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Hamiltonicity in directed Toeplitz graphs $T_n\langle 1, 3; 1, t\rangle$

by
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Abstract

A square matrix of order n is called a Toeplitz matrix if it has constant values along all diagonals parallel to the main diagonal. A directed Toeplitz graph $T_n\langle s_1, \dots, s_k; t_1, \dots, t_l\rangle$ with vertices $1, 2, \dots, n$, where the edge (i, j) occurs if and only if $j - i = s_p$ or $i - j = t_q$ for some $1 \le p \le k$ and $1 \le q \le l$, is a digraph whose adjacency matrix is a Toeplitz matrix. In this paper, we study hamiltonicity in directed Toeplitz graphs $T_n\langle 1, 3; 1, t\rangle$. We obtain new results and improve existing results on $T_n\langle 1, 3; 1, t\rangle$.

Key Words: Adjacency matrix, Toeplitz graph, Hamiltonian graph, length of an edge.
2010 Mathematics Subject Classification: Primary 05C20; Secondary 05C45.

1 Introduction

Let G be a finite vertex-labeled graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. A graph G' is called a *subgraph* of G if $V(G') \subset V(G)$ and $E(G') \subset E(G)$. If $E(G) = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\}$, where $v_i \ne v_j$ for all distinct i, j, then G is called a *cycle*. A cycle minus one edge is called a *path*. A cycle that visits each vertex of a graph H is called hamiltonian, and H is then called a *hamiltonian graph*. We consider here simple graphs, as multiple edges and loops play no role in hamiltonicity. The *adjacency matrix* $A = (a_{ij})_{n\times n}$ of G is the matrix in which $a_{ij} = 1$ if v_i is adjacent to v_j in G, and $a_{ij} = 0$ otherwise. The main diagonal is zero, i.e., $a_{ii} = 0$ as G has no loop.

A *Toeplitz matrix*, named so after Otto Toeplitz (1881-1940), is a square matrix which has constant values along all diagonals parallel to the main diagonal. The main diagonal of a Toeplitz adjacency matrix of order n will be labeled 0. The $n-1$ diagonals above and below the main diagonal will be labeled $1, 2, \dots, n-1$. Let $s_1, s_2, \dots, s_k$ be the upper diagonals containing ones and $t_1, t_2, \dots, t_l$ be the lower diagonals containing ones, such that $0 < s_1 < s_2 < \dots < s_k < n$ and $0 < t_1 < t_2 < \dots < t_l < n$. Then, the corresponding Toeplitz graph will be denoted by $T_n\langle s_1, s_2, \dots, s_k; t_1, t_2, \dots, t_l\rangle$. That is, $T_n\langle s_1, s_2, \dots, s_k; t_1, t_2, \dots, t_l\rangle$ is the graph with vertices $1, 2, \dots, n$, in which the edge (i, j) occurs, if and only if $j - i = s_p$ or $i - j = t_q$ for some p and q ($1 \le p \le k$, $1 \le q \le l$), see an example in Figure 1. The edges of $T_n\langle s_1, s_2, \dots, s_k; t_1, t_2, \dots, t_l\rangle$ are of two types: *increasing edges* (u, v), for which $u < v$, and *decreasing edges* (u, v), where $u > v$. We define the *length* of an edge (u, v) to be $|u - v|$. Note that any increasing edge has length s_p for some p, and any decreasing edge has length t_q for some q. If the Toeplitz matrix is symmetric, then $s_i = t_i$ for all i, so the corresponding Toeplitz graph is undirected and can be denoted as $T_n\langle s_1, \dots, s_k\rangle$. Hamiltonicity results obtained in the undirected case for a Toeplitz graph have a direct impact on the directed case. Hamiltonicity of $T_n\langle s_1, s_2, \ldots, s_k\rangle$ means hamiltonicity of $T_n\langle s_1, \ldots, s_k; t_1, \ldots, t_l\rangle$.

Remark that $T_n\langle s_1, \ldots, s_i; t_1, \ldots, t_j\rangle$ and $T_n\langle t_1, \ldots, t_j; s_1, \ldots, s_i\rangle$ are obtained from each other by reversing the orientation of all edges.

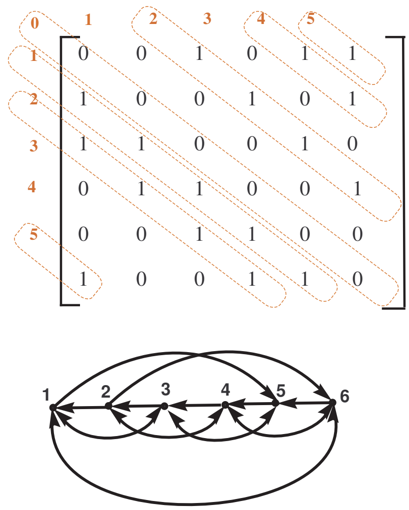

Figure 1: Toeplitz graph $T_6\langle 2,4,5;1,2,5\rangle$

Properties of Toeplitz graphs, such as colourability, planarity, bipartiteness, connectivity, cycle discrepancy, edge irregularity strength, decomposition, labeling, and metric dimension have been studied in [1]-[6], [8]-[12], [14]-[15], and [24] . Hamiltonian properties of Toeplitz graphs were first investigated by R. van Dal et al. in [7] and then studied in [13, 23, 25], while the hamiltonicity in directed Toeplitz graphs was first studied by S. Malik and T. Zamfirescu in [22], by S. Malik in [16], by S. Malik and A.M. Qureshi in