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Infinite series containing central binomial coefficients

by

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Abstract

Two classes of infinite series involving central binomial coefficients are evaluated in closed forms, including a solution to the problem proposed recently by Ribeiro (2018).

Key Words: Infinite series, binomial coefficient, the Γ-function, generating function, partial fraction decomposition.
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1 Introduction and motivation

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ stand, respectively, for the sets of real numbers, natural numbers and integers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The aim of this paper is to evaluate, for $m \in \mathbb{N}_0$, $\lambda \in \mathbb{N}$ and $\lambda \in \frac{1}{2} + \mathbb{Z}$, the infinite series $T_m(1, \lambda)$ in closed forms, where $T_m(x, \lambda)$ is defined by

$$T_m(x,\lambda) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n} x^{n+\lambda}}{4^n (n+\lambda)^{m+1}}.$$

This is inspired by the following problem proposed recently by Ribeiro [4], who asks to prove the formula:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n+1)^3} = \frac{\pi^3}{48} + \frac{\pi \ln^2 2}{4}. \tag{1.1}$$

We shall organize the paper as follows. In the next section, we shall first derive a closed form expression for the generating function of $T_m(x, \lambda)$ by iterating integrals, and then establish, by means of partial fractions and the Γ-function expansions, two main theorems for infinite series $T_m(1, \lambda)$ and $T_m(1, \frac{1}{2} + \mu)$ with $m \in \mathbb{N}_0$, $\lambda \in \mathbb{N}$ and $\mu \in \mathbb{Z}$, which express these two infinite series as finite convolutions concerning the Bell polynomials. As applications, the paper will end with Section 3, where several closed formulae are presented for specific triplet $\{m, \lambda, \mu\}$.

2 Main results and proofs

Define the infinite series by

$$f_\lambda(x) := \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{n+\lambda}}{4^n}.$$

According to the binomial series

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x)^n = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{4^n},$$

we have that

$$f_\lambda(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{n+\lambda}}{4^n} = \frac{x^\lambda}{\sqrt{1-x}}.$$

For $\lambda > 0$, it is trivial to see that

$$T_0(x,\lambda) = \int_0^x \frac{f_\lambda(x_0)}{x_0} dx_0 = \sum_{n=0}^{\infty} \frac{\binom{2n}{n} x^{n+\lambda}}{4^n (n+\lambda)},$$

$$T_1(x,\lambda) = \int_0^x \frac{T_0(x_1,\lambda)}{x_1} dx_1 = \int_0^x \frac{dx_1}{x_1} \int_0^{x_1} \frac{f_\lambda(x_0)}{x_0} dx_0,$$

$$T_2(x,\lambda) = \int_0^x \frac{T_1(x_2,\lambda)}{x_2} dx_2 = \int_0^x \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_1}{x_1} \int_0^{x_1} \frac{f_\lambda(x_0)}{x_0} dx_0.$$

Proceeding by induction, we can show that

$$\begin{aligned} T_m(x,\lambda) &= \int_0^x \frac{T_{m-1}(x_m,\lambda)}{x_m} dx_m \\ &= \int_0^x \frac{dx_m}{x_m} \int_0^{x_m} \frac{dx_{m-1}}{x_{m-1}} \cdots \int_0^{x_2} \frac{dx_1}{x_1} \int_0^{x_1} \frac{f_\lambda(x_0)}{x_0} dx_0, \end{aligned}$$

where the integration domain is determined by

$$0 \le x_0 \le x_1 \le x_2 \le \cdots \le x_m \le x.$$

We can reformulate, by reversing the integral order, the multiple integral as

$$T_m(x,\lambda) = \int_0^x \frac{f_\lambda(x_0)}{x_0} dx_0 \int_{x_0}^x \frac{dx_1}{x_1} \cdots \int_{x_{m-2}}^x \frac{dx_{m-1}}{x_{m-1}} \int_{x_{m-1}}^x \frac{dx_m}{x_m}.$$

