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The distribution of powers modulo q

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Abstract

Let $\delta, \delta_1, \delta_2$ be any real numbers with $0 < \delta, \delta_1, \delta_2 \leq 1$, and $q \geq 2$ be an integer, $h, k, l > 1$ be any fixed non-zero pairwise distinct integers. In the present paper we use some estimates of exponential sums to study the distribution of integer powers modulo q. Define

$$N_{h,k,l,\delta_1,\delta_2}(q) = \#\Big\{a : 0 < a \leq q, (a, q) = 1, a \in \mathcal{A}_{h,k,\delta_1}(q) \cap \mathcal{A}_{k,l,\delta_2}(q)\Big\},$$

where

$$\mathcal{A}_{h,k,\delta}(q) = \Big\{a : 0 < a \leq q, (a, q) = 1, \Big|\Big\{\frac{a^h}{q}\Big\} - \Big\{\frac{a^k}{q}\Big\}\Big| < \delta\Big\}.$$

We derive asymptotic formulas for $N_{h,k,l,\delta_1,\delta_2}(q)$.

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1 Introduction

Let $q > 1$ be an integer. If a is an integer coprime to q, we denote by $\overline{a}$ the natural number less than q such that $a\overline{a} \equiv 1(\text{mod } q)$, and $\overline{a}$ is called the inverse of a. Several number theorists studied the distribution of an integer and its inverse. Related works can be found in [2, 3, 6, 8, 11].

Let p be an odd prime. For any fixed real number $0 < \delta < 1$, we define

$$S(p, \delta) = \#\Big\{a : 0 \leq a \leq p, |a - \bar{a}| < \delta p\Big\}.$$

Zhang [10] studied the limit distribution properties of

$$\frac{1}{p-1} S(p, \delta),$$

and derived that

$$S(p, \delta) = \delta(2 - \delta)p + O(p^{\frac{1}{2}} \ln^2(p)).$$

Moreover, for any fixed integer k, Zhang [11] studied the asymptotic properties of $S_k(p, \delta)$ with

$$S_k(p, \delta) = \#\Big\{a : 0 \leq a \leq p, \Big|\Big\{\frac{a^k}{p}\Big\} - \Big\{\frac{\overline{a}^k}{p}\Big\}\Big| < \delta\Big\},$$

and obtained that

$$S_k(p,\delta) = \delta(2-\delta)p + O_k(p^{\frac{1}{2}}\ln^2(p)).$$

In [7], Xu studied the distribution of the difference of an integer and its m-th power modulo q over incomplete intervals. Let λ, δ be any real numbers with $0 < \lambda, \delta \leq 1$, $q > \max\{[\frac{1}{\lambda}], [\frac{1}{\delta}]\}$ and $m \geq 2$ be integers. Let P be the parallelogram with vertices $(0,-\delta), (\lambda, \lambda-\delta), (\lambda, \lambda+\delta)$ and $(0,\delta)$. Xu defined

$$S_{m,q,\lambda,\delta} = \#\{a : 1 \leq a \leq \lambda q, (a,q) = 1, |a - (a^m)_q| \leq \delta q\},$$

where $(a)_q$ denotes the integer b with $1 \leq b \leq q$ such that $b \equiv a (\text{mod } q)$ for any integer q and proved the formula

$$S_{m,q,\lambda,\delta} = \phi(q) A_{P\cap[0,1]^2} + O(m^{\omega(q)+\frac{1}{2}} q^{\frac{1}{2}} d(q) \ln^3 q),$$

where $A_{P\cap[0,1]^2}$ denotes the area of $P \cap [0,1]^2$. Zhang [10] proved an asymptotic formula for the cardinality of $S_{m,q,\lambda,\delta}$ in a special case:

$$S_{\phi(q)-1,q,1,\delta} = \delta(2-\delta)\phi(q) + O(q^{\frac{1}{2}} d^2(q) \ln^3 q).$$

We study the distribution of integer powers modulo q. Let integer $q \geq 2$, $h,k,l \geq 2$ be any fixed non-zero pairwise distinct integers, $0 < \delta, \delta_1, \delta_2 \leq 1$ be real numbers. We define

$$N_{h,k,l,\delta_1,\delta_2}(q) = \#\left\{a : 0 < a \leq q, (a,q) = 1, a \in \mathcal{A}_{h,k,\delta_1}(q) \cap \mathcal{A}_{k,l,\delta_2}(q)\right\},$$

where

$$\mathcal{A}_{h,k,\delta}(q) = \left\{a : 0 < a \leq q, (a,q) = 1, \left|\left\{\frac{a^h}{q}\right\} - \left\{\frac{a^k}{q}\right\}\right| < \delta\right\},$$

and $\{x\} = x - [x]$ denotes the fractional part of x, $[x]$ denotes the integral part of x; thus $q\{a^k/q\}$ is the least positive residue mod q of a^k. Li [4] proved an asymptotic formula of $\mathcal{A}_{h,k,\delta}(q)$:

$$\#\mathcal{A}_{h,k,\delta}(q) = \delta(2-\delta)\phi(q) + O(q^{\frac{1}{2}+\varepsilon}) \tag{1.1}$$

where $q = p^\alpha$ in his master thesis. In this paper, we prove the following asymptotic formula for $N_{h,k,l,\delta_1,\delta_2}(q)$.

Theorem 1. *Let integer $q \geq 2$, $h,k,l \geq 2$ be any fixed non-zero pairwise distinct integers, δ_1, δ_2 be real numbers with $0 < \delta_1 \leq \delta_2 \leq 1$, we have asymptotic estimations:*

$$N_{h,k,l,\delta_1,\delta_2}(q) = \begin{cases} \left(4\delta_1\delta_2 - 2\delta_2^2\delta_1 - \delta_1^2\delta_2 - \frac{\delta_1^3}{3}\right)\phi(q) + O(q^{\frac{2}{3}+\varepsilon}), & \text{if } \delta_1 + \delta_2 \leq 1; \\ \left(\delta_1 + \delta_2 - \delta_1^2 - \delta_2^2 + 2\delta_1\delta_2 \right. \\ \qquad \left. + \frac{\delta_2^3}{3} - \delta_2^2\delta_1 - \frac{1}{3}\right)\phi(q) + O(q^{\frac{2}{3}+\varepsilon}), & \text{if } \delta_1 + \delta_2 > 1. \end{cases}$$

In particular, taking $\delta_1 = \delta_2 = \delta$, from Theorem 1 we get an asymptotical formula in the following corollary:

Corollary 1. *Let integer $q \geq 2$, $h, k, l > 1$ be any fixed non-zero pairwise distinct integers, let δ be a real number with $0 < \delta \leq 1$, we obtain*

$$N_{h,k,l,\delta,\delta}(q) = \begin{cases} \Big(4\delta^2 - \frac{10\delta^3}{3}\Big)\phi(q) + O(q^{\frac{2}{3}+\varepsilon}), \ \text{if } \delta \leq \frac{1}{2}; \\ \\ \Big(2\delta - \frac{2\delta^3}{3} - \frac{1}{3}\Big)\phi(q) + O(q^{\frac{2}{3}+\varepsilon}), \ \text{if } \delta > \frac{1}{2} \ . \end{cases}$$

Taking that $\delta_1 = \delta, \delta_2 = 1$, we get the following corollary:

Corollary 2. *Let integer $q \geq 2$, $h, k, l > 1$ be any fixed non-zero pairwise distinct integers, let δ be a real number with $0 < \delta \leq 1$, we have*

$$N_{h,k,l,\delta,1}(q) = \delta(2-\delta)\phi(q) + O(q^{\frac{2}{3}+\varepsilon}).$$

Remark. In fact, when $l = k$, we see that $N_{h,k,l,\delta,\delta}(q)$ is the cardinality of $\mathcal{A}_{h,k,\delta}(q)$. Furthermore, when $l = h$, by using a similar method in the proof of Theorem 1 and the following upper bound of two-term exponential sum

$$\max_{gcd(u_1,u_2,q)=g} \left| \sum_{a=1}^{q}{}' e\left(\frac{u_1 a^{m_1} + u_2 a^{m_2}}{q}\right) \right| \leq g^{\frac{1}{2}} q^{\frac{1}{2}+\varepsilon}$$

from the case $t = 2$ of Lemma 3, we get a corresponding result

$$N_{h,k,l,\delta_1,\delta_2}(q) = \delta'(2-\delta')\phi(q) + O(q^{\frac{1}{2}+\varepsilon}),$$

where $\delta' = \min\{\delta_1, \delta_2\}$. We also derive a sharper asymptotic formula as follows

$$N_{h,k,l,\delta,1}(q) = \delta(2-\delta)\phi(q) + O(q^{\frac{1}{2}+\varepsilon}).$$

2 Technical Lemmas

We need the following technical lemmas to prove our theorem.

