing α times, while the second product, regarded as a series in q, counts partitions into distinct parts from $T = \{f(2n-1) : n \in \mathbb{N}\}$. Therefore, we obtain

$$a_S(n) = \sum_{0 \leq k \leq n} c_{S,\alpha}(k) d_{T,1}(n-k), \tag{3.3}$$

where $c_{S,\alpha}(n)$ is the number of partitions of n into parts from S with no part appearing α times, while $d_{T,1}(n)$ is the number of partitions of n into distinct parts from T (we set $c_{S,\alpha}(0) = d_{T,1}(0) = 1$). It is now clear that $a_S(n) \geq 0$ for any $n \geq 1$, and that $a_S(n) > 0$ for large enough n. ∎

Combinatorial proof of Theorem 2. As the sets $S = \{n^k : n \in \mathbb{N}\}$ satisfy the hypotheses of Theorem 3 for any $k \geq 2$ (and, in fact, for any $k \geq 1$) the inequalities (1.3) follow as an easy consequence of Theorem 3. The asymptotics of both $c_{S,\alpha}(k)$ and $d_{T,1}(k)$ can be obtained as special cases of the work of Liardet and Thomas, see Theorem 14.2 in [17]. Since the asymptotics will explicitly follow from our analytic proof of Theorem 2, we leave it as an exercise to the interested reader to derive asymptotics for $a_S(n)$ using (3.3) and the results of [17], and to thus prove the equidistribution. ∎

4 Strategy of the analytic approach

Having discussed the combinatorial approach to Theorems 2 and 3, we will focus from now on solely on the analytic aspects and on partitions into kth powers. For this, we recall the first representation of the product $G_k(q)$ given in (2.3), which says that

$$G_k(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2^k n^k})^2}{(1-q^{2^{k+1} n^k})(1-q^{n^k})} = \sum_{n=0}^{\infty} a_k(n) q^n.$$

Our objective is to compute asymptotics for the coefficients $a_k(n)$ and to prove that they are positive as $n \to \infty$.

4.1 Meinardus's Theorem

The reader familiar with asymptotics of infinite product generating functions might have already recognized the similarity between the infinite product expression for $G_k(q)$ and the one studied by Meinardus in [18], which, on writing $q = e^{-\tau}$ with $\mathrm{Re}(\tau) > 0$, is of the form

$$F(q) = \prod_{n=1}^{\infty} (1-q^n)^{-a_n} = \sum_{n=0}^{\infty} r(n) q^n,$$

with $a_n \geq 0$. Under certain assumptions on which we do not elaborate now, Meinardus found asymptotic formulas for the coefficients $r(n)$. More precisely, if the Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad (s = \sigma + it)$$

converges for $\sigma > \alpha > 0$ and admits a meromorphic continuation to the region $\sigma > -c_0$ ($0 < c_0 < 1$), region in which $D(s)$ is holomorphic everywhere except for a simple pole at $s = \alpha$ with residue A, then the following holds.

Theorem 4 (Andrews [1, Ch. 6], cf. Meinardus [18])**.** *As $n \to \infty$, we have*

$$r(n) = cn^{\kappa} \exp\left(n^{\frac{\alpha}{\alpha+1}} \left(1 + \frac{1}{\alpha}\right) (A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}} \right) (1 + O(n^{-\kappa_1})),$$

where

$$\begin{aligned} c &= e^{D'(0)} \left(2\pi(\alpha+1)\right)^{-\frac{1}{2}} \left(A\Gamma(\alpha+1)\zeta(\alpha+1)\right)^{\frac{1-2D(0)}{2+2\alpha}}, \\ \kappa &= \frac{2D(0)-2-\alpha}{2(\alpha+1)}, \\ \kappa_1 &= \frac{\alpha}{\alpha+1} \min\left\{ \frac{c_0}{\alpha} - \frac{\delta}{4}, \frac{1}{2} - \delta \right\}, \end{aligned}$$

with $\delta > 0$ arbitrary.

Remark 1. We also refer the reader to the work of Debruyne and Tenenbaum [11], who, under some assumptions, investigated how the saddle-point method (see also Section 7.1) can be used to derive asymptotic information about $p_\Lambda(n)$ solely from the analytic properties of the associated Dirichlet series $L_\Lambda(s) = \sum_{n \in \Lambda} n^{-s}$, for certain sets $\Lambda \subseteq \mathbb{N}$. Their results generalize and simplify several aspects from [6], [12] and [13]; e.g., they can be used to compute some constants that are not explicit in the asymptotics of $p_{\mathcal{A}_f}(n)$ from [12].

Returning to our argument, if we write $\tau = y - 2\pi ix$, an application of Cauchy's Theorem gives

$$r(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{F(q)}{q^{n+1}}dq = e^{ny}\int_{-\frac{1}{2}}^{\frac{1}{2}} F(e^{-y+2\pi ix})e^{-2\pi inx}dx, \tag{4.1}$$

where $\mathcal{C}$ is the (positively oriented) circle of radius e^{-y} around the origin. Meinardus found the estimate stated in Theorem 4 by splitting the integral from (4.1) into two integrals evaluated over $|x| \le y^{\beta}$ and over $y^{\beta} \le |x| \le \frac{1}{2}$, for a certain choice of β in terms of α, and by showing that the former integral gives the main contribution for the coefficients $r(n)$, while the latter is only an error term.

The positivity condition $a_n \ge 0$ is, however, essential in Meinardus's proof and, as one can readily note, this is not satisfied by the factors from the product $G_k(q)$. For this reason, we have to modify the argument using the circle method and Wright's modular transformations [26] for partitions into kth powers. This will show that the integral over $y^{\beta} \le |x| \le \frac{1}{2}$ does not contribute to the main term.

On comparing with what was done for the case $k = 2$, the reader might notice that, up to this point, the strategy described here is analogous to that from [8]. The essential difference is that, in the case $k = 2$, a numerical check ([8, Lemma 5]) had to be carried out in order to prove a certain estimate ([8, Lemma 6]). This numerical check was rather technical and certainly cannot be carried out for all $k \ge 2$. In the present paper, we show how to avoid it by using a bound on Gauss sums due to Banks and Shparlinski [5] and by modifying a certain step in the argument from [8]. It is precisely this step that allows a significantly simpler proof and, at the same time, a generalization to any $k \ge 2$.