For the last expression of $T_m(x,\lambda)$, evaluating the right most integral for x_m

$$\int_{x_{m-1}}^x \frac{dx_m}{x_m} = \ln x - \ln x_{m-1},$$

then the penultimate one for x_{m-1}

$$\int_{x_{m-2}}^x \frac{\ln x - \ln x_{m-1}}{x_{m-1}} dx_{m-1} = \frac{(\ln x - \ln x_{m-2})^2}{2!},$$

until the second integral for x_1

$$\int_{x_0}^x \frac{(\ln x - \ln x_1)^{m-1}}{(m-1)!x_1} dx_1 = \frac{(\ln x - \ln x_0)^m}{m!};$$

we reduce finally $T_m(x,\lambda)$ to the following single integral expression:

$$T_m(x,\lambda)=\int_0^x \frac{(\ln x-\ln x_0)^m}{m!}\frac{f_\lambda(x_0)}{x_0}dx_0.$$

Next, consider the generating function

$$\begin{aligned}\sum_{m=0}^{\infty}T_m(x,\lambda)y^m&=\int_0^x\frac{f_\lambda(x_0)}{x_0}\sum_{m=0}^{\infty}\frac{(\ln x-\ln x_0)^m y^m}{m!}dx_0\\&=\int_0^x\frac{f_\lambda(x_0)}{x_0}e^{y(\ln x-\ln x_0)}dx_0\\&=\int_0^x\Big(\frac{x}{x_0}\Big)^y\frac{x_0^{\lambda-1}}{\sqrt{1-x_0}}dx_0.\end{aligned}$$

When $x=1$, by evaluating the beta integral

$$\int_0^1 x_0^{\lambda-y-1}(1-x_0)^{-\frac12}dx_0=B\Big(\frac12,\lambda-y\Big)=\frac{\sqrt{\pi}\,\Gamma(\lambda-y)}{\Gamma(\frac12+\lambda-y)},$$

we obtain the following closed expression in terms of the Γ-function quotient.

Lemma 1. *Let y be an indeterminate and λ a real number subject to $\lambda\notin\mathbb{Z}\backslash\mathbb{N}_0$. Then we have the generating function*

$$\sum_{m=0}^{\infty}T_m(1,\lambda)y^m=\frac{\sqrt{\pi}\,\Gamma(\lambda-y)}{\Gamma(\frac12+\lambda-y)}.$$

Remark. The last formula is, in fact, a formal power series identity for the indeterminates y and $\lambda\in\mathbb{R}$ with $\lambda\notin\mathbb{Z}\backslash\mathbb{N}_0$, where the domain for λ has been extended by analytic continuation from its initial condition $\lambda>0$.

2.1 Evaluation of $T_m(1,\lambda)$ with $\lambda\in\mathbb{N}$

By making use of the recurrence relation of the Γ-function, we can rewrite the generating function as

$$\sum_{m=0}^{\infty}T_m(1,\lambda)y^m=\sqrt{\pi}\frac{(1-y)_{\lambda-1}}{(\frac12-y)_\lambda}\times\frac{\Gamma(1-y)}{\Gamma(\frac12-y)},\tag{2.1}$$

where the shifted factorial is given by

$$(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)}=\begin{cases}1, & n=0;\\ x(x+1)\cdots(x+n-1), & n\in\mathbb{N}.\end{cases}$$

Recall the Γ-function expansions (cf. Rainville [3, §9] and Chu [2])

$$\Gamma(1-y)=\exp\Big\{\sum_{i=1}^{\infty}\frac{\sigma_i}{i}y^i\Big\},\tag{2.2}$$

$$\Gamma(\tfrac12-y)=\sqrt{\pi}\exp\Big\{\sum_{i=1}^{\infty}\frac{\tau_i}{i}y^i\Big\};\tag{2.3}$$

where the sequences $\{\sigma_i, \tau_i\}$ are defined by the Riemann zeta function

$$\sigma_1 = \gamma, \qquad \sigma_m = \zeta(m), \qquad m = 2, 3, \cdots;$$
$$\tau_1 = \gamma + 2\ln 2, \qquad \tau_m = (2^m - 1)\zeta(m), \quad m = 2, 3, \cdots;$$

with γ being the usual Euler–Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{1}{k} - \ln n\right\}.$$