Lemma 1. *Let p be a prime and u be an integer with $(u, p) = 1$, then for any integers $k \geq 2$ and $\alpha \geq 2$, there holds*

$$\left| \sum_{\substack{a=1 \\ p \nmid a}}^{p^\alpha} e\left(\frac{ua^k}{p^\alpha}\right) \right| \leq kp^{\frac{\alpha}{2}}.$$

Proof. This can be easily obtained from [1, formula 0.5]. ∎

Lemma 2. *Let $q > 1$ be an integer, $k \geq 2$ be a fixed positive integer. For any integer u such that $(u, q) = 1$, we have the following estimation*

$$\left| \sum_{a=1}^{q}{}' e\Big(\frac{ua^k}{q}\Big) \right| \leq k^{\omega(q)} q^{\frac{1}{2}},$$

where $\omega(q)$ denotes distinct prime divisors of q.

Proof. Let $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime factor decomposition of q, we first note that

$$\sum_{a=1}^{q}{}' e\Big(\frac{ua^k}{q}\Big) = \prod_{i=1}^{r} \Bigg(\sum_{a=1}^{p_i^{\alpha_i}}{}' e\Big(\frac{u(\frac{q}{p_i^{\alpha_i}})^{k-1} a^k}{p_i^{\alpha_i}}\Big)\Bigg). \tag{2.1}$$

In fact, if m and n pass through a reduced residue system modulo q_1 and q_2 respectively, $mq_2 + nq_1$ passes through a reduced system modulo q_1q_2, so one can write

$$\begin{aligned} \sum_{a=1}^{q}{}' e\Big(\frac{ua^k}{q}\Big) &= \sum_{m=1}^{q_1}{}' \sum_{n=1}^{q_2}{}' e\Big(\frac{u(mq_2 + nq_1)^k}{q_1q_2}\Big) \\ &= \sum_{m=1}^{q_1}{}' e\Big(\frac{u(mq_2)^k}{q_1q_2}\Big) \sum_{n=1}^{q_2}{}' e\Big(\frac{u(nq_1)^k}{q_1q_2}\Big) \\ &= \sum_{m=1}^{q_1}{}' e\Big(\frac{uq_2^{k-1}m^k}{q_1}\Big) \sum_{n=1}^{q_2}{}' e\Big(\frac{uq_1^{k-1}n^k}{q_2}\Big). \end{aligned}$$

This yields the identity (2.1). By Lemma 1 we can get Lemma 2. □

Lemma 3. *For a positive integer q, let $m_1, \dots, m_t$ be $t \geq 2$ non-zero fixed pairwise distinct integers. Then the bound*

$$\max_{gcd(u_1,\dots,u_t,q)=g} \left| \sum_{a=1}^{q}{}' e\left(\frac{u_1a^{m_1} + \cdots + u_ta^{m_t}}{q}\right)\right| \leq g^{\frac{1}{t}} q^{1-\frac{1}{t}+\varepsilon}$$

holds.

Proof. See [5, Lemma 1]. □

Lemma 4. *Let λ be a real constant with $0 < \lambda \leq 1$, $q > [\frac{1}{\lambda}]$ be any integer and r be an integer with $1 \leq r \leq q$. For any nonnegative integer l, we have the estimate*

$$\sum_{a=1}^{[\lambda q]} a^l e\left(\frac{ra}{q}\right) \begin{cases} = \frac{(\lambda q)^{l+1}}{l+1} + O((\lambda q)^l), & \text{if } q \mid r; \\ \ll \frac{(\lambda q)^l}{|\sin \frac{\pi r}{q}|}, & \text{if } q \nmid r. \end{cases}$$

Proof. See [7, Lemma 3]. □

Lemma 5. *Let $q \geq 2$ be an integer, $k > 1$ be a fixed positive integer. Then we have*

$$\sum_{r=1}^{q-1} \left| \sum_{a=1}^{q}{}' e\left(\frac{ra^h}{q}\right)\right| \frac{1}{|\sin \frac{\pi r}{q}|} = O(q^{\frac{3}{2}+\varepsilon}), \tag{2.2}$$

$$\sum_{s=1}^{q-1} \left| \sum_{a=1}^{q}{}' e\left(\frac{sa^k}{q}\right)\right| \frac{1}{|\sin \frac{\pi s}{q}|} = O(q^{\frac{3}{2}+\varepsilon}), \tag{2.3}$$

$$\sum_{m=1}^{q-1}\sum_{\substack{s=1\\ s\neq m}}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{(s-m)a^k}{q}\right)\right|\frac{1}{|\sin\frac{\pi m}{q}||\sin\frac{\pi s}{q}||\sin\frac{\pi(m-s)}{q}|}=O(q^{\frac{7}{2}+\varepsilon}), \tag{2.4}$$

and

$$\sum_{m=1}^{q-1}\sum_{\substack{s=1\\ s\neq m}}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{(s-m)a^k}{q}\right)\right|\frac{1}{|\sin\frac{\pi m}{q}|^2|\sin\frac{\pi s}{q}|}=O(q^{\frac{7}{2}+\varepsilon}). \tag{2.5}$$

Proof. We prove (2.2), and the others can be obtained by the same methods. We begin with the estimates of formula (2.2), by Lemma 2 and the Jordan inequality

$$\frac{2}{\pi}\leq\frac{\sin x}{x}\qquad(|x|\leq\frac{\pi}{2}), \tag{2.6}$$

one can write

$$\sum_{r=1}^{q-1}\frac{1}{|\sin\frac{\pi r}{q}|}\leq q\sum_{r=1}^{[\frac{q-1}{2}]}\frac{1}{r}+q\sum_{r=[\frac{q+1}{2}]}^{q-1}\frac{1}{q-r}\ll q\ln q.$$

Now we have

$$\begin{aligned}\sum_{r=1}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h}{q}\right)\right|\frac{1}{|\sin\frac{\pi r}{q}|}=&\sum_{\substack{r=1\\(r,q)=1}}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h}{q}\right)\right|\frac{1}{|\sin\frac{\pi r}{q}|}\\&+\sum_{\substack{r=1\\(r,q)>1}}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h}{q}\right)\right|\frac{1}{|\sin\frac{\pi r}{q}|}.\end{aligned} \tag{2.7}$$

For the first term of (2.7), from Lemma 2 and the Jordan inequality (2.6), we have

$$\sum_{\substack{r=1\\(r,q)=1}}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h}{q}\right)\right|\frac{1}{|\sin\frac{\pi r}{q}|}\ll q^{\frac{3}{2}}h^{\omega(q)}\ln^2 q. \tag{2.8}$$

where we used the bound $\omega(q)\ll\frac{\ln q}{\ln\ln q}$. For the second term of (2.7), from Lemma 2 and the Jordan inequality (2.6), we have

$$\begin{aligned}\sum_{\substack{r=1\\(r,q)>1}}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h}{q}\right)\right|\frac{1}{|\sin\frac{\pi r}{q}|}&=\sum_{\substack{d|q\\d>1}}d\sum_{\substack{r=1\\d|r,(\frac{r}{d},\frac{q}{d})=1}}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{r/da^h}{q/d}\right)\right|\frac{1}{|\sin\frac{\pi r}{q}|}\\&\ll q^{\frac{1}{2}}\sum_{\substack{d|q\\d>1}}d^{\frac{1}{2}}h^{\omega(q/d)}\sum_{\substack{r=1\\d|r}}^{q-1}\frac{1}{|\sin\frac{\pi r}{q}|}\\&\ll q^{\frac{1}{2}}\sum_{\substack{d|q\\d>1}}d^{\frac{1}{2}}h^{\omega(q/d)}\sum_{\substack{r=1\\d|r}}^{[\frac{2q-2}{d}]}\frac{1}{|\sin\frac{\pi r}{q/d}|}\end{aligned}$$

$$\ll q^{\frac{3}{2}} h^{\omega(q)} \ln^2 q. \tag{2.9}$$

This completes the proof. ∎

Lemma 6. *Let integer $q \geq 2$, $h, k > 1$ be any fixed non-zero pairwise distinct integers. Then we have*