4.2 Two estimates

Keeping the notation introduced in the previous subsection and writing $q = e^{-\tau}$, with $\tau = y - 2\pi ix$ and $y > 0$, we recall that

$$G_k(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2^k n^k})^2}{(1-q^{2^{k+1}n^k})(1-q^{n^k})}. \tag{4.2}$$

Let $s = \sigma + it$ and

$$D_k(s) = \sum_{n=1}^{\infty}\frac{1}{n^{ks}} + \sum_{n=1}^{\infty}\frac{1}{(2^{k+1}n^k)^s} - 2\sum_{n=1}^{\infty}\frac{1}{(2^k n^k)^s} = (1 + 2^{-s(k+1)} - 2^{1-sk})\zeta(ks),$$

which is convergent for $\sigma > \frac{1}{k} = \alpha$, has a meromorphic continuation to $\mathbb{C}$ and a simple pole at $s = \frac{1}{k}$ with residue $A = \frac{1}{k} \cdot 2^{-\frac{k+1}{k}}$.

If $\mathcal{C}$ is the (positively oriented) circle of radius e^{-y} around the origin, Cauchy's Theorem tells us that

$$a_k(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{G_k(q)}{q^{n+1}}dq = e^{ny}\int_{-\frac{1}{2}}^{\frac{1}{2}} G_k(e^{-y+2\pi ix})e^{-2\pi inx}dx, \tag{4.3}$$

for $n > 0$. Set

$$\beta = 1 + \frac{\alpha}{2}\left(1 - \frac{\delta}{2}\right), \quad \text{with } 0 < \delta < \frac{2}{3}, \tag{4.4}$$

so that

$$\frac{3k+1}{3k} < \beta < \frac{2k+1}{2k}, \tag{4.5}$$

and rewrite

$$a_k(n) = I_k(n) + J_k(n),$$

where

$$I_k(n) = e^{ny} \int_{-y^\beta}^{y^\beta} G_k(q) e^{-2\pi i n x} dx \qquad \text{and} \qquad J_k(n) = e^{ny} \int_{y^\beta \le |x| \le \frac{1}{2}} G_k(q) e^{-2\pi i n x} dx.$$

As already mentioned, the idea is that the main contribution for $a_k(n)$ is given by $I_k(n)$, which we are able to estimate using standard integration techniques. Showing that $J_k(n)$ is an error term will prove to be much trickier.

5 The main term $I_k(n)$

In this section, we prove the following estimate.

Lemma 1. *If* $|x| \le \frac{1}{2}$ *and* $|\mathrm{Arg}\,(\tau)| \le \frac{\pi}{4}$, *then*

$$G_k(e^{-\tau}) = 2^{-\frac{k-1}{2}} \exp\left(A\Gamma\left(\frac{1}{k}\right) \zeta\left(1 + \frac{1}{k}\right) \tau^{-\frac{1}{k}} + O(y^{c_0}) \right)$$

holds uniformly in x *as* $y \to 0$, *for any* $0 < c_0 < 1$.

Proof. By taking logarithms in (4.2), we obtain

$$\mathrm{Log}\,(G_k(e^{-\tau})) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left(e^{-kn^k\tau} + e^{-2^{k+1}kn^k\tau} - 2e^{-2^k kn^k\tau} \right).$$

Using the Mellin inversion formula (see, e.g., [4, p. 54]) we get

$$e^{-\tau} = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \tau^{-s} \Gamma(s) ds$$

for $\mathrm{Re}(\tau) > 0$ and $\sigma_0 > 0$, thus

$$\begin{aligned} \mathrm{Log}\,(G_k(e^{-\tau})) &= \frac{1}{2\pi i} \int_{\alpha+1-i\infty}^{\alpha+1+i\infty} \Gamma(s) \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} ((kn^k\tau)^{-s} + (2^{k+1}kn^k\tau)^{-s} - 2(2^k kn^k\tau)^{-s}) ds \\ &= \frac{1}{2\pi i} \int_{\frac{k+1}{k} - i\infty}^{\frac{k+1}{k} + i\infty} \Gamma(s) D_k(s) \zeta(s+1) \tau^{-s} ds. \end{aligned} \tag{5.1}$$

By assumption,

$$|\tau^{-s}| = |\tau|^{-\sigma} e^{t \cdot \mathrm{Arg}\,(\tau)} \le |\tau|^{-\sigma} e^{\frac{\pi}{4}|t|}.$$

Well-known results (see, e.g., [2, Corollary 1.4.4] and [23, Ch. 5.1]) state that the bounds

$$D_k(s) = O(|t|^{c_1}), \quad \zeta(s+1) = O(|t|^{c_2}), \quad \Gamma(s) = O\big(e^{-\frac{\pi|t|}{2}} |t|^{c_3}\big)$$

hold uniformly in σ for $-c_0 \leq \sigma \leq \frac{k+1}{k} = \alpha + 1$ as $|t| \to \infty$, for any $0 < c_0 < 1$ and for some $c_1, c_2, c_3 > 0$, which means that we may shift the path of integration from $\sigma = \alpha + 1$ to $\sigma = -c_0$ (we impose the condition $c_0 < 1$ to avoid the poles of $\Gamma(s)$). A quick computation gives $D_k(0) = 0$ and $D_k'(0) = -\frac{(k-1)\log 2}{2}$. The integrand in (5.1) has poles at $s = \frac{1}{k}$ and $s = 0$, with residues equal to $\tau^{-\frac{1}{k}}$ and $-\frac{(k-1)\log 2}{2}$ respectively, whereas the remaining integral equals

$$\frac{1}{2\pi i}\int_{-c_0-i\infty}^{-c_0+i\infty} \tau^{-s}\Gamma(s)D(s)\zeta(s+1)ds \ll |\tau|^{c_0}\int_0^{\infty} t^{c_1+c_2+c_3}e^{-\frac{\pi t}{4}}dt$$
$$\ll |\tau|^{c_0} = |y - 2\pi i x|^{c_0} \leq (\sqrt{2}y)^{c_0},$$

since (again by the assumption)

$$\frac{2\pi|x|}{y} = \tan(|\mathrm{Arg}\,(\tau)|) \leq \tan\left(\frac{\pi}{4}\right) = 1.$$

Integration along the shifted contour now gives

$$\mathrm{Log}\,(G_k(e^{-\tau})) = \left(A\Gamma\left(\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\tau^{-\frac{1}{k}} - \frac{(k-1)\log 2}{2}\right) + O(y^{c_0}),$$

which concludes the proof. ∎

Commentary. This part is a straightforward generalization of [8, Lemma 1]. We thought it best for the reader to keep the reasoning here as close as possible to that presented in [8, §3.2]. On replacing $k = 2$, the proof of [8, Lemma 1] can be easily traced back.