Define further the θ-sequence by

$$\theta_k = \frac{\sigma_k - \tau_k}{k}$$

which can be expressed explicitly as

$$\theta_1 = -2\ln 2 \quad \text{and} \quad \theta_k = \frac{2-2^k}{k}\zeta(k) \quad \text{for} \quad k = 2, 3, \cdots.$$

Then we can expand the Γ-function quotient

$$\sqrt{\pi}\frac{\Gamma(1-y)}{\Gamma(\frac{1}{2}-y)} = \exp\left\{\sum_{k=1}^{\infty}\theta_k y^k\right\} = \sum_{n=0}^{\infty} y^n B_n(\theta_1, \theta_2, \cdots, \theta_n), \tag{2.4}$$

where the Bell polynomials are defined by

$$B_n(x_1, x_2, \cdots, x_n) := \sum_{\substack{k_1+2k_2+\cdots+nk_n=n \\ k_i\geq 0:\ i=1,2,\cdots,n}} \frac{x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{2.5}$$

For the quotient of shifted factorials, we can decompose it by partial fractions

$$\frac{(1-y)_{\lambda-1}}{(\frac{1}{2}-y)_\lambda} = \sum_{i=1}^{\lambda}\frac{\alpha_i}{i-\frac{1}{2}-y},$$

where the coefficients α_i is given explicitly by

$$\alpha_i = (-1)^{\lambda-i}\binom{\lambda-1}{i-1}\binom{i-\frac{3}{2}}{\lambda-1}.$$

By extracting the coefficient of y^k

$$\rho_k(\lambda) := [y^k]\frac{(1-y)_{\lambda-1}}{(\frac{1}{2}-y)_\lambda} = [y^k]\sum_{i=1}^{\lambda}\frac{\alpha_i}{i-\frac{1}{2}}\sum_{k=0}^{\infty}\Big(\frac{y}{i-\frac{1}{2}}\Big)^k$$

we get

$$\rho_k(\lambda) = \sum_{i=1}^{\lambda}\frac{(-1)^{\lambda-i}}{(i-\frac{1}{2})^{k+1}}\binom{\lambda-1}{i-1}\binom{i-\frac{3}{2}}{\lambda-1}. \tag{2.6}$$

Combining (2.1) with (2.4) and (2.6), we find the explicit formula for $T_m(1,\lambda)$.

Theorem 2 ($m \in \mathbb{N}_0$ and $\lambda \in \mathbb{N}$)**.** *Let B_n and $\rho_k(\lambda)$ be defined respectively by* (2.5) *and* (2.6)*. Then there holds the summation formula*

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+\lambda)^{m+1}} = \sum_{k=0}^{m} \rho_k(\lambda) B_{m-k}(\theta_1, \theta_2, \cdots, \theta_{m-k}).$$

2.2 Evaluation of $T_m(1, \frac{1}{2} + \mu)$ with $\mu \in \mathbb{Z}$

Replacing λ by $\mu + \frac{1}{2}$ in Lemma 1 with $\mu \in \mathbb{Z}$, we get

$$\sum_{m=0}^{\infty} T_m(1, \tfrac{1}{2} + \mu) y^m = \sqrt{\pi} \frac{\Gamma(\frac{1}{2} + \mu - y)}{\Gamma(1 + \mu - y)} = \sqrt{\pi} \frac{(\frac{1}{2} - y)_\mu}{(1 - y)_\mu} \times \frac{\Gamma(\frac{1}{2} - y)}{\Gamma(1 - y)}.$$