$$\sum_{r=1}^{q-1}\sum_{s=1}^{q-1}\left|\sum_{a=1}^{q}{}' e\left(\frac{ra^h + sa^k}{q}\right)\right| \frac{1}{|\sin\frac{\pi r}{q}||\sin\frac{\pi s}{q}|} = O(q^{\frac{5}{2}+\varepsilon}), \tag{2.10}$$

$$\sum_{\substack{r=1\\ r\neq q-s}}^{q-1}\sum_{s=1}^{q-1}\left|\sum_{a=1}^{q}{}' e\left(\frac{ra^h + sa^k}{q}\right)\right| \frac{1}{|\sin\frac{\pi s}{q}||\sin\frac{\pi(r+s)}{q}|} = O(q^{\frac{5}{2}+\varepsilon}), \tag{2.11}$$

$$\sum_{m=1}^{q-1}\left|\sum_{a=1}^{q}{}' e\left(\frac{ma^h - ma^k}{q}\right)\right| \frac{1}{|\sin\frac{\pi m}{q}|} = O(q^{\frac{3}{2}+\varepsilon}), \tag{2.12}$$

$$\sum_{m=1}^{q-1}\sum_{r=1}^{q-1}\left|\sum_{a=1}^{q}{}' e\left(\frac{(m+r)a^h - ma^k}{q}\right)\right| \frac{1}{|\sin\frac{\pi m}{q}||\sin\frac{\pi r}{q}|} = O(q^{\frac{5}{2}+\varepsilon}), \tag{2.13}$$

$$\sum_{m=1}^{q-1}\sum_{s=1}^{q-1}\left|\sum_{a=1}^{q}{}' e\left(\frac{ma^h + (s-m)a^k}{q}\right)\right| \frac{1}{|\sin\frac{\pi m}{q}||\sin\frac{\pi s}{q}|} = O(q^{\frac{5}{2}+\varepsilon}), \tag{2.14}$$

$$\sum_{m=1}^{q-1}\sum_{\substack{r=1\\ r\neq q-m}}^{q-1}\sum_{\substack{s=1\\ s\neq m}}^{q-1}\left|\sum_{a=1}^{q}{}' e\left(\frac{(m+r)a^h + (s-m)a^k}{q}\right)\right| \times \frac{1}{|\sin\frac{\pi m}{q}||\sin\frac{\pi s}{q}||\sin\frac{\pi(r+s)}{q}|} = O(q^{\frac{7}{2}+\varepsilon}), \tag{2.15}$$

and

$$\sum_{m=1}^{q-1}\sum_{\substack{r=1\\ r\neq q-m}}^{q-1}\sum_{\substack{s=1\\ s\neq m}}^{q-1}\left|\sum_{a=1}^{q}{}' e\left(\frac{(m+r)a^h + (s-m)a^k}{q}\right)\right| \times \frac{1}{|\sin\frac{\pi m}{q}||\sin\frac{\pi s}{q}||\sin\frac{\pi r}{q}|} = O(q^{\frac{7}{2}+\varepsilon}), \tag{2.16}$$

Proof. We give the detailed proof of (2.15), and the others can be proved by same ways. For the case when $t = 2$ of Lemma 3 and formula (2.6), one has

$$\sum_{m=1}^{q-1}\sum_{\substack{r=1\\ r\neq q-m}}^{q-1}\sum_{\substack{s=1\\ s\neq m}}^{q-1}\left|\sum_{a=1}^{q}{}' e\left(\frac{(m+r)a^h + (s-m)a^k}{q}\right)\right| \frac{1}{|\sin\frac{\pi m}{q}||\sin\frac{\pi s}{q}||\sin\frac{\pi(r+s)}{q}|}$$

$$\leq q^{\frac{1}{2}+\varepsilon}\sum_{m=1}^{q-1}\sum_{\substack{r=1\\ r\neq q-m}}^{q-1}\sum_{\substack{s=1\\ s\neq m}}^{q-1}\frac{(m+r, s-m, q)^{\frac{1}{2}}}{|\sin\frac{\pi m}{q}||\sin\frac{\pi s}{q}||\sin\frac{\pi(r+s)}{q}|}$$

$$\leq q^{\frac{1}{2}+\varepsilon}\sum_{d|q}d^{\frac{1}{2}}\sum_{\substack{m=1\\ r\neq q-m, s\neq m\\ d|(m+r), d|(s-m)}}^{q-1}\sum_{r=1}^{q-1}\sum_{s=1}^{q-1}\frac{1}{|\sin\frac{\pi m}{q}||\sin\frac{\pi s}{q}||\sin\frac{\pi(r+s)}{q}|}$$

$$\leq q^{\frac{1}{2}+\varepsilon}\sum_{d|q}d^{\frac{1}{2}}\sum_{m=1}^{q-1}\sum_{r=1}^{q-1}\sum_{\substack{s=1\\ r\neq q-s\\ d|(r+s)}}^{q-1}\frac{1}{|\sin\frac{\pi m}{q}||\sin\frac{\pi s}{q}||\sin\frac{\pi(r+s)}{q}|}$$

$$\leq q^{\frac{1}{2}+\varepsilon}\left(\sum_{m=1}^{q-1}\frac{1}{|\sin\frac{\pi m}{q}|}\right)^2\sum_{d|q}d^{\frac{1}{2}}\sum_{h=1}^{[\frac{2q-2}{d}]}\frac{1}{|\sin\frac{\pi h}{q/d}|}\quad\ll q^{\frac{7}{2}+\varepsilon}.$$

□

Lemma 7. *Let integer $q\geq 2$, $h,k,l>1$ be any fixed non-zero pairwise distinct integers. Then we have*

$$\sum_{r=1}^{q-1}\sum_{s=1}^{q-1}\sum_{f=1}^{q-1}\left|\sum_{a=1}^{q}{}' e\left(\frac{ra^h+sa^k+fa^l}{q}\right)\right|\frac{1}{|\sin\frac{\pi r}{q}||\sin\frac{\pi f}{q}||\sin\frac{\pi s}{q}|}=O(q^{\frac{11}{3}+\varepsilon}),\tag{2.17}$$

$$\sum_{r=1}^{q-1}\sum_{\substack{s=1\\ s\neq q-f}}^{q-1}\sum_{f=1}^{q-1}\left|\sum_{a=1}^{q}{}' e\left(\frac{ra^h+sa^k+fa^l}{q}\right)\right|\frac{1}{|\sin\frac{\pi r}{q}||\sin\frac{\pi f}{q}||\sin\frac{\pi(s+f)}{q}|}=O(q^{\frac{11}{3}+\varepsilon}),\tag{2.18}$$

Proof. Applying the case when $t=3$ of Lemma 3 and (2.6) we have

$$\sum_{r=1}^{q-1}\sum_{\substack{s=1\\ s\neq q-f}}^{q-1}\sum_{f=1}^{q-1}\left|\sum_{a=1}^{q}{}' e\left(\frac{ra^h+sa^k+fa^l}{q}\right)\right|\frac{1}{|\sin\frac{\pi r}{q}||\sin\frac{\pi f}{q}||\sin\frac{\pi(s+f)}{q}|}$$

$$\leq q^{\frac{2}{3}+\varepsilon}\sum_{r=1}^{q-1}\sum_{\substack{s=1\\ s\neq q-f}}^{q-1}\sum_{f=1}^{q-1}\frac{(r,s,f,q)^{\frac{1}{3}}}{|\sin\frac{\pi r}{q}||\sin\frac{\pi f}{q}||\sin\frac{\pi(s+f)}{q}|}$$

$$\leq q^{\frac{2}{3}+\varepsilon}\sum_{d|q}d^{\frac{1}{3}}\sum_{\substack{r=1\\ d|r, d|f, d|s\\ s\neq q-f}}^{q-1}\sum_{s=1}^{q-1}\sum_{f=1}^{q-1}\frac{1}{|\sin\frac{\pi r}{q}||\sin\frac{\pi f}{q}||\sin\frac{\pi(s+f)}{q}|}$$

$$\leq q^{\frac{2}{3}+\varepsilon}\sum_{d|q}d^{\frac{1}{3}}\left(\sum_{h=1}^{[\frac{q-1}{d}]}\frac{1}{|\sin\frac{\pi h}{q/d}|}\right)^2\sum_{l=1}^{[\frac{2q-2}{d}]}\frac{1}{|\sin\frac{\pi l}{q/d}|}\ll q^{\frac{11}{3}+\varepsilon}.$$