6 The error term $J_k(n)$

This section is dedicated to proving that $J_k(n)$ does not contribute to the main term of the coefficients $a_k(n)$. More precisely, we prove the following estimate.

Lemma 2. *There exists $\varepsilon > 0$ such that, as $y \to 0$,*

$$G_k(e^{-\tau}) = O\left(\exp\left(A\Gamma\left(\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)y^{-\frac{1}{k}} - cy^{-\varepsilon}\right)\right) \tag{6.1}$$

holds uniformly in x with $y^{\beta} \leq |x| \leq \frac{1}{2}$, for some $c > 0$.

The proof is slightly more involved and will come in several steps. We start by describing the setup needed to apply the circle method.

6.1 Circle method

Inspired by Wright [26], we consider the Farey dissection of order $\left\lfloor y^{-\frac{k}{k+1}}\right\rfloor$ of the circle $\mathcal{C}$ over which we integrate in (4.3). We further distinguish two kinds of arcs:

(i) major arcs, denoted $\mathfrak{M}_{a,b}$, such that $b \leq y^{-\frac{1}{k+1}}$;

(ii) minor arcs, denoted $\mathfrak{m}_{a,b}$, such that $y^{-\frac{1}{k+1}} < b \leq y^{-\frac{k}{k+1}}$.

In what follows, we express any $\tau \in \mathfrak{M}_{a,b} \cup \mathfrak{m}_{a,b}$ in the form

$$\tau = y - 2\pi i x = \tau' - 2\pi i \frac{a}{b}, \tag{6.2}$$

with $\tau' = y - 2\pi i x'$. From basics of Farey theory it follows that

$$|x'| \leq \frac{y^{\frac{k}{k+1}}}{b}. \tag{6.3}$$

For a neat introduction to Farey fractions and the circle method, the reader is referred to [4, Ch. 5.4].

6.2 Modular transformations

Recalling the definition of $H_k(q)$, we can rewrite (4.2) as

$$G_k(q) = \frac{H_k(q) H_k(q^{2^{k+1}})}{H_k(q^{2^k})^2}. \tag{6.4}$$

In order to obtain more information about $G_k(q)$, we would next like to use Wright's transformation law [26, Theorem 4] for the generating function $H_k(q)$ of partitions into kth powers.

Before doing so, we need to introduce a bit of notation. In what follows, $0 \leq a < b$ are assumed to be coprime positive integers (with the requirement that $a = 0$ if and only if $b = 1$), with b_1 the least positive integer such that $b \mid b_1^2$ and $b = b_1 b_2$. First, set

$$j = j(k) = 0, \qquad \omega_{a,b} = 1$$

if k is even, and

$$j = j(k) = \frac{(-1)^{\frac{1}{2}(k+1)}}{(2\pi)^{k+1}} \Gamma(k+1)\zeta(k+1), \qquad \omega_{a,b} = \exp\left(\pi \left(\frac{1}{b^2} \sum_{h=1}^{b} h d_h - \frac{1}{4}(b - b_2)\right)\right)$$

if k is odd, where $0 \leq d_h < b$ is defined by the congruence

$$a h^2 \equiv d_h \pmod{b}$$

and where, for $1 \leq r \leq k$, we set

$$\mu_{h,r} = \begin{cases} \frac{d_h}{b} & \text{if } r \text{ is odd,} \\ \frac{b - d_h}{b} & \text{if } r \text{ is even,} \end{cases}$$

for $d_h \neq 0$. If $d_h = 0$, we set $\mu_{h,r} = 1$. Further, let

$$S_k(a,b) = \sum_{n=1}^{b} \exp\left(\frac{2\pi i a n^k}{b}\right) \tag{6.5}$$

be the so-called *Gauss sum* (of order k), and

$$\Lambda_{a,b} = \frac{\Gamma\left(1+\frac{1}{k}\right)}{b} \sum_{m=1}^{\infty} \frac{S_k(ma,b)}{m^{1+\frac{1}{k}}}. \tag{6.6}$$

Finally, put

$$C_{a,b} = \left(\frac{b_1}{2\pi}\right)^{\frac{k}{2}} \omega_{a,b}$$

and

$$P_{a,b}(\tau') = \prod_{h=1}^{b} \prod_{r=1}^{k} \prod_{\ell=0}^{\infty} \left(1 - g(h,\ell,r)\right)^{-1},$$

with

$$g(h,\ell,r) = \exp\left(\frac{(2\pi)^{\frac{k+1}{k}} (\ell + \mu_{h,r})^{\frac{1}{k}} e^{\frac{\pi i}{2k}(2r+k+1)}}{b\sqrt[k]{\tau'}} - \frac{2\pi i h}{b}\right).$$

Having introduced all the required objects, we can now state Wright's modular transformation [26, Theorem 4], which says, in our notation, that

$$H_k(q) = H_k\left(e^{\frac{2\pi i a}{b} - \tau'}\right) = C_{a,b}\sqrt{\tau'} e^{j\tau'} \exp\left(\frac{\Lambda_{a,b}}{\sqrt[k]{\tau'}}\right) P_{a,b}(\tau'). \tag{6.7}$$

On combining (6.4) and (6.7) we obtain, for some positive constant C that can be made explicit if necessary,

$$G_k(q) = Ce^{j\tau'} \exp\left(\frac{\lambda_{a,b}}{\sqrt[k]{\tau'}}\right) \frac{P_{a,b}(\tau') P'_{a,b}(2^{k+1}\tau')}{P''_{a,b}(2^k\tau')^2}, \tag{6.8}$$

where

$$P'_{a,b} = P_{\frac{2^{k+1}a}{(b,2^{k+1})}, \frac{b}{(b,2^{k+1})}}, \quad P''_{a,b} = P_{\frac{2^k a}{(b,2^k)}, \frac{b}{(b,2^k)}}$$

and

$$\lambda_{a,b} = \Lambda_{a,b} + 2^{-\frac{k+1}{k}} \Lambda_{\frac{2^{k+1}a}{(2^{k+1},b)}, \frac{b}{(2^{k+1},b)}} - \Lambda_{\frac{2^k a}{(2^k,b)}, \frac{b}{(2^k,b)}}. \tag{6.9}$$

6.3 Gauss sums

As we shall soon see, a crucial step in our proof is finding an upper bound for $\mathrm{Re}(\lambda_{a,b})$ or, what is equivalent, a bound for $|\lambda_{a,b}|$. This is given by the following sharp estimate found by Banks and Shparlinski [5] for the Gauss sums defined in (6.5).

