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Some remarks on totally positive algebraic integers

by
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Abstract

In this paper we prove several results concerning the set $\mathbb{A}_+$ of totally positive algebraic integers. We prove that the set $\mathbb{A}_+$ is dense in the set of positive real numbers. We explicitly construct an infinite family of cubic polynomials, which are minimal polynomials of totally positive algebraic integers, and use it to show that the distance between a totally positive algebraic integer and one of its conjugates can be arbitrarily small. Finally, we employ a new method to construct, for any prime $p \geq 3$, a monic, integer, irreducible polynomial of degree $p-1$, with all roots positive.

Key Words: Totally positive algebraic integer, Schur–Siegel–Smyth trace problem.
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1 Introduction

Let $\mathbb{A}_+$ be the set of totally positive algebraic integers, i.e. the set of algebraic integers such that all their conjugates are positive real numbers. For $\alpha \in \mathbb{A}_+$, denote by f_α the minimal polynomial of α, and let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ denote the conjugates of α, where $d = \deg(f_\alpha)$. The trace of α is then defined as $tr(\alpha) = \alpha_1 + \alpha_2 + \ldots + \alpha_d$. We also define $m_\alpha = \min_{i \neq j} |\alpha_i - \alpha_j|$, and $M_\alpha = \max_{i,j} |\alpha_i - \alpha_j|$.

A polynomial with real coefficients is called real rooted if all its roots are real (some authors prefer the term hyperbolic for this type of (univariate) polynomials). Real rooted polynomials arise in many areas of mathematics, including combinatorics, probability, PDEs and optimization (the excellent survey [10] contains many useful references), and have been extensively studied (see [5] for a survey of recent results). Totally real algebraic integers often appear naturally as eigenvalues of matrices with integer entries (see [11] p. 262 and [12]). An important class of real rooted polynomials is the set of monic, integer and irreducible polynomials, with all roots positive, i.e. the set of minimal polynomials of totally positive algebraic integers. There are several open questions concerning totally positive algebraic integers, the most famous one being the Schur-Siegel-Smith trace problem [2], that we recall below.

Conjecture. *(The Schur-Siegel-Smyth trace problem). Fix $\rho < 2$. Then, there are only finitely many $\alpha \in \mathbb{A}_+$ with* $\dfrac{tr(\alpha)}{\deg(\alpha)} \leq \rho$.

This conjecture has a long history, and many mathematicians (including Schur [13], Siegel [15], Smyth [16], [17], McKee and Smyth [8], Serre, Aguirre and Peral [1], Flammang [3], [4] , Liang and Wu [6]) have worked on it, improving on earlier results. The best result to date has been obtained by Wang, Wu and Wu [19], where this conjecture was proved for $\rho < 1.793145$. For a generalization of this conjecture, and related results, we refer the reader to [9] and [18]. In many of these problems is it necessary to have a good supply of (families of) minimal polynomials of elements of $\mathbb{A}_+$ in explicit form, in order to perform various computations using their coefficients. Unfortunately, these families of minimal polynomials of totally positive algebraic integers are rather scarce in the literature. Often, the totally positive algebraic integers that appear in literature are affine transformations of the roots of the Chebyshev polynomial of the first kind.

The set $\mathbb{A}_+$ of totally positive algebraic integers is obviously closed under addition and multiplication. Next, we investigate some of the properties of this set.

A Pisot number is a real algebraic integer greater than 1, all of whose conjugates have absolute value less than 1. Siegel [14] proved that the smallest Pisot number is the positive solution of the equation $x^3 - x - 1 = 0$.

Unlike the set of Pisot numbers, the set $\mathbb{A}_+$ does not have a smallest element. This can be easily seen by considering the equation $x^2 - 2nx + 1 = 0$, for integer $n > 1$. If we denote by $x_{1,n} < x_{2,n}$ its roots, we have that $x_{1,n} = n - \sqrt{n^2 - 1} \in \mathbb{A}_+$, and $x_{1,n} \to 0$, as $n \to \infty$. Since $x_{2,n} - x_{1,n} = (n + \sqrt{n^2 - 1}) - (n - \sqrt{n^2 - 1}) = 2\sqrt{n^2 - 1}$, this shows that the distance between consecutive roots of the minimal polynomial of an element of $\mathbb{A}_+$ can be arbitrarily large.

On the other hand, it follows from the proof of Proposition 2 that the distance between consecutive roots of the minimal polynomial f_α of a totally positive algebraic integer α can be arbitrarily small. We have thus proved

Proposition 1. *For $\alpha \in \mathbb{A}_+$, m_α can be arbitrarily small, and M_α can be arbitrarily large.*

The next result of this paper concerns the set $\mathbb{A}_+$. More precisely, we prove the following.

Theorem 1. *The set $\mathbb{A}_+$ of totally positive algebraic integers is dense in the set of positive real numbers.*

In the last section of the paper we construct explicitly two families of minimal polynomials of totally positive algebraic integers.