Similarly, we can get the formula (2.17). □

Lemma 8. *Let $\delta_1, \delta_2 \in (0,1]$ be real constants, integer $q \geq 2$, $h,k,l > 1$ be any fixed non-zero pairwise distinct integers, we have*

$$\sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \sideset{}{'}\sum_{c=1}^{q} \sideset{}{'}\sum_{\substack{d=1 \\ a^h \equiv b \pmod q \\ a^k \equiv c \pmod q \\ a^l \equiv d \pmod q \\ b>c, c>d}}^{q} 1 = \frac{\phi(q)}{6} + O(q^{\frac{2}{3}+\varepsilon}).$$

Proof. By the trigonometric identity

$$\sum_{a=1}^{q} e\Big(\frac{ma}{q}\Big) = \begin{cases} n, & \text{if } q \mid m; \\ 0, & \text{if } q \nmid m, \end{cases}$$

we write

$$\begin{aligned}
\sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \sideset{}{'}\sum_{c=1}^{q} \sideset{}{'}\sum_{\substack{d=1 \\ a^h \equiv b \pmod q \\ a^k \equiv c \pmod q \\ a^l \equiv d \pmod q \\ b>c, c>d}}^{q} 1 &= \frac{1}{q^3} \sum_{r=1}^{q} \sum_{s=1}^{q} \sum_{f=1}^{q} \sum_{\alpha=1}^{q} \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=1}^{\beta-1} \\
&\quad \times \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \sideset{}{'}\sum_{c=1}^{q} \sideset{}{'}\sum_{\substack{d=1 \\ a^h \equiv b \pmod q \\ a^k \equiv c \pmod q \\ a^l \equiv d \pmod q}}^{q} e\left(\frac{r(b-\alpha)+s(c-\beta)+f(d-\gamma)}{q}\right) \\
&= \frac{1}{q^3} \sum_{r=1}^{q} \sum_{s=1}^{q} \sum_{f=1}^{q} \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \sideset{}{'}\sum_{c=1}^{q} \sideset{}{'}\sum_{\substack{d=1 \\ a^h \equiv b \pmod q \\ a^k \equiv c \pmod q \\ a^l \equiv d \pmod q}}^{q} e\left(\frac{rb+sc+fd}{q}\right) \\
&\quad \times \sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right) \sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right) \sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right) \\
&= \frac{1}{q^3} \sum_{r=1}^{q} \sum_{s=1}^{q} \sum_{f=1}^{q} \sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h+sa^k+fa^l}{q}\right) \\
&\quad \times \sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right) \sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right) \sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right) \\
&= \frac{\phi(q)}{q^3} \sum_{\alpha=1}^{q} \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=1}^{\beta-1} 1 + \frac{1}{q^3} \sum_{s=1}^{q-1} \sum_{f=1}^{q-1} \sideset{}{'}\sum_{a=1}^{q} e\left(\frac{sa^k+fa^l}{q}\right) \\
&\quad \times \sum_{\alpha=1}^{q} \sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right) \sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right)
\end{aligned}$$

$$+\frac{1}{q^3}\sum_{r=1}^{q-1}\sum_{f=1}^{q-1}{\sum_{a=1}^{q}}' e\left(\frac{ra^h+fa^l}{q}\right)\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1}\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right)$$

$$+\frac{1}{q^3}\sum_{r=1}^{q-1}\sum_{s=1}^{q-1}{\sum_{a=1}^{q}}' e\left(\frac{ra^h+sa^k}{q}\right)\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)(\beta-1)$$

$$+\frac{1}{q^3}\sum_{f=1}^{q-1}{\sum_{a=1}^{q}}' e\left(\frac{fa^l}{q}\right)\sum_{\alpha=1}^{q}\sum_{\beta=1}^{\alpha-1}\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right)$$

$$+\frac{1}{q^3}\sum_{s=1}^{q-1}{\sum_{a=1}^{q}}' e\left(\frac{sa^k}{q}\right)\sum_{\alpha=1}^{q}\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)(\beta-1)$$

$$+\frac{1}{q^3}\sum_{r=1}^{q-1}{\sum_{a=1}^{q}}' e\left(\frac{ra^h}{q}\right)\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1}(\beta-1)$$

$$+\frac{1}{q^3}\sum_{r=1}^{q-1}\sum_{s=1}^{q-1}\sum_{f=1}^{q-1}{\sum_{a=1}^{q}}' e\left(\frac{ra^h+sa^k+fa^l}{q}\right)$$

$$\times\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right),$$

from Lemma 4 we have

$$\sum_{\alpha=1}^{q}\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right) \ll \frac{q}{|\sin\frac{\pi f}{q}|}\left(\frac{1}{|\sin\frac{\pi(s+f)}{q}|}+\frac{1}{|\sin\frac{\pi s}{q}|}\right),$$

$$\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1}\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right) \ll \frac{q}{|\sin\frac{\pi f}{q}||\sin\frac{\pi r}{q}|},$$

$$\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)(\beta-1) \ll \frac{q}{|\sin\frac{\pi r}{q}||\sin\frac{\pi s}{q}|},$$

$$\sum_{\alpha=1}^{q}\sum_{\beta=1}^{\alpha-1}\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right) \ll \frac{q^2}{|\sin\frac{\pi f}{q}|},$$

$$\sum_{\alpha=1}^{q}\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)(\beta-1) \ll \frac{q^2}{|\sin\frac{\pi s}{q}|},$$

$$\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1}(\beta-1) \ll \frac{q^2}{|\sin\frac{\pi r}{q}|},$$

and

$$\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right)$$

$$\ll \frac{1}{|\sin\frac{\pi f}{q}||\sin\frac{\pi s}{q}|}\left(\frac{1}{|\sin\frac{\pi r}{q}|}+\frac{1}{|\sin\frac{\pi(r+s)}{q}|}\right)$$
$$+\frac{1}{|\sin\frac{\pi f}{q}||\sin\frac{\pi(s+f)}{q}|}\left(\frac{1}{|\sin\frac{\pi r}{q}|}+\frac{1}{|\sin\frac{\pi(r+s+f)}{q}|}\right).$$

This follows that

$$\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c,c>d}}^{q} 1=\frac{\phi(q)}{6}+O(\Sigma_1)+O(\Sigma_2)+O(\Sigma_3)+O(\Sigma_4)+O(\Sigma_5)$$
$$+O(\Sigma_6)+O(\Sigma_7)+O(\Sigma_8)+O(\Sigma_9)+O(\Sigma_{10})+O(\Sigma_{11}),$$

where

$$\Sigma_1=\frac{1}{q^3}\sum_{s=1}^{q-1}\sum_{f=1}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{sa^k+fa^l}{q}\right)\right|\frac{q}{|\sin\frac{\pi f}{q}||\sin\frac{\pi s}{q}|}$$

$$\Sigma_2=\frac{1}{q^3}\sum_{s=1}^{q-1}\sum_{f=1}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{sa^k+fa^l}{q}\right)\right|\frac{q}{|\sin\frac{\pi f}{q}||\sin\frac{\pi(s+f)}{q}|}$$

$$\Sigma_3=\frac{1}{q^3}\sum_{r=1}^{q-1}\sum_{f=1}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h+fa^l}{q}\right)\right|\frac{q}{|\sin\frac{\pi f}{q}||\sin\frac{\pi r}{q}|}$$

$$\Sigma_4=\frac{1}{q^3}\sum_{r=1}^{q-1}\sum_{s=1}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h+sa^k}{q}\right)\right|\frac{q}{|\sin\frac{\pi r}{q}||\sin\frac{\pi s}{q}|}$$

$$\Sigma_5=\frac{1}{q^3}\sum_{f=1}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{fa^l}{q}\right)\right|\frac{q^2}{|\sin\frac{\pi f}{q}|}$$

$$\Sigma_6=\frac{1}{q^3}\sum_{s=1}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{sa^k}{q}\right)\right|\frac{q^2}{|\sin\frac{\pi s}{q}|}$$

$$\Sigma_7=\frac{1}{q^3}\sum_{r=1}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h}{q}\right)\right|\frac{q^2}{|\sin\frac{\pi r}{q}|}$$

$$\Sigma_8=\frac{1}{q^3}\sum_{r=1}^{q-1}\sum_{s=1}^{q-1}\sum_{f=1}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h+sa^k+fa^l}{q}\right)\right|\frac{1}{|\sin\frac{\pi f}{q}||\sin\frac{\pi s}{q}||\sin\frac{\pi r}{q}|}$$

$$\Sigma_9=\frac{1}{q^3}\sum_{r=1}^{q-1}\sum_{s=1}^{q-1}\sum_{f=1}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h+sa^k+fa^l}{q}\right)\right|\frac{1}{|\sin\frac{\pi f}{q}||\sin\frac{\pi s}{q}||\sin\frac{\pi(r+s)}{q}|}$$

$$\Sigma_{10}=\frac{1}{q^3}\sum_{r=1}^{q-1}\sum_{\substack{s=1\\ s\neq q-f}}^{q-1}\sum_{f=1}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h+sa^k+fa^l}{q}\right)\right|\frac{1}{|\sin\frac{\pi f}{q}||\sin\frac{\pi(s+f)}{q}||\sin\frac{\pi r}{q}|}$$

$$\Sigma_{11} = \frac{1}{q^3}\sum_{r=1}^{q-1}\sum_{\substack{s=1\\ s\neq q-f}}^{q-1}\sum_{f=1}^{q-1}\left|\sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ra^h+sa^k+fa^l}{q}\right)\right|$$
$$\times \frac{1}{|\sin\frac{\pi f}{q}||\sin\frac{\pi(s+f)}{q}||\sin\frac{\pi(r+s+f)}{q}|}.$$