Theorem 2 ([5, Theorem 1]). *For any coprime positive integers a, b with $b \geq 2$ and any $k \geq 2$, we have*

$$|S_k(a,b)| \leq \mathcal{A} b^{1-\frac{1}{k}}, \tag{6.10}$$

where $\mathcal{A} = 4.709236\ldots$.

The constant $\mathcal{A}$ is known as *Stechkin's constant*. Stechkin [22] conjectured in 1975 that the quantity

$$\mathcal{A} = \sup_{b,n\geq 2} \max_{(a,b)=1} \frac{|S_k(a,b)|}{b^{1-\frac{1}{k}}}$$

is finite, this being proven in 1991 by Shparlinski [21]. In the absence of any effective bounds on the sums $S_k(a,b)$, the precise value of $\mathcal{A}$ remained a mystery until 2015 when, using the work of Cochrane and Pinner [9] on Gauss sums with prime moduli and that of Cohn and Elkies [10] on lower bounds for the center density in the sphere packing problem, Banks and Shparlinski [5] were finally able to determine it. Coming back to our problem, we can now prove the following estimate.

Lemma 3. *If $0 \leq a < b$ are coprime integers with $b \geq 2$ and $\mathcal{A}$ is Stechkin's constant, we have*

$$|\lambda_{a,b}| < 4\mathcal{A}\cdot\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)b^{-\frac{1}{k}}\sum_{d|b}\frac{1}{d}.$$

Proof. Let us first give a bound for $|\Lambda_{a,b}|$. If we recall (6.6) and write $\Lambda_{a,b} = \Gamma\left(1+\frac{1}{k}\right)\Lambda^*_{a,b}$, we have, on using the fact that $S_k(ma,b) = dS_k\left(\frac{ma}{d},\frac{b}{d}\right)$ to prove the second equality below, and on replacing $m \mapsto md$ and $d \mapsto \frac{b}{d}$ to prove the third and fourth respectively,

$$\begin{aligned}
\Lambda^*_{a,b} &= \frac{1}{b}\sum_{m=1}^{\infty}\frac{S_k(ma,b)}{m^{1+\frac{1}{k}}} = \frac{1}{b}\sum_{d|b}\sum_{\substack{m\geq 1\\(m,b)=d}}\frac{dS_k\left(\frac{ma}{d},\frac{b}{d}\right)}{m^{1+\frac{1}{k}}} = \frac{1}{b}\sum_{d|b} d\sum_{\substack{m\geq 1\\(m,b/d)=1}}\frac{S_k\left(ma,\frac{b}{d}\right)}{(md)^{1+\frac{1}{k}}}\\
&= \frac{1}{b}\sum_{d|b} d^{-\frac{1}{k}}\sum_{\substack{m\geq 1\\(m,b/d)=1}}\frac{S_k\left(ma,\frac{b}{d}\right)}{m^{1+\frac{1}{k}}} = \frac{1}{b}\sum_{d|b}\left(\frac{b}{d}\right)^{-\frac{1}{k}}\sum_{\substack{m\geq 1\\(m,d)=1}}\frac{S_k(ma,d)}{m^{1+\frac{1}{k}}}\\
&= \frac{1}{b^{1+\frac{1}{k}}}\sum_{d|b} d^{\frac{1}{k}}\sum_{\substack{m\geq 1\\(m,d)=1}}\frac{S_k(ma,d)}{m^{1+\frac{1}{k}}}.
\end{aligned}$$

Using the previous identity, the definition of $\Lambda_{a,b}$ and estimate (6.10), we obtain

$$\begin{aligned}
|\Lambda_{a,b}| &\leq \frac{\Gamma\left(1+\frac{1}{k}\right)}{b^{1+\frac{1}{k}}}\sum_{d|b} d^{\frac{1}{k}}\sum_{\substack{m\geq 1\\(m,d)=1}}\frac{|S_k(ma,d)|}{m^{1+\frac{1}{k}}} \leq \frac{\Gamma\left(1+\frac{1}{k}\right)}{b^{1+\frac{1}{k}}}\sum_{d|b} d^{\frac{1}{k}}\sum_{\substack{m\geq 1\\(m,d)=1}}\frac{\mathcal{A}d^{1-\frac{1}{k}}}{m^{1+\frac{1}{k}}}\\
&\leq \frac{\mathcal{A}\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)}{b^{1+\frac{1}{k}}}\sum_{d|b} d = \frac{\mathcal{A}\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)}{b^{1+\frac{1}{k}}}\sum_{d|b}\frac{b}{d}\\
&= \mathcal{A}\cdot\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)b^{-\frac{1}{k}}\sum_{d|b}\frac{1}{d}.
\end{aligned}$$

The claim follows on applying this bound and using the modulus inequality in (6.9). ∎

Remark 2. The fact that $S_k(a,b) \ll b^{1-\frac{1}{k}}$ was known; see, for instance, [24, Theorem 4.2]. This means that instead of the Stechkin constant $\mathcal{A}$ in Lemma 3, we would have a constant $\mathcal{A}_k$ depending on k, which would already be enough for our purposes, as it will be revealed shortly in the proof of Lemma 2. While a universal bound like Stechkin's constant $\mathcal{A}$ is not needed for the proof and a constant depending on k would suffice, we thought it instructive to present this elegant argument and to point out a remarkable connection between power partitions and the sphere packing problem.

6.4 Final estimates

We are now getting closer to our purpose and we only need a few last steps before giving the proof of Lemma 2. Let us begin by estimating the factors of the form $P_{a,b}$ appearing in (6.8).