2 Preliminary results

In this section we include a characterisation of the coefficients of a real rooted polynomial, all of whose roots are positive. This result will be useful in our investigations, and in particular, will be used in the proof of Theorem 1.

Lemma 1. *Let $f \in \mathbb{R}[x]$ be a real rooted monic polynomial. Then all its roots are positive if and only if f has the form $f = x^n - a_{n-1}x^{n-1} + \cdots + (-1)^n a_0$, with $a_i > 0$, for all $0 \le i \le n - 1$.*

Proof. The direct implication follows immediately, using Vieta's relations.

For the converse, let $f = x^n - a_{n-1}x^{n-1} + \cdots + (-1)^n a_0$, with $a_i > 0$, for all $0 \le i \le n-1$. Let $g(x) = f(-x)$. Then

$$g(x) = (-x)^n - a_{n-1}(-x)^{n-1} + \cdots + (-1)^n a_0 = (-1)^n[x^n + a_{n-1}x^{n-1} + \cdots + a_0] \quad (2.1)$$

The polynomial $g(x)$ is also real rooted, and its roots are of the form $-\alpha$, where α is a root of $f(x)$.

From (2.1) it follows that all roots of $g(x)$ are negative, therefore all roots of $f(x)$ are positive, which completes the proof of this lemma.

□

3 Proof of Theorem 1

Surprinsingly, it's enough to use elements of $\mathbb{A}_+$ of degree 2 to prove this density result.

Proof. Fix $\alpha, \beta > 0$, with $\alpha < \beta$. We show that there exist positive integers a and b, such that the polynomial $x^2 - ax + b$ is real rooted, irreducible and one of its roots, say $x_1 = \frac{a-\sqrt{a^2-4b}}{2}$ is between α and β.

Take $a > 2\beta$. Then $a > 2\beta > 2\alpha > \alpha$, so $a\alpha - \alpha^2 > 0$.

For $a > 2\beta$, we have the following equivalences:

$$\begin{aligned} & a\alpha - \alpha^2 < b < a\beta - \beta^2 \quad (3.1) \\ \iff \quad & a^2 - (a-2\alpha)^2 < 4b < a^2 - (a-2\beta)^2 \\ \iff \quad & (a-2\beta)^2 < a^2 - 4b < (a-2\alpha)^2 \\ \iff \quad & a - 2\beta < \sqrt{a^2-4b} < a - 2\alpha \\ \iff \quad & 2\alpha < a - \sqrt{a^2-4b} < 2\beta \\ \iff \quad & \alpha < x_1 < \beta \end{aligned}$$

So it's enough to show that there exist positive integers a and b such that relation (3.1) holds, and $a^2 - 4b$ is not a perfect square.

Let $l := (a\beta - \beta^2) - (a\alpha - \alpha^2) = (\beta - \alpha)(a - (\alpha + \beta))$.

For $a > \alpha + \beta + \frac{2}{\beta-\alpha}$, we have $l > 2$.

Next, we show that we can find an integer b in the interval $I := (a\alpha - \alpha^2, a\beta - \beta^2)$ such that $a^2 - 4b$ is not a square. Suppose, for the sake of contradiction, that for all $b \in I$, $a^2 - 4b$ is a square.

We know that for $a > \alpha + \beta + \frac{2}{\beta-\alpha}$, the interval I, of length $l > 2$, contains at least two consecutive integers b, say b_1 and $b_1 + 1$. We obtain

$$a^2 - 4b_1 = k_1^2$$

$$a^2 - 4(b_1 + 1) = k_2^2$$

for some positive integers k_1 and k_2. Subtracting, we derive $k_1^2 = k_2^2 + 4$. This gives $(k_1 - k_2)(k_1 + k_2) = 4$, and since $k_1 + k_2 > k_1 - k_2$, the only possibility is $k_1 + k_2 = 4$ and $k_1 - k_2 = 1$, which would imply $2k_1 = 5$, a contradiction with the fact that k_1 is an integer.

This completes the proof of theorem 1.

□

4 Minimal polynomials of totally positive algebraic integers

In this section, first we construct an infinite family of cubics (depending on three parameters) which are minimal polynomials of totally positive algebraic integers. Moreover, we specify intervals (depending on the coefficients) in which the roots lie. We believe that this construction can be generalised to all degrees, but so far we have not been able to do this.