By Lemma 5, 6 and 7, Lemma 8 is easily deduced. ∎

Lemma 9. *Let $\delta_1, \delta_2 \in (0,1]$ be real constants, integer $q \geq 2$, $h, k, l > 1$ be any fixed non-zero pairwise distinct integers, we have*

$$\sum_{m=1}^{q-1}\sum_{t_1=1}^{[\delta_1 q]}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c, c>d}}^{q} e\left(\frac{m(b-c-t_1)}{q}\right) \tag{2.19}$$
$$= \left(\frac{\delta_1}{3} - \frac{\delta_1^2}{2} + \frac{\delta_1^3}{6}\right)\phi(q)q + O(q^{\frac{5}{3}+\varepsilon}).$$

Proof. The summation (2.19) can be rewritten by

$$\sum_{m=1}^{q-1}\sum_{t_1=1}^{[\delta_1 q]}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c, c>d}}^{q} e\left(\frac{m(b-c-t_1)}{q}\right)$$
$$= \sum_{m=1}^{q-1}\sum_{t_1=1}^{[\delta_1 q]} e\left(\frac{-mt_1}{q}\right)\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c, c>d}}^{q} e\left(\frac{m(b-c)}{q}\right). \tag{2.20}$$

For the inner summation in (2.20), using the trigonometric identity we can write

$$\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c, c>d}}^{q} e\left(\frac{m(b-c)}{q}\right) = \sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{b-1}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q}}^{c-1} e\left(\frac{m(b-c)}{q}\right)$$
$$= \frac{1}{q^3}\sum_{r=1}^{q}\sum_{s=1}^{q}\sum_{f=1}^{q}\left(\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q}}^{q} e\left(\frac{(m+r)b+(s-m)c+fd}{q}\right)\right)$$

$$\times \left(\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right) \sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right) \sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right) \right).$$

Now we separate the summation over r, s and f into the following fifteen cases:

i) $r = q-m\ ,\ s = m\ ,\ f = q$;

ii) $r = q\ ,\ s = q\ ,\ f = q$;

iii) $1 \leq r \leq q-1\ ,\ s = q\ ,\ f = q$;

iv) $1 \leq r \leq q-1\ ,\ s = q\ ,\ 1 \leq f \leq q-1$;

v) $1 \leq r \leq q-1\ ,\ s = m\ ,\ 1 \leq f \leq q-1$;

vi) $r = q\ ,\ 1 \leq s \leq q-1\ ,\ s \neq m\ ,\ 1 \leq f \leq q-1$;

vii) $r = q\ ,\ s = q\ ,\ 1 \leq f \leq q-1$;

viii) $r = q\ ,\ 1 \leq s \leq q-1\ ,\ f = q$;

ix) $1 \leq r \leq q-1, r \neq q-m\ ,\ 1 \leq s \leq q-1\ s \neq m\ ,\ f = q$;

x) $1 \leq r \leq q-1, r \neq q-m\ ,\ s = m\ ,\ f = q$;

xi) $1 \leq r \leq q-1\ , r \neq q-m\ ,\ s = m\ ,\ 1 \leq f \leq q-1$;

xii) $1 \leq r \leq q-1, r \neq q-m\ ,\ 1 \leq s \leq q-1\ ,\ s \neq m\ ,\ 1 \leq f \leq q-1$;

xiii) $r = q-m\ ,\ s = m\ ,\ 1 \leq f \leq q-1$;

xiv) $r = q-m\ ,\ 1 \leq s \leq q-1\ ,\ s \neq m\ ,\ f = q$;

xv) $r = q-m\ ,\ 1 \leq s \leq q-1\ ,\ s \neq m\ ,\ 1 \leq f \leq q-1$.

It yields

$$\begin{aligned} &\mathop{\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' \sum_{c=1}^{q}{}' \sum_{d=1}^{q}{}'}_{\substack{a^h \equiv b \pmod q \\ a^k \equiv c \pmod q \\ a^l \equiv d \pmod q}} e\left(\frac{(m+r)b + (s-m)c) + fd}{q}\right) \\ &= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9 + T_{10} \\ &\quad + T_{11} + T_{12} + T_{13} + T_{14} + T_{15} \end{aligned}$$

where

$$T_1 = \frac{\phi(q)}{q^3} \sum_{\alpha=1}^{q} e\left(\frac{m\alpha}{q}\right) \sum_{\beta=1}^{\alpha-1} e\left(\frac{-m\beta}{q}\right)(\beta-1)$$

$$T_2 = \frac{1}{q^3}\left(\mathop{\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' \sum_{c=1}^{q}{}'}_{\substack{a^h \equiv b \pmod q \\ a^k \equiv c \pmod q}} e\left(\frac{mb-mc}{q}\right)\right)\left(\sum_{\alpha=1}^{q}\sum_{\beta=1}^{\alpha-1}(\beta-1)\right)$$

$$T_3 = \frac{1}{q^3}\sum_{r=1}^{q-1}\left(\mathop{\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' \sum_{c=1}^{q}{}'}_{\substack{a^h \equiv b \pmod q \\ a^k \equiv c \pmod q}} e\left(\frac{(m+r)b-mc}{q}\right)\right)\left(\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1}(\beta-1)\right)$$

$$T_4 = \frac{1}{q^3}\sum_{s=1}^{q-1}\left(\sideset{}{'}\sum_{\substack{a=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q}}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q} e\left(\frac{mb+(s-m)c}{q}\right)\right)\left(\sum_{\alpha=1}^{q}\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)(\beta-1)\right)$$

$$T_5 = \frac{1}{q^3}\sum_{\substack{r=1\\ r\neq q-m}}^{q-1}\sum_{\substack{s=1\\ s\neq m}}^{q-1}\left(\sideset{}{'}\sum_{\substack{a=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q}}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q} e\left(\frac{(m+r)b+(s-m)c}{q}\right)\right)$$

$$\times\left(\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)(\beta-1)\right)$$

$$T_6 = \frac{1}{q^3}\sum_{\substack{r=1\\ r\neq q-m}}^{q-1}\left(\sideset{}{'}\sum_{\substack{a=1\\ a^h\equiv b \pmod q}}^{q}\sideset{}{'}\sum_{b=1}^{q} e\left(\frac{(m+r)b}{q}\right)\right)$$

$$\times\left(\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-m\beta}{q}\right)(\beta-1)\right)$$

$$T_7 = \frac{1}{q^3}\sum_{\substack{s=1\\ s\neq m}}^{q-1}\left(\sideset{}{'}\sum_{\substack{a=1\\ a^k\equiv c \pmod q}}^{q}\sideset{}{'}\sum_{b=1}^{q} e\left(\frac{(s-m)c}{q}\right)\right)$$

$$\times\left(\sum_{\alpha=1}^{q} e\left(\frac{-m\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)(\beta-1)\right)$$

$$T_8 = \frac{1}{q^3}\sum_{f=1}^{q-1}\left(\sideset{}{'}\sum_{\substack{a=1\\ a^l\equiv d \pmod q}}^{q}\sideset{}{'}\sum_{d=1}^{q} e\left(\frac{fd}{q}\right)\right)\sum_{\alpha=1}^{q} e\left(\frac{m\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-m\beta}{q}\right)\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right)$$

$$T_9 = \frac{1}{q^3}\sum_{f=1}^{q}\left(\sideset{}{'}\sum_{\substack{a=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q}}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{d=1}^{q} e\left(\frac{mb-mc+fd}{q}\right)\right)\sum_{\alpha=1}^{\beta-1}\sum_{\beta=1}^{\alpha-1}\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right)$$