Lemma 4. *If $\tau \in \mathfrak{M}_{a,b} \cup \mathfrak{m}_{a,b}$, then*

$$\log|P_{a,b}(\tau')| \ll b \quad \text{as } y \to 0.$$

Proof. Using (6.3) and letting $y \to 0$, we have

$$|\tau'|^{1+\frac{1}{k}} = (y^2 + 4\pi^2 x'^2)^{\frac{k+1}{2k}} \leq \left(y^2 + \frac{4\pi^2 y^{\frac{2k}{k+1}}}{b^2}\right)^{\frac{k+1}{2k}} \leq \frac{c_4 y}{b^{\frac{k+1}{k}}} = \frac{c_4 \mathrm{Re}\,(\tau')}{b^{\frac{k+1}{k}}},$$

for some $c_4 > 0$. Thus, [26, Lemma 4] gives

$$|g(h,\ell,r)| \leq e^{-c_5(\ell+1)^{\frac{1}{k}}},$$

with $c_5 = \frac{4\sqrt[k]{2\pi}}{kc_4}$, which in turn leads to

$$|\log|P_{a,b}(\tau')|| \leq \sum_{h=1}^{b}\sum_{r=1}^{k}\sum_{\ell=1}^{\infty} |\log(1-g(h,\ell,r))| \leq kb\sum_{\ell=1}^{\infty}\left|\log\left(1-e^{-c_5(\ell+1)^{\frac{1}{k}}}\right)\right| \ll b,$$

concluding the proof. $\square$

The next result gives a bound for $G_k(q)$ on the minor arcs. As it is an immediate consequence of replacing $a = \frac{1}{k}$, $b = \frac{1}{k+1}$, $c = 2^{k-1}$, $\gamma = \varepsilon$ and $N = y^{-1}$ in [26, Lemma 17], we omit its proof.

Lemma 5. *If $\varepsilon > 0$ and $\tau \in \mathfrak{m}_{a,b}$, then*

$$|\mathrm{Log}\,(G(q))| \ll_\varepsilon y^{\frac{k2^{k-1}-k-1}{k(k+1)}-\varepsilon}.$$

Remark 3. Note that $k2^{k-1} > k+1$ for any $k \geq 2$, therefore the exponent of y in Lemma 5 is positive for a small enough choice of $\varepsilon > 0$.

At last, we need the following estimate, a modified version of [8, Lemma 6].

Lemma 6. *If $0 \le a < b$ are coprime integers with $b \ge 2$ and $x \notin \mathbb{Q}$, as $y \to 0$ we have, for some $c > 0$,*

$$\mathrm{Re}\left(\frac{\lambda_{a,b}}{\sqrt[k]{\tau'}}\right) \le \frac{\lambda_{0,1} - c}{\sqrt[k]{y}}. \tag{6.11}$$

Proof. Note that

$$\lambda_{0,1} = 2^{-\frac{k+1}{k}} \Lambda_{0,1} = 2^{-\frac{k+1}{k}} \Gamma\left(1 + \frac{1}{k}\right) \zeta\left(1 + \frac{1}{k}\right). \tag{6.12}$$

Writing $\tau' = y + ity$ for some $t \in \mathbb{R}$, we have

$$\begin{aligned}\mathrm{Re}\left(\frac{\lambda_{a,b}}{\sqrt[k]{\tau'}}\right) &= \frac{1}{\sqrt[k]{y}}\mathrm{Re}\left(\frac{\lambda_{a,b}}{\sqrt[k]{1+it}}\right) = \frac{1}{\sqrt[k]{y}}\mathrm{Re}\left(\frac{\lambda_{a,b}}{\sqrt[2k]{1+t^2}e^{\frac{i}{k}\arctan t}}\right) \\ &= \frac{1}{\sqrt[k]{y}\sqrt[2k]{1+t^2}}\left(\cos\left(\frac{\arctan t}{k}\right)\mathrm{Re}\left(\lambda_{a,b}\right) + \sin\left(\frac{\arctan t}{k}\right)\mathrm{Im}\left(\lambda_{a,b}\right)\right).\end{aligned}$$

Taking absolute values, we obtain

$$\left|\mathrm{Re}\left(\frac{\lambda_{a,b}}{\sqrt[k]{\tau'}}\right)\right| \le \frac{2|\lambda_{a,b}|}{\sqrt[k]{y}\sqrt[2k]{1+t^2}}. \tag{6.13}$$

Denoting $f_k(t) = \frac{1}{\sqrt[2k]{1+t^2}}$, we see that $f_k(t) \to 0$ as $|t| \to \infty$. Note now that the choice of x is independent from that of y, and recall from (6.2) that $\tau' = y - 2\pi i x'$, with $x' = x - \frac{a}{b}$, hence $t = -\frac{x'}{2\pi y}$. The assumption $x \notin \mathbb{Q}$ implies $x' \neq 0$, hence $|t| \to \infty$ and, consequently, $f_k(t) \to 0$ as $y \to 0$. The existence of a constant $c > 0$ such that inequality (6.11) holds as $y \to 0$ follows now on invoking (6.12), (6.13) and Lemma 3. □

6.5 Estimate for $J_k(n)$

We are now equipped with all the machinery needed for Lemma 2.

Proof of Lemma 2. If $\tau \in \mathfrak{m}_{a,b}$, the statement holds trivially by an application of Lemma 5. According to Remark 3, for any small enough $\varepsilon > 0$, the exponent of y in the estimate from Lemma 5 is positive. To conclude the claim, we note that as $y \to 0$, a negative power of y dominates any positive power of y (hence the error term in Lemma 2 absorbs that from Lemma 5).

Let us therefore assume now that $\tau \in \mathfrak{M}_{a,b}$, and we first consider the behavior near 0, corresponding to $a = 0$, $b = 1$, $\tau = \tau' = y - 2\pi i x$. Writing $y^\beta = y^{\frac{2k+1}{2k} - \varepsilon}$ with $\varepsilon > 0$ (here we use the second inequality from (4.5)) and setting $b = 1$ in (6.3), we have

$$y^{\frac{2k+1}{2k} - \varepsilon} \le |x| = |x'| \le y^{\frac{k}{k+1}}. \tag{6.14}$$

By (6.8) we get

$$G_k(q) = Ce^{j\tau} \exp\left(\frac{\lambda_{0,1}}{\sqrt[k]{\tau}}\right) \frac{P_{0,1}(\tau)P_{0,1}(2^{k+1}\tau)}{P_{0,1}(2^k\tau)^2}$$

for some $C > 0$. Thus, by Lemma 4 we obtain

$$\log |G_k(q)| = \frac{\lambda_{0,1}}{\sqrt[k]{|\tau|}} + jy + O(1).$$

Using (6.14) to prove the first inequality below and expanding into Taylor series to prove the second, we have, on letting $y \to 0$,

$$\frac{1}{\sqrt[k]{|\tau|}} = \frac{1}{\sqrt[k]{y}} \frac{1}{\left(1 + \frac{4\pi^2 x^2}{y^2}\right)^{\frac{1}{2k}}} \le \frac{1}{\sqrt[k]{y}} \frac{1}{\left(1 + 4\pi^2 y^{\frac{1}{k} - 2\varepsilon}\right)^{\frac{1}{2k}}} \le \frac{1}{\sqrt[k]{y}} \left(1 - c_6 y^{\frac{1}{k} - 2\varepsilon}\right)$$

for some $c_6 > 0$, and this concludes the proof in this case.