Analogously, we have the expansion

$$\sqrt{\pi} \frac{\Gamma(\frac{1}{2} - y)}{\Gamma(1 - y)} = \pi \exp\Big\{ -\sum_{k=1}^{\infty} \theta_k y^k \Big\} = \pi \sum_{n=0}^{\infty} y^n B_n(-\theta_1, -\theta_2, \cdots, -\theta_n), \tag{2.7}$$

and the partial fraction decompositions

$$\boxed{\mu \geq 0} \qquad \frac{(\frac{1}{2} - y)_\mu}{(1 - y)_\mu} = 1 + \sum_{i=1}^{\mu} \frac{\beta_i}{i - y},$$

$$\boxed{\mu < 0} \qquad \frac{(\frac{1}{2} - y)_\mu}{(1 - y)_\mu} = \frac{(y)_{-\mu}}{(\frac{1}{2} + y)_{-\mu}} = 1 + \sum_{i=1}^{-\mu} \frac{\gamma_i}{i - \frac{1}{2} + y};$$

where

$$\beta_i = (-1)^{\mu - i - 1} \binom{\mu - 1}{i - 1} \binom{i - \frac{1}{2}}{\mu} \mu,$$

$$\gamma_i = (-1)^{\mu - i} \binom{-\mu - 1}{i - 1} \binom{i - \frac{1}{2}}{-\mu} \mu.$$

Denote by $\delta_{i,j}$ the usual Kronecker delta with $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$. Extracting the coefficients of y^k from the factorial quotients

$$\phi_k(\mu) := [y^k] \frac{(\frac{1}{2} - y)_\mu}{(1 - y)_\mu} = [y^k] \Big\{ 1 + \sum_{i=1}^{\mu} \beta_i \sum_{k=0}^{\infty} \frac{y^k}{i^{k+1}} \Big\} = \delta_{0,k} + \sum_{i=1}^{\mu} \frac{\beta_i}{i^{k+1}},$$

$$\psi_k(\mu) := [y^k] \frac{(y)_{-\mu}}{(\frac{1}{2} + y)_{-\mu}} = [y^k] \Big\{ 1 + \sum_{i=1}^{-\mu} \gamma_i \sum_{k=0}^{\infty} \frac{(-y)^k}{(i - \frac{1}{2})^{k+1}} \Big\} = \delta_{0,k} - \sum_{i=1}^{-\mu} \frac{\gamma_i}{(\frac{1}{2} - i)^{k+1}};$$

we can express them in binomial sums

$$\phi_k(\mu) = \delta_{0,k} - \sum_{i=1}^{\mu} \frac{(-1)^{\mu - i}}{i^k} \binom{\mu}{i} \binom{i - \frac{1}{2}}{\mu}, \tag{2.8}$$

$$\psi_k(\mu) = \delta_{0,k} - \sum_{i=1}^{-\mu} \frac{(-1)^{\mu - i}}{(\frac{1}{2} - i)^k} \binom{-\mu - 1}{i - 1} \binom{i - \frac{3}{2}}{-\mu - 1}. \tag{2.9}$$

Summing up, we have established the following summation formulae.

Theorem 3 ($m \in \mathbb{N}_0$ and $\mu \in \mathbb{Z}$)**.** *Let* B_n*,* $\phi_k(\mu)$ *and* $\psi_k(\mu)$ *be defined respectively by* (2.5), (2.8) *and* (2.9)*. Then the following formulae hold:*

$$\boxed{\mu \geq 0} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+\mu+\frac{1}{2})^{m+1}} = \pi \sum_{k=0}^{m} \phi_k(\mu) B_{m-k}(-\theta_1, -\theta_2, \cdots, -\theta_{m-k}),$$

$$\boxed{\mu < 0} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+\mu+\frac{1}{2})^{m+1}} = \pi \sum_{k=0}^{m} \psi_k(\mu) B_{m-k}(-\theta_1, -\theta_2, \cdots, -\theta_{m-k}).$$

3 Examples

Based on Theorems 2 and 3, we have devised appropriately *Mathematica* commands that can be utilized to evaluate $T_m(1, \lambda)$ and $T_m(1, \frac{1}{2}+\mu)$ for any specific triplet $\{m, \lambda, \mu\}$ with $m, \lambda \in \mathbb{N}$ and $\mu \in \mathbb{Z}$. Several elegant formulae are highlighted below as exemplification.

The informed reader may notice that the series corresponding to $m = 0$ are not recorded because in this case, the series can simply be evaluated by the well–known theorem of Gauss for the ${}_2F_1(1)$-series (cf. Bailey[1, §1.3]). In addition, we point out that most of the identities given below don't seem to have appeared previously, except for the two series labeled by $\mu = 0$ in Example 2 and Example 4, where the former can be located in Zucker [5, Equation 2.16], while the latter confirms the formula (1.1) proposed by Ribeiro [4].