$$T_{10} = \frac{1}{q^3}\sum_{\substack{r=1\\ r\neq q-m}}^{q-1}\sum_{f=1}^{q-1}\left(\sideset{}{'}\sum_{\substack{a=1\\ a^h\equiv b \pmod q\\ a^l\equiv d \pmod q}}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{d=1}^{q} e\left(\frac{(m+r)b+fd}{q}\right)\right)$$

$$\times\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-m\beta}{q}\right)\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right)$$

$$T_{11} = \frac{1}{q^3}\sum_{r=1}^{q-1}\sum_{f=1}^{q-1}\left(\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^l\equiv d \pmod q}}^{q} e\left(\frac{(m+r)b-mc+fd}{q}\right)\right)$$
$$\times \sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1}\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right)$$

$$T_{12} = \frac{1}{q^3}\sum_{\substack{s=1\\ s\neq m}}^{q-1}\sum_{f=1}^{q-1}\left(\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{\substack{c=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q}}^{q} e\left(\frac{mb+(s-m)c+fd}{q}\right)\right)$$
$$\times \sum_{\alpha=1}^{q}\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right)$$

$$T_{13} = \frac{1}{q^3}\sum_{f=1}^{q-1}\left(\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{\substack{b=1\\ a^h\equiv b \pmod q\\ a^l\equiv d \pmod q}}^{q} e\left(\frac{(m+r)b+fd}{q}\right)\right)$$
$$\times \sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-m\beta}{q}\right)\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right)$$

$$T_{14} = \frac{1}{q^3}\sum_{\substack{r=1\\ r\neq q-m}}^{q-1}\sum_{\substack{s=1\\ s\neq m}}^{q-1}\sum_{f=1}^{q-1}\left(\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{\substack{c=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q}}^{q} e\left(\frac{(m+r)b+(s-m)c+fd}{q}\right)\right)$$
$$\times \sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right)$$

$$T_{15} = \frac{1}{q^3}\sum_{\substack{s=1\\ s\neq m}}^{q-1}\sum_{f=1}^{q-1}\left(\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{\substack{b=1\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q}}^{q} e\left(\frac{(s-m)c+fd}{q}\right)\right)$$
$$\times \sum_{\alpha=1}^{q} e\left(\frac{m\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-s\beta}{q}\right)\sum_{\gamma=1}^{\beta-1} e\left(\frac{-f\gamma}{q}\right).$$

Hence the first case of (2.20) is

$$\sum_{m=1}^{q-1}\sum_{t_1=1}^{[\delta_1 q]} e\left(\frac{-mt_1}{q}\right)T_1$$

$$
\begin{aligned}
&= \frac{\phi(q)}{q^3}\sum_{m=1}^{q-1}\sum_{t_1=1}^{[\delta_1 q]} e\left(\frac{-mt_1}{q}\right)\left(\sum_{\alpha=1}^{q} e\left(\frac{m\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-m\beta}{q}\right)(\beta-1)\right)\\
&= \frac{\phi(q)}{q^2}\sum_{t_1=1}^{[\delta_1 q]}\sum_{\substack{\alpha=1\\ \alpha-\beta-t_1\equiv 0 \pmod q}}^{q}\sum_{\beta=1}^{\alpha-1}(\beta-1) - \frac{\delta_1\phi(q)}{q^2}\sum_{\alpha=1}^{q}\sum_{\beta=1}^{\alpha-1}(\beta-1)\\
&= \frac{\phi(q)}{q^2}\sum_{t_1=1}^{[\delta_1 q]}\sum_{\substack{\alpha=1\\ \alpha-\beta=t_1}}^{q}\sum_{\beta=1}^{\alpha-1}(\beta-1) - \frac{\delta_1 q\phi(q)}{6} + O(q)\\
&= \frac{\phi(q)}{q^2}\sum_{t_1=1}^{[\delta_1 q]}\frac{(q-t_1)(q-t_1-1)}{2} - \frac{\delta_1 q\phi(q)}{6} + O(q)\\
&= \left(\frac{\delta_1}{3} - \frac{\delta_1^2}{2} + \frac{\delta_1^3}{6}\right)\phi(q)q + O(q).
\end{aligned}
$$

By the estimation formulas in Lemma 5, 6 and 7, we can get the other cases corresponding to the error term of (2.19) are $O(q^{\frac{5}{3}+\varepsilon})$. This completes the proofs of Lemma 9. ∎

Lemma 10. *Let $\delta_1, \delta_2 \in (0,1]$ be real constants, integer $q \geq 2$, $h,k,l > 1$ be any fixed non-zero pairwise distinct integers, we have*

$$
\sum_{m=1}^{q-1}\sum_{n=1}^{q-1}\sum_{t_1=1}^{[\delta_1 q]}\sum_{t_2=1}^{[\delta_2 q]}\sum_{\substack{a=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c,c>d}}^{q}{}' \sum_{b=1}^{q}{}' \sum_{c=1}^{q}{}' \sum_{d=1}^{q}{}' e\left(\frac{m(b-c-t_1)+n(c-d-t_2)}{q}\right)
$$

$$
= \begin{cases} \left(\frac{\delta_1\delta_2}{6} - \frac{\delta_1^3\delta_2}{6} - \frac{\delta_2^3\delta_1}{6}\right)\phi(q)q^2 + O(q^{\frac{8}{3}+\varepsilon}), & \text{if } \delta_1+\delta_2 \leq 1;\\ \left(\frac{\delta_1}{2} + \frac{\delta_2}{2} - \frac{\delta_1^2}{2} - \frac{\delta_2^2}{2} - \frac{5\delta_1\delta_2}{6} + \frac{\delta_1^3}{6} + \frac{\delta_2^3}{6}\right. & \\ \quad \left. + \frac{\delta_1^2\delta_2}{2} + \frac{\delta_1\delta_2^2}{2} - \frac{\delta_1^3\delta_2}{6} - \frac{\delta_2^3\delta_1}{6} - \frac{1}{6}\right)\phi(q)q^2 + O(q^{\frac{8}{3}+\varepsilon}), & \text{if } \delta_1+\delta_2 > 1. \end{cases} \tag{2.21}
$$

Proof. We denote the left-hand side of (2.21) by R, then

$$
\begin{aligned}
R = &\sum_{m=1}^{q-1}\sum_{n=1}^{q-1}\sum_{t_1=1}^{[\delta_1 q]}\sum_{t_2=1}^{[\delta_2 q]} e\left(-\frac{mt_1+nt_2}{q}\right)\\
&\times \sum_{\substack{a=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c,c>d}}^{q}{}' \sum_{b=1}^{q}{}' \sum_{c=1}^{q}{}' \sum_{d=1}^{q}{}' e\left(\frac{m(b-c)+n(c-d)}{q}\right).
\end{aligned} \tag{2.22}
$$

For the inner summation in the right-hand side of (2.22), recalling the trigonometric identity we get

$$\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c,c>d}}^{q} e\left(\frac{m(b-c)+n(c-d)}{q}\right)$$

$$=\frac{1}{q^3}\sum_{r=1}^{q}\sum_{u=1}^{q}\sum_{v=1}^{q}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q}}^{q} e\left(\frac{(m+r)b+(n-m+u)c+(v-n)d)}{q}\right)$$

$$\times\left(\sum_{\alpha=1}^{q} e\left(\frac{-r\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{-u\beta}{q}\right)\sum_{\gamma=1}^{\beta-1} e\left(\frac{-v\gamma}{q}\right)\right).$$

We separate the summation over r, u and v into the following fifteen cases:

i) $r=q-m$, $u=m-n$ and $v=n$;

ii) $r=q$, $u=q$ and $v=q$;

iii) $1\le r\le q-1$, $u=q$ and $v=q$;

iv) $1\le r\le q-1$, $u=q$ and $1\le v\le q-1$;

v) $1\le r\le q-1$, $v=q$ and $1\le u\le q-1$;

vi) $r=q$, $1\le u\le q-1$ and $1\le v\le q-1$;

vii) $r=q$, $u=q$ and $1\le v\le q-1$;

viii) $r=q$, $1\le u\le q-1$ and $v=q$;

ix) $1\le r\le q-1, r\ne q-m$, $1\le u\le q-1$ $u\ne m-n$ and $v=n$;

x) $1\le r\le q-1, r\ne q-m$, $u=m-n$ and $v=n$;

xi) $1\le r\le q-1$,$r\ne q-m$, $u=m-n$ and $1\le v\le q-1$, $v\ne n$;

xii) $1\le r\le q-1, r\ne q-m$,$1\le u\le q-1$,$u\ne m-n$,$1\le v\le q-1$,$v\ne n$;

xiii) $r=q-m$, $u=m-n$ and $1\le v\le q-1$,$v\ne n$;

xiv) $r=q-m$, $1\le u\le q-1$, $u\ne m-n$ and $v=n$;

xv) $r=q-m$, $1\le u\le q-1$, $u\ne m-n$ and $1\le v\le q-1, v\ne n$;