To complete the proof, let $\tau \in \mathfrak{M}_{a,b}$, with $2 \le b \le y^{-\frac{1}{k+1}}$. We distinguish two cases. First, let us deal with the case when $x \notin \mathbb{Q}$. By (6.8) and Lemma 4 we obtain

$$\log |G_k(q)| = \operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[k]{\tau'}}\right) + jy + O\left(y^{-\frac{1}{k+1}}\right) = \operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[k]{\tau'}}\right) + O\left(y^{-\frac{1}{k+1}}\right) \tag{6.15}$$

as $y \to 0$. Since by Lemma 6 there exists $c_7 > 0$ such that

$$\operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[k]{\tau'}}\right) \le \frac{\lambda_{0,1} - c_7}{\sqrt[k]{y}}, \tag{6.16}$$

we infer from (6.16) that, as $y \to 0$, we have

$$\log |G_k(q)| \le \frac{\lambda_{0,1} - c_8}{\sqrt[k]{y}}$$

for some $c_8 > 0$ and the proof is concluded under the assumption that $x \notin \mathbb{Q}$.

Finally, assume that $x = \frac{a}{b}$, that is, $x' = 0$ and $\tau = y - 2\pi i \frac{a}{b}$. We claim that the estimate (6.1) is satisfied with the same implied constant, call it C_1. Suppose by sake of contradiction that this is not the case. Then there exist infinitely small values of $y > 0$ for which

$$|G_k(e^{-\tau})| \ge C_2 \exp\left(\frac{\lambda_{0,1}}{\sqrt[k]{y}} - c y^{-\varepsilon}\right),$$

with $C_2 > C_1$. However, we can pick now $x' \notin \mathbb{Q}$ infinitely small and set $\tau_1 = y - 2\pi i \left(x' + \frac{a}{b}\right)$. For a fixed choice of y, we have $t \to 0$ as $x' \to 0$; thus, by the same calculation done in the proof of Lemma 6, we obtain

$$\operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[k]{\tau_1'}}\right) \to \operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[k]{y}}\right) = \operatorname{Re}\left(\frac{\lambda_{a,b}}{\sqrt[k]{\tau'}}\right), \tag{6.17}$$

since $f_k(t) \to 1$. On noting that $\operatorname{Re}(\tau_1') = \operatorname{Re}(\tau) = y$, while clearly all factors of the form $|P_{a,b}(k\tau_1')|$ tend to $|P_{a,b}(k\tau')|$ as $x' \to 0$, we obtain a contradiction, in the sense that, on one hand, (6.15) and (6.17) yield

$$|G_k(e^{-\tau_1})| \to |G_k(e^{-\tau})|$$

as $x' \to 0$, whereas on the other, for a sufficiently small choice of $y > 0$, we have

$$|G_k(e^{-\tau})| - |G_k(e^{-\tau_1})| \geq (C_2 - C_1) \exp\left(\frac{\lambda_{0,1}}{\sqrt[k]{y}} - cy^{-\varepsilon}\right),$$

quantity which gets arbitrarily large for sufficiently small choices of $y > 0$. ∎

Commentary. It is in this part where our proof differs substantially from that given in [8] in the case $k = 2$. More precisely, [8, Lemma 5] was needed to prove the inequality (6.11) for all values of y, inequality which was then used in the estimates made in the proof of [8, Lemma 2], the equivalent of Lemma 2 from the present paper. However, we are only interested in establishing the estimates from Lemma 2 on letting $y \to 0$, which is why we only need the bound (6.11) to hold as $y \to 0$. The argument presented in Lemma 6 further tells us that, in order for this to happen, the estimate (6.10), obtained using the bound on Gauss sums found by Banks and Shparlinski [5], is enough. As a consequence, we can avoid the rather involved numerical check done in [8, Lemma 5], a check which we would, in fact, not even be able to implement for all values $k \geq 2$. In particular, the present argument gives a simplified proof of the results from [8].

7 Analytic proof of Theorem 2

In this section we give the analytic proof of Theorem 2. Having already proven the two estimates from Lemmas 1–2, the rest is only a matter of careful computations. The reader is reminded that, because of the reformulation from Section 2.1, what we are interested in is computing asymptotics for the coefficients

$$a_k(n) = e^{ny} \int_{-\frac{1}{2}}^{\frac{1}{2}} G_k(e^{-y+2\pi ix}) e^{-2\pi inx} dx. \tag{7.1}$$

7.1 Saddle-point method

Recall that, as defined in Section 4.2, we denote $\alpha = \frac{1}{k}$ and $A = \frac{1}{k} \cdot 2^{-\frac{k+1}{k}}$, notation which we keep, for simplicity, in what follows. Before delving into the proof, we make a particular choice for y as a function of n. More precisely, let

$$y = n^{-\frac{1}{\alpha+1}} (A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}. \tag{7.2}$$

The reason for this choice of y is motivated by the *saddle-point method.* As the maximum absolute value of the integrand from (7.1) occurs for $x = 0$, around which point Lemma 1 tells us that the integrand is well approximated by

$$\exp(A\Gamma(\alpha)\zeta(\alpha+1)y^{-\alpha} + ny),$$

the saddle-point method suggests maximizing this expression, that is, solving

$$\frac{d}{dy}(\exp(A\Gamma(\alpha)\zeta(\alpha+1)y^{-\alpha} + ny)) = 0,$$

which leads to the value of y from (7.2).