Example 1 ($m = 1$ in Theorem 2: $1 \leq \lambda \leq 5$)**.**

$$\boxed{\lambda = 1} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+1)^2} = 4 - 4\ln 2.$$

$$\boxed{\lambda = 2} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+2)^2} = \frac{4}{9}(5 - 6\ln 2).$$

$$\boxed{\lambda = 3} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+3)^2} = \frac{8}{225}(47 - 60\ln 2).$$

$$\boxed{\lambda = 4} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+4)^2} = \frac{16}{3675}(319 - 420\ln 2).$$

$$\boxed{\lambda = 5} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+5)^2} = \frac{64}{99225}(1879 - 2520\ln 2).$$

Example 2 ($m = 1$ in Theorem 3: $-3 \le \mu \le 3$)**.**

$$\boxed{\mu = -3} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(2n-5)^2} = \frac{4\pi}{15}.$$

$$\boxed{\mu = -2} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(2n-3)^2} = \frac{\pi}{3}.$$

$$\boxed{\mu = -1} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(2n-1)^2} = \frac{\pi}{2}.$$

$$\boxed{\mu = 0} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(2n+1)^2} = \frac{\pi}{2}\ln 2.$$

$$\boxed{\mu = 1} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(2n+3)^2} = \frac{\pi}{8}\Big\{2\ln 2 - 1\Big\}.$$

$$\boxed{\mu = 2} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(2n+5)^2} = \frac{\pi}{64}\{12\ln 2 - 7\}.$$

$$\boxed{\mu = 3} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(2n+7)^2} = \frac{\pi}{384}\Big\{60\ln 2 - 37\Big\}.$$

Example 3 ($m = 2$ in Theorem 2: $1 \le \lambda \le 5$)**.**

$$\boxed{\lambda = 1} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+1)^3} = \frac{1}{3}\Big\{24 - \pi^2 - 24\ln 2 + 12\ln^2 2\Big\}.$$

$$\boxed{\lambda = 2} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+2)^3} = \frac{2}{27}\Big\{56 - 3\pi^2 - 60\ln 2 + 36\ln^2 2\Big\}.$$

$$\boxed{\lambda = 3} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+3)^3} = \frac{8}{3375}\Big\{1307 - 75\pi^2 - 1410\ln 2 + 900\ln^2 2\Big\}.$$

$$\boxed{\lambda = 4} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+4)^3} = \frac{16}{385875}\Big\{62098 - 3675\pi^2 - 66990\ln 2 + 44100\ln^2 2\Big\}.$$

$$\boxed{\lambda = 5} \qquad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(n+5)^3} = \frac{32}{31255875}\Big\{2197133 - 132300\pi^2 - 2367540\ln 2 + 1587600\ln^2 2\Big\}.$$

Example 4 ($m = 2$ in Theorem 3: $-3 \leq \mu \leq 3$)**.**

$$\boxed{\mu = -3} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n-5)^3} = \frac{\pi}{225} \Big\{ 60 \ln 2 - 47 \Big\}.$$

$$\boxed{\mu = -2} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n-3)^3} = \frac{\pi}{18} \Big\{ 6 \ln 2 - 5 \Big\}.$$

$$\boxed{\mu = -1} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n-1)^3} = \frac{\pi}{2} \Big\{ \ln 2 - 1 \Big\}.$$

$$\boxed{\mu = 0} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n+1)^3} = \frac{\pi}{48} \Big\{ \pi^2 + 12 \ln^2 2 \Big\}.$$

$$\boxed{\mu = 1} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n+3)^3} = \frac{\pi}{96} \Big\{ \pi^2 - 6 - 12 \ln 2 + 12 \ln^2 2 \Big\}.$$

$$\boxed{\mu = 2} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n+5)^3} = \frac{\pi}{256} \Big\{ 2\pi^2 - 11 - 28 \ln 2 + 24 \ln^2 2 \Big\}.$$

$$\boxed{\mu = 3} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n+7)^3} = \frac{\pi}{4608} \Big\{ 30\pi^2 - 155 - 444 \ln 2 + 360 \ln^2 2 \Big\}.$$