On the one hand, we are going to use V for the first case of R as above, which is

$$V=\sum_{m=1}^{q-1}\sum_{n=1}^{q-1}\sum_{t_1=1}^{[\delta_1 q]}\sum_{t_2=1}^{[\delta_2 q]} e\left(-\frac{mt_1+nt_2}{q}\right)$$

$$\times\sum_{\alpha=1}^{q} e\left(\frac{m\alpha}{q}\right)\sum_{\beta=1}^{\alpha-1} e\left(\frac{(n-m)\beta}{q}\right)\sum_{\gamma=1}^{\beta-1} e\left(\frac{n\gamma}{q}\right)$$

$$= \frac{\phi(q)}{q^3} \sum_{m=1}^{q-1} \sum_{n=1}^{q-1} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sum_{\alpha=1}^{q} \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=1}^{\beta-1} e\left(\frac{m(\alpha-\beta-t_1)+n(\beta+\gamma-t_2)}{q}\right)$$

$$= \frac{\phi(q)}{q} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sum_{\substack{\alpha=1 \\ \alpha-\beta-t_1\equiv 0 \pmod q \\ \beta+\gamma-t_2\equiv 0 \pmod q}}^{q} \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=1}^{\beta-1} 1$$

$$- \frac{\phi(q)}{q^3} \sum_{m=1}^{q-1} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sum_{\alpha=1}^{q} \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=1}^{\beta-1} e\left(\frac{m(\alpha-\beta-t_1)}{q}\right)$$

$$- \frac{\phi(q)}{q^3} \sum_{n=1}^{q-1} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sum_{\alpha=1}^{q} \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=1}^{\beta-1} e\left(\frac{n(\beta+\gamma-t_2)}{q}\right)$$

$$- \frac{\phi(q)}{q^3} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sum_{\alpha=1}^{q} \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=1}^{\beta-1} 1$$

$$= \frac{\phi(q)}{q} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sum_{\substack{\alpha=1 \\ \alpha-\beta=t_1 \\ \beta-\gamma=t_2}}^{q} \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=1}^{\beta-1} 1 - \frac{\phi(q)}{q^2} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sum_{\substack{\alpha=1 \\ \alpha-\beta=t_1}}^{q} \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=1}^{\beta-1} 1$$

$$- \frac{\phi(q)}{q^2} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sum_{\alpha=1}^{q} \sum_{\substack{\beta=1 \\ \beta-\gamma=t_2}}^{\alpha-1} \sum_{\gamma=1}^{\beta-1} 1 + \frac{\phi(q)}{q^3} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sum_{\alpha=1}^{q} \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=1}^{\beta-1} 1$$

$$= \frac{\phi(q)}{q} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} (q-t_1-t_2) - \frac{\phi(q)}{q^2} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \frac{(q-t_1)(q-t_1-1)}{2}$$

$$- \frac{\phi(q)}{q^2} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sum_{\alpha=1}^{q} \frac{(q-t_2)(q-t_2-1)}{2} + \frac{\delta_1\delta_2\phi(q)q^2}{6} + O(q^2).$$

After simple calculations one can write:

- If $\delta_1+\delta_2 \le 1$,

$$V = \left(\frac{\delta_1\delta_2}{6} - \frac{\delta_1^3\delta_2}{6} - \frac{\delta_2^3\delta_1}{6}\right)\phi(q)q^2 + O(q^2).$$

- If $\delta_1+\delta_2 > 1$,

$$V = \frac{\phi(q)}{q} \sum_{t_1=1}^{q-[\delta_2 q]} \sum_{t_2=1}^{[\delta_2 q]} (q-t_1-t_2) + \frac{\phi(q)}{q} \sum_{t_1=[\delta_2 q]}^{[\delta_1 q]} \sum_{t_1=1}^{q-t_1} (q-t_1-t_2)$$
$$- \left(\frac{\delta_1\delta_2}{2} - \frac{\delta_1^2\delta_2}{2} + \frac{\delta_1^3\delta_2}{6}\right)\phi(q)q^2 - \left(\frac{\delta_1\delta_2}{2} - \frac{\delta_2^2\delta_1}{2} + \frac{\delta_2^3\delta_1}{6}\right)\phi(q)q^2$$
$$+ \frac{\delta_1\delta_2}{6}\phi(q)q^2 + O(q^2)$$

$$= \Big(\frac{\delta_1}{2} + \frac{\delta_2}{2} - \frac{\delta_1^2}{2} - \frac{\delta_2^2}{2} - \frac{5\delta_1\delta_2}{6} + \frac{\delta_1^3}{6} + \frac{\delta_2^3}{6} + \frac{\delta_1^2\delta_2}{2} + \frac{\delta_1\delta_2^2}{2} - \frac{\delta_1^3\delta_2}{6} - \frac{\delta_2^3\delta_1}{6} - \frac{1}{6}\Big)\phi(q)q^2 + O(q^2).$$

On the other hand, by the same methods of Lemma 8, 9, we obtain that the error term of R is $O(q^{\frac{8}{3}+\varepsilon})$. This complete the proof of Lemma 10. □

3 Proof of Theorem 1

By the definition, $N_{h,k,l,\delta_1,\delta_2}(q)$ can be expressed as the following form

$$N_{h,k,l,\delta_1,\delta_2}(q) = \sideset{}{'}\sum_{\substack{a=1 \\ |\{\frac{a^h}{q}\}-\{\frac{a^k}{q}\}|<\delta_1 \\ |\{\frac{a^k}{q}\}-\{\frac{a^l}{q}\}|<\delta_2}}^{q} 1 = \sideset{}{'}\sum_{\substack{a=1 \\ |q\{\frac{a^h}{q}\}-q\{\frac{a^k}{q}\}|<\delta_1 q \\ |q\{\frac{a^k}{q}\}-q\{\frac{a^l}{q}\}|<\delta_2 q}}^{q} 1 = \sum_{t_1=0}^{[\delta_1 q]}\sum_{t_2=0}^{[\delta_2 q]}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1 \\ a^h\equiv b \pmod q \\ a^k\equiv c \pmod q \\ a^l\equiv d \pmod q \\ |b-c|=t_1, |c-d|=t_2}}^{q} 1$$

$$= \sum_{t_1=1}^{[\delta_1 q]}\sum_{t_2=1}^{[\delta_2 q]}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1 \\ a^h\equiv b \pmod q \\ a^k\equiv c \pmod q \\ a^l\equiv d \pmod q \\ |b-c|=t_1, |c-d|=t_2}}^{q} 1 + \sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1 \\ a^h\equiv b \pmod q \\ a^k\equiv c \pmod q \\ a^l\equiv d \pmod q \\ b=c, c=d}}^{q} 1$$

$$+ \sum_{t_1=1}^{[\delta_1 q]}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1 \\ a^h\equiv b \pmod q \\ a^k\equiv c \pmod q \\ a^l\equiv d \pmod q \\ |b-c|=t_1, c=d}}^{q} 1 + \sum_{t_2=1}^{[\delta_2 q]}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1 \\ a^h\equiv b \pmod q \\ a^k\equiv c \pmod q \\ a^l\equiv d \pmod q \\ b=c, |c-d|=t_2}}^{q} 1$$

$$= S^{(1)} + S^{(2)} + S^{(3)} + S^{(4)}. \tag{3.1}$$

We begin with the estimation of $S^{(1)}$. It is necessarily noticeable that the constraint conditions $|b-c| = t_1, |c-d| = t_2$ can be discussed in eight cases as follows:

i) $b > c\ ,\ c > d$;
ii) $b < c\ ,\ c < d$;
iii) $b > c\ , c < d\ ,\ b > d$;
iv) $b < c\ , c > d\ ,\ b > d$;
v) $b > c\ , c < d\ ,\ b < d$;
vi) $b < c\ , c > d\ ,\ b < d$;
vii) $b = d\ , b > c$;
viii) $b = d\ , b < c$.