7.2 Completing the proof

We have now all ingredients necessary to conclude the proof of Theorem 2. The proof merely consists of a skillful computation, which can be carried out in two ways. Since Lemma 1 and Lemma 2 are completely analogous to the two estimates found by Meinardus (combined in the *Hilfssatz* from [18, p. 390]), one way is to follow his approach and carry out the same computations done in [18, pp. 392–394]. The second way is slightly more explicit and is based entirely on the computation done in the proof of the case $k = 2$ from [8, pp. 139–141]. For sake of completeness and for comparison with the corresponding computation, we will sketch in what follows the main steps of the argument, while leaving some details and technicalities as an exercise for the interested reader.

Analytic proof of Theorem 2. We begin by proving the inequalities (1.3). By Lemma 2 and (7.2) we have

$$\begin{aligned} J_k(n) &= e^{ny} \int_{y^\beta \le |x| \le \frac{1}{2}} G(e^{-y+2\pi i x}) e^{-2\pi i n x} dx \\ &= e^{ny} \int_{y^\beta \le |x| \le \frac{1}{2}} O\left(\exp(y^{-\alpha} A\Gamma(\alpha)\zeta(\alpha+1) - cy^{-\varepsilon})\right) dx \\ &= e^{ny} \cdot O\left(\exp(y^{-\alpha} A\Gamma(\alpha)\zeta(\alpha+1) - cy^{-\varepsilon})\right) \\ &= O\left(\exp\left(n^{\frac{\alpha}{\alpha+1}}\left(1+\frac{1}{\alpha}\right)(A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}} - C_1 n^{\varepsilon_1}\right)\right) \end{aligned}$$

as $n \to 0$, with $\varepsilon_1 = \frac{k\varepsilon}{k+1} > 0$ and some $C_1 > 0$.

We now compute the main asymptotic contribution. Let $n \ge n_1$ be large enough so that $y^{\beta-1} \le \frac{1}{2\pi}$. This choice allows us to apply Lemma 1, as it ensures $|x| \le \frac{1}{2}$ and $|\mathrm{Arg}\,(\tau)| \le \frac{\pi}{4}$. From Lemma 1 we get

$$I_k(n) = \frac{e^{ny}}{2^{\frac{k-1}{2}}} \int_{-y^\beta}^{y^\beta} \exp\left(A\Gamma(\alpha)\zeta(\alpha+1)\tau^{-\alpha} + O(y^\varepsilon) - 2\pi i n x\right) dx. \tag{7.3}$$

Writing

$$\tau^{-\alpha} = \frac{1}{\sqrt[k]{\tau}} = \frac{1}{\sqrt[k]{y}} + \left(\frac{1}{\sqrt[k]{\tau}} - \frac{1}{\sqrt[k]{y}}\right),$$

we can further express (7.3) as

$$\begin{aligned} I_k(n) &= \frac{e^{ny}}{2^{\frac{k-1}{2}}} \int_{-y^\beta}^{y^\beta} \exp\left(\frac{B}{\sqrt[k]{y}} + B\left(\frac{1}{\sqrt[k]{\tau}} - \frac{1}{\sqrt[k]{y}}\right) - 2\pi i n x + O(y^{c_0})\right) dx \\ &= \mathcal{E} \int_{-y^\beta}^{y^\beta} \exp\left(\frac{B}{\sqrt[k]{y}}\left(\frac{1}{\sqrt[k]{1-\frac{2\pi i x}{y}}} - 1\right) - 2\pi i n x + O(y^{c_0})\right) dx, \end{aligned}$$

where we set $B = A\Gamma(\alpha)\zeta(\alpha+1)$ and $\mathcal{E} = 2^{-\frac{k-1}{2}} \exp\left(\left(1+\frac{1}{\alpha}\right) n^{\frac{\alpha}{\alpha+1}} (\alpha B)^{\frac{1}{\alpha+1}}\right)$ for simplicity. On denoting $u = -\frac{2\pi x}{y}$, we obtain

$$I_k(n) = \frac{y\,\mathcal{E}}{2\pi} \int_{-2\pi y^{\beta-1}}^{2\pi y^{\beta-1}} \exp\left(\frac{B}{\sqrt[k]{y}}\left(\frac{1}{\sqrt[k]{1+iu}} - 1\right) + inuy + O(y^{c_0})\right) dx. \tag{7.4}$$

We have the Taylor series expansion

$$\frac{1}{\sqrt[k]{1+iu}} = 1 - \frac{iu}{k} - \frac{(k+1)u^2}{2k^2} + O(|u|^3),$$

from where, on recalling that $|u| \leq 2\pi y^{\beta-1}$ and using (7.2) to compute $B = kny^{1+\frac{1}{k}}$, it follows that

$$\begin{aligned} B\frac{1}{\sqrt[k]{y}}\left(\frac{1}{\sqrt[k]{1+iu}} - 1\right) + inuy &= -\frac{Biu}{k\sqrt[k]{y}} + inuy - \frac{(k+1)Bu^2}{2k^2\sqrt[k]{y}} + O\left(\frac{|u|^3}{\sqrt[k]{y}}\right) \\ &= -\frac{(k+1)Bu^2}{2k^2\sqrt[k]{y}} + O\left(n^{\frac{1}{k+1}(1+3k(1-\beta))}\right). \end{aligned}$$

For an appropriate constant C_2, we may then change the integral from (7.4), which we denote by $\mathcal{I}$, into

$$\begin{aligned} \mathcal{I} &= \int_{|u|\leq C_2} \exp\left(-\frac{(k+1)Bu^2}{2k^2\sqrt[k]{y}} + O\left(y^{c_0} + \frac{|u|^3}{\sqrt[k]{y}}\right)\right) du \\ &= \int_{|u|\leq C_2} \exp\left(-\frac{(k+1)Bu^2}{2k^2\sqrt[k]{y}} + O\Big(n^{-\frac{kc_0}{k+1}} + n^{\frac{1+3k(1-\beta)}{k+1}}\Big)\right) du \\ &= \int_{|u|\leq C_2} \exp\left(-\frac{(k+1)Bu^2}{2k^2\sqrt[k]{y}}\right)\left[1 + \left(\exp\left(O\Big(n^{-\frac{kc_0}{k+1}} + n^{\frac{1+3k(1-\beta)}{k+1}}\Big)\right) - 1\right)\right] du. \end{aligned}$$