Example 5 ($m = 3$ in Theorem 2: $1 \leq \lambda \leq 5$)**.**

$$\boxed{\lambda = 1} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (n+1)^4} = \frac{1}{3} \left\{ \begin{array}{c} 48 - 2\pi^2 - 48 \ln 2 + 2\pi^2 \ln 2 \\ +24 \ln^2 2 - 8 \ln^3 2 - 12\zeta(3) \end{array} \right\}.$$

$$\boxed{\lambda = 2} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (n+2)^4} = \frac{2}{81} \left\{ \begin{array}{c} 328 - 15\pi^2 - 336 \ln 2 + 18\pi^2 \ln 2 \\ +180 \ln^2 2 - 72 \ln^3 2 - 108\zeta(3) \end{array} \right\}.$$

$$\boxed{\lambda = 3} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (n+3)^4} = \frac{4}{50625} \left\{ \begin{array}{c} 76684 - 3525\pi^2 - 78420 \ln 2 - 27000\zeta(3) \\ +42300 \ln^2 2 + 4500\pi^2 \ln 2 - 18000 \ln^3 2 \end{array} \right\}.$$

$$\boxed{\lambda = 4} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (n+4)^4} = \frac{8}{40516875} \left\{ \begin{array}{c} 25545482 - 1172325\pi^2 - 26081160 \ln 2 - 9261000\zeta(3) \\ +14067900 \ln^2 2 + 1543500\pi^2 \ln 2 - 6174000 \ln^3 2 \end{array} \right\}.$$

$$\boxed{\lambda = 5} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (n+5)^4} = \frac{32}{9845600625} \left\{ \begin{array}{c} 1357207508 - 62147925\pi^2 - 1384193790 \ln 2 - 500094000\zeta(3) \\ +83349000\pi^2 \ln 2 + 745775100 \ln^2 2 - 333396000 \ln^3 2 \end{array} \right\}.$$

Example 6 ($m = 3$ in Theorem 3: $-3 \le \mu \le 3$)**.**

$$\boxed{\mu = -3} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n-5)^4} = \frac{\pi}{6750} \Big\{ 75\pi^2 + 900 \ln^2 2 - 1410 \ln 2 + 1307 \Big\}.$$

$$\boxed{\mu = -2} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n-3)^4} = \frac{\pi}{216} \Big\{ 3\pi^2 + 36 \ln^2 2 - 60 \ln 2 + 56 \Big\}.$$

$$\boxed{\mu = -1} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n-1)^4} = \frac{\pi}{48} \Big\{ \pi^2 + 12 \ln^2 2 - 24 \ln 2 + 24 \Big\}.$$

$$\boxed{\mu = 0} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n+1)^4} = \frac{\pi}{48} \Big\{ \pi^2 \ln 2 + 4 \ln^3 2 + 6\zeta(3) \Big\}.$$

$$\boxed{\mu = 1} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n+3)^4} = \frac{\pi}{192} \left\{ \begin{array}{c} 2\pi^2 \ln 2 - \pi^2 - 6 + 12\zeta(3) \\ +8 \ln^3 2 - 12 \ln^2 2 - 12 \ln 2 \end{array} \right\}.$$

$$\boxed{\mu = 2} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n+5)^4} = \frac{\pi}{3072} \left\{ \begin{array}{c} 24\pi^2 \ln 2 - 14\pi^2 - 57 + 144\zeta(3) \\ +96 \ln^3 2 - 168 \ln^2 2 - 132 \ln 2 \end{array} \right\}.$$

$$\boxed{\mu = 3} \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n (2n+7)^4} = \frac{\pi}{55296} \left\{ \begin{array}{c} 360\pi^2 \ln 2 - 222\pi^2 - 769 + 2160\zeta(3) \\ +1440 \ln^3 2 - 2664 \ln^2 2 - 1860 \ln 2 \end{array} \right\}.$$

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