Let

$$S^{(1)} = S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 + S_8, \tag{3.2}$$

where $S_i (i = 1, 2, \ldots, 8)$ represent the summation mentioned above over the constraint conditions of the first, second, ..., eighth case, respectively.

We consider S_1 firstly. From the trigonometric identity one can get

$$\begin{aligned}
S_1 &= \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \sideset{}{'}\sum_{c=1}^{q} \sideset{}{'}\sum_{\substack{d=1 \\ a^h \equiv b \pmod q \\ a^k \equiv c \pmod q \\ a^l \equiv d \pmod q \\ b-c=t_1, c-d=t_2}}^{q} 1 \\
&= \frac{1}{q^2} \sum_{m=1}^{q} \sum_{n=1}^{q} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \sideset{}{'}\sum_{c=1}^{q} \sideset{}{'}\sum_{\substack{d=1 \\ a^h \equiv b \pmod q \\ a^k \equiv c \pmod q \\ a^l \equiv d \pmod q \\ b>c, c>d}}^{q} e\left(\frac{m(b-c-t_1) + n(c-d-t_2)}{q}\right) \\
&= \frac{1}{q^2} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \sideset{}{'}\sum_{c=1}^{q} \sideset{}{'}\sum_{\substack{d=1 \\ a^h \equiv b \pmod q \\ a^k \equiv c \pmod q \\ a^l \equiv d \pmod q \\ b>c, c>d}}^{q} 1 \\
&\quad + \frac{1}{q^2} \sum_{m=1}^{q-1} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \sideset{}{'}\sum_{c=1}^{q} \sideset{}{'}\sum_{\substack{d=1 \\ a^h \equiv b \pmod q \\ a^k \equiv c \pmod q \\ a^l \equiv d \pmod q \\ b>c, c>d}}^{q} e\left(\frac{m(b-c-t_1)}{q}\right) \\
&\quad + \frac{1}{q^2} \sum_{n=1}^{q-1} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \sideset{}{'}\sum_{c=1}^{q} \sideset{}{'}\sum_{\substack{d=1 \\ a^h \equiv b \pmod q \\ a^k \equiv c \pmod q \\ a^l \equiv d \pmod q \\ b>c, c>d}}^{q} e\left(\frac{n(c-d-t_2)}{q}\right) \\
&\quad + \frac{1}{q^2} \sum_{m=1}^{q-1} \sum_{n=1}^{q-1} \sum_{t_1=1}^{[\delta_1 q]} \sum_{t_2=1}^{[\delta_2 q]} \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \sideset{}{'}\sum_{c=1}^{q} \sideset{}{'}\sum_{\substack{d=1 \\ a^h \equiv b \pmod q \\ a^k \equiv c \pmod q \\ a^l \equiv d \pmod q \\ b>c, c>d}}^{q} e\left(\frac{m(b-c-t_1) + n(c-d-t_2)}{q}\right).
\end{aligned}$$

Note that Lemma 8, 9 and 10 imply the following results.

- If $\delta_1+\delta_2\le 1$,

$$S_1=\Big(\delta_1\delta_2-\frac{\delta_1^2\delta_2}{2}-\frac{\delta_2^2\delta_1}{2}\Big)\phi(q)+O(q^{\frac{2}{3}+\varepsilon}).$$

- If $\delta_1+\delta_2>1$,

$$S_1=\Big(\frac{\delta_1}{2}+\frac{\delta_2}{2}-\frac{\delta_1^2}{2}-\frac{\delta_2^2}{2}+\frac{\delta_1^3}{6}+\frac{\delta_2^3}{6}-\frac{1}{6}\Big)\phi(q)+O(q^{\frac{2}{3}+\varepsilon}).$$

We only need to consider S_1, and $S_2,\dots,S_6$ can be deduced by the same ways. Hence

$$S_2=\begin{cases}\Big(\delta_1\delta_2-\frac{\delta_1^2\delta_2}{2}-\frac{\delta_2^2\delta_1}{2}\Big)\phi(q)+O(q^{\frac{2}{3}+\varepsilon}), & \text{if } \delta_1+\delta_2\le 1;\\ \Big(4\delta_1\delta_2-2\delta_2^2\delta_1-\delta_1^2\delta_2-\frac{\delta_1^3}{3}\Big)\phi(q)+O(q^{\frac{2}{3}+\varepsilon}), & \text{if } \delta_1+\delta_2>1.\end{cases}$$

$$S_3=S_4=\begin{cases}\Big(\frac{\delta_2^2}{2}-\frac{\delta_2^3}{3}\Big)\phi(q)+O(q^{\frac{2}{3}+\varepsilon}), & \text{if } \delta_1>\delta_2;\\ \Big(\delta_1\delta_2-\frac{\delta_1^2}{2}-\frac{\delta_2^2\delta_1}{2}+\frac{\delta_1^3}{6}\Big)\phi(q)+O(q^{\frac{2}{3}+\varepsilon}), & \text{if } \delta_1\le\delta_2.\end{cases}$$

$$S_5=S_6=\begin{cases}\Big(\delta_1\delta_2-\frac{\delta_2^2}{2}-\frac{\delta_1^2\delta_2}{2}+\frac{\delta_2^3}{6}\Big)\phi(q)+O(q^{\frac{2}{3}+\varepsilon}), & \text{if } \delta_1>\delta_2;\\ \Big(\frac{\delta_1^2}{2}-\frac{\delta_1^3}{3}\Big)\phi(q)+O(q^{\frac{2}{3}+\varepsilon}), & \text{if } \delta_1\le\delta_2.\end{cases}$$

We estimate S_7 now. Let $\delta=\min\{\delta_1,\delta_2\}$. One can write

$$\begin{aligned}S_7&=\sum_{t=1}^{[\delta q]}\sum_{\substack{a=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b=d,\,b-c=t}}^{q}{}'\sum_{b=1}^{q}{}'\sum_{c=1}^{q}{}'\sum_{d=1}^{q}{}' 1\\ &=\frac{1}{q^2}\sum_{t=1}^{[\delta q]}\sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{\substack{a=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c}}^{q}{}'\sum_{b=1}^{q}{}'\sum_{c=1}^{q}{}'\sum_{d=1}^{q}{}' e\Big(\frac{m(b-d)+n(b-c-t)}{q}\Big)\\ &=\frac{1}{q^2}\sum_{t=1}^{[\delta q]}\sum_{\substack{a=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c}}^{q}{}'\sum_{b=1}^{q}{}'\sum_{c=1}^{q}{}'\sum_{d=1}^{q}{}' 1\end{aligned}$$

$$+\frac{1}{q^2}\sum_{t=1}^{[\delta q]}\sum_{m=1}^{q-1}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c}}^{q} e\left(\frac{m(b-d)}{q}\right)$$

$$+\frac{1}{q^2}\sum_{t=1}^{[\delta q]}\sum_{n=1}^{q-1}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c}}^{q} e\left(\frac{n(b-c-t)}{q}\right)$$

$$+\frac{1}{q^2}\sum_{t=1}^{[\delta q]}\sum_{m=1}^{q-1}\sum_{n=1}^{q-1}\sideset{}{'}\sum_{a=1}^{q}\sideset{}{'}\sum_{b=1}^{q}\sideset{}{'}\sum_{c=1}^{q}\sideset{}{'}\sum_{\substack{d=1\\ a^h\equiv b \pmod q\\ a^k\equiv c \pmod q\\ a^l\equiv d \pmod q\\ b>c}}^{q} e\left(\frac{m(b-d)+n(b-c-t)}{q}\right)$$

$$=\frac{1}{q^2}\Big(\sum_{t=1}^{[\delta q]} S_7^{(1)}+S_7^{(2)}+S_7^{(3)}+S_7^{(4)}\Big).$$

It is easy to observe that the order of the main terms and error terms of $S_7^{(1)}, S_7^{(3)}$ and $S_7^{(4)}$ are the same as the main terms and the error terms in Lemma 8, 9 and 10 respectively, and the order of $S_7^{(2)}$ is equivalent to that of $S_7^{(3)}$. Therefore, we have $S_7 \ll q^{\frac{2}{3}+\varepsilon}$. Similarly, we also have $S_8 \ll q^{\frac{2}{3}+\varepsilon}$. By the same methods of S_1, we can prove that $S^{(2)}, S^{(3)}$ and $S^{(4)}$ are $O(q^{\frac{2}{3}+\varepsilon})$. Without loss of generality, assuming $\delta_1 \le \delta_2$, from (3.1) and (3.2), we derive the desired result. This completes the proof of the Theorem 1.

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