From the first inequality in (4.5), we see that $1 + 3k(1-\beta) < 0$, and thus

$$\exp\left(O\Big(n^{-\frac{kc_0}{k+1}} + n^{\frac{1+3k(1-\beta)}{k+1}}\Big)\right) - 1 = \exp\left(O\Big(n^{-\frac{kc_0}{k+1}} + n^{-\frac{1}{6}+\frac{\delta}{4}}\Big)\right) - 1 = O(n^{-\kappa}),$$

where $\kappa = \frac{1}{k+1}\min\left\{kc_0, \frac{1}{2} - \frac{3\delta}{4}\right\}$. We further get, on using (4.4) when changing the limits of integration,

$$\begin{aligned} \mathcal{I} &= \int_{|u|\leq C_2} \exp\left(-\frac{(k+1)Bu^2}{2k^2\sqrt[k]{y}}\right)(1 + O(n^{-\kappa}))du \\ &= c(n)\int_{|v|\leq C_3\cdot n^{\frac{\delta}{4(k+1)}}} e^{-v^2}(1 + O(n^{-\kappa}))dv, \end{aligned} \tag{7.5}$$

where $c(n) = \sqrt{\frac{2k}{k+1}}(\alpha B n^{\alpha})^{-\frac{1}{2(\alpha+1)}}$ and $C_3 > 0$ is a constant. By letting $n \to \infty$, and turning the integral from (7.5) into a Gauss integral, we obtain

$$\mathcal{I} = c(n)\sqrt{\pi}(1 + O(n^{-\kappa_1})), \tag{7.6}$$

where $\kappa_1 = \frac{1}{k+1}\min\left\{kc_0 - \frac{\delta}{4}, \frac{1}{2} - \delta\right\}$. Putting together (7.4), (7.5) and (7.6) we see that, as predicted by Meinardus (Theorem 3), the main asymptotic contribution for our coefficients is given by

$$a_k(n) \sim Cn^{-\frac{\alpha+2}{2(\alpha+1)}} \exp\left(n^{\frac{\alpha}{\alpha+1}}\left(1 + \frac{1}{\alpha}\right)(A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{\alpha+1}}\right), \tag{7.7}$$

where

$$C=\frac{1}{\sqrt{2^k(\alpha+1)\pi}}(A\Gamma(\alpha+1)\zeta(\alpha+1))^{\frac{1}{2(\alpha+1)}}.$$

This shows that the inequalities in (1.3) are true for $n\to\infty$. The proof is completed either by adding the estimate (7.7) for $a_k(n)=(-1)^n(p_k(0,2;n)-p_k(1,2;n))$ to that obtained by Wright for $p_k(n)=p_k(0,2;n)+p_k(1,2;n)$ (see [26, Theorem 2]), or by invoking the work of Zhou [27, Corollary 1.2]. ∎

8 Open questions

It would be of interest to see if Theorems 1 and 2 admit analogues for moduli $m\geq 3$, as another conjecture formulated in the unpublished manuscript of Bringmann and Mahlburg [7] states the following.

Conjecture 1 (Bringmann–Mahlburg, 2012)**.** *As $n\to\infty$, we have*

$$p_2(0,3;n)\sim p_2(1,3;n)\sim p_2(2,3;n)\sim\frac{p_2(n)}{3} \tag{8.1}$$

and

$$\begin{cases} p_2(0,3;n)>p_2(1,3;n)>p_2(2,3;n) & \text{if } n\equiv 0 \pmod 3,\\ p_2(1,3;n)>p_2(2,3;n)>p_2(0,3;n) & \text{if } n\equiv 1 \pmod 3,\\ p_2(2,3;n)>p_2(0,3;n)>p_2(1,3;n) & \text{if } n\equiv 2 \pmod 3.\end{cases} \tag{8.2}$$

Indeed, the work of Zhou [27] proves the equidistribution statement from (8.1). However, while the inequalities (8.2) hold for small values of n, numerical experiments reveal the fact that the pattern loses its structure as n grows larger, and that the signs of the inequalities change. In this regard, the following variant of this question seems more reasonable.

Let $S\subseteq\mathbb{N}$. If for any $n\in\mathbb{N}$ we arrange $p_S(a,m;n)$ in non-increasing order, we obtain an m-tuple $(p_S(i_0,m;n),\dots,p_S(i_{m-1},m;n))$, with $\{i_0,i_1,\dots,i_{m-1}\}=\{0,1,\dots,m-1\}$. We define a sequence $\{u_n\}_{n\geq 1}$ of such ordered m-tuples by setting $u_n=(i_0,i_1,\dots,i_{m-1})$, with an increasing cyclic ordering of the arguments in case equalities occur.

Question 1. For the set $S=\{n^k : n\in\mathbb{N}\}$ of perfect kth powers or, more generally, for sets $S=\{f(n)\}_{n\geq 1}$ of polynomial functions as those in [17] and [20], does the sequence $\{u_n\}_{n\geq 1}$ become periodic?

Question 2. If so, is the statement of Question 1 true for *all* $m\geq 3$?

Finally, we note that, although we could not directly apply Meinardus's Theorem to our problem, we did end up nevertheless with the two similar estimates, obtaining the asymptotics that his theorem would have heuristically predicted. This naturally leads to the following question.

Question 3. Can Meinardus's Theorem be strengthened so as to deal with a more general class of infinite product generating functions than that studied in [18]?

Acknowledgement. *The author would like to thank Kathrin Bringmann for proposing the question in which this paper found its original inspiration, Ef Sofos for suggesting the generalization addressed in this paper during a talk given by the author at the Max Planck Institute for Mathematics in Bonn, and Pieter Moree for the invitation to give that talk. In particular, the author is grateful to Igor Shparlinski for pointing out bound* (6.10)*. Part of the work was supported by the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013)/ERC Grant agreement no. 335220 — AQSER. A significant review completing this work was carried out during a postdoctoral stay of the author at the Max Planck Institute for Mathematics. The author would like to acknowledge the hospitality of the staff and the inspiring atmosphere. Last but not least, the author is thankful to the anonymous referees for their thorough reading of this paper, for indicating several references of which the author was not aware, and for the thoughtful remarks made on improving the presentation and correcting certain computations (in particular, for suggesting the combinatorial argument discussed in Section 3).*

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Received: 27.02.2022
Revised: 08.09.2022
Accepted: 15.09.2022

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