Proposition 2. *Let p be a prime, and let u be a positive integer. Then for all large enough positive integers n, the polynomial $f_n = x^3 - p^u n^2 x^2 + p^{u+1} nx - p$ is irreducible, and all its roots are positive.*

Proof. Fix a prime p and a positive integer u. Let $f_n = x^3 - p^u n^2 x^2 + p^{u+1} nx - p$. The polynomial f_n is irreducible, from Eisenstein's criterion. One has

$$f_n(0) = -p < 0$$

$$f_n(\frac{1}{n}) = \frac{1}{n^3} + p^{u+1} - p^u - p > 0$$

$$f_n(\frac{1}{\sqrt{n}}) = \frac{1}{n\sqrt{n}} - p^u n + p^{u+1}\sqrt{n} - p < 0, \text{ for } n \to \infty$$

$$f_n(n^2) = (1 - p^u)n^6 + p^{u+1} n^3 - p < 0, \text{ for } n \to \infty$$

$$f_n(p^u n^2) = p^{u+1} n (p^u n^2) - p > 0$$

So for all large enough n, f_n has three distinct positive roots, say $x_1(n) \in (0, \frac{1}{n})$, $x_2(n) \in (\frac{1}{n}, \frac{1}{\sqrt{n}})$ and $x_3(n) \in (n^2, p^u n^2)$. In particular, $|x_1(n) - x_2(n)| < \frac{1}{\sqrt{n}} \longrightarrow 0$, as $n \to \infty$.

□

Next, we present a new method for explicitly constructing minimal polynomials of totally positive algebraic integers. Using this method, for any prime $p \geq 3$ we explicitly construct a monic integer polynomial of degree $p-1$ which is irreducible, and has all roots positive, i.e. it is the minimal polynomial of a totally positive algebraic integer. For this construction, we use a theorem of Malo [7] on real rooted polynomials, which we recall below.

Theorem (Malo). *If the roots of both of the real polynomials*

$$f = a_0 + a_1 z + \ldots + a_m z^m \quad \textit{and} \quad g = b_0 + b_1 z + \ldots + b_n z^n$$

are all real and those of the former have the same sign (all positive or all negative), then the roots of the polynomial

$$f \odot g := a_0 b_0 + a_1 b_1 z + \ldots + a_k b_k z^k,$$

where $k = \min(m, n)$, are all real. Moreover, they are all different if $a_0 b_0 \neq 0$ and $n \leq m$.

Proposition 3. *Let $p \geq 3$ be a prime, and let $a_{p-j} := \frac{1}{p}\binom{p}{j-1}^2(p-j+1)$, for all $0 \leq j \leq p-2$. Then the polynomial*

$$T_p = x^{p-1} - \binom{p-1}{1}a_{p-2}x^{p-2} + \binom{p-1}{2}a_{p-3}x^{p-3} - \ldots + (-1)^{p-1}p$$

is the minimal polynomial of a totally positive algebraic integer.

Proof. For $n \geq 1$, let f_n denote the polynomial $(x-1)^n$. Obviously, the roots of the polynomial

$$f_p = (x-1)^p = x^p - \binom{p}{1}x^{p-1} + \binom{p}{2}x^{p-2} - \ldots + (-1)^p$$

are all real and have the same sign.

Using Malo's theorem, we deduce that the roots of the polynomial

$$g_p := f_p \odot f_p = x^p + \binom{p}{1}^2 x^{p-1} + \binom{p}{2}^2 x^{p-2} + \ldots + 1$$

are all real. Therefore, the roots of the derivative of g_p are all real. One can write

$$g_p' = p\big[x^{p-1} + a_{p-2}x^{p-2} + a_{p-3}x^{p-3} + \ldots + a_0\big]$$

where

$$a_{p-(j+1)} := \frac{1}{p}\binom{p}{j}^2(p-j) \in \mathbb{Z}, \text{ since } p|\binom{p}{j}, \text{ for all } 1 \leq j \leq p-1$$

We obtain

$$\frac{g_p'}{p} = x^{p-1} + a_{p-2}x^{p-2} + a_{p-3}x^{p-3} + \ldots + a_0$$

with $p|a_j$, for all $0 \leq j \leq p-2$. Furthermore, $a_0 = p$.

The roots of the polynomial

$$f_{p-1} = (x-1)^{p-1} = x^{p-1} - \binom{p-1}{1}x^{p-2} + \binom{p-1}{2}x^{p-3} - \ldots + (-1)^{p-1}$$

are all real and have the same sign.

It then follows from Malo's theorem that all the roots of the polynomial

$$T_p := \frac{g_p'}{p} \odot f_{p-1} = x^{p-1} - \binom{p-1}{1}a_{p-2}x^{p-2} + \binom{p-1}{2}a_{p-3}x^{p-3} - \ldots + (-1)^{p-1}p \in \mathbb{Z}[X]$$

are real. Lemma (1) now shows that all the roots of the polynomial T_p are positive. Finally, the polynomial T_p is p-Eisenstein, and therefore irreducible, so it is the minimal polynomial of a totally positive algebraic integer.

∎

We conclude the paper by asking the following question, related to the structure of the set $\mathbb{A}_+$: what conditions should be satisfied by $d \geq 2$ disjoint (finite) intervals of positive real numbers, I_j, with $1 \leq j \leq d$, in order to guarantee the existence of a totally positive algebraic integer α, such that its conjugates $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_d$ belong to the specified intervals, e.g. $\alpha_j \in I_j$, for all $1 \leq j \leq d$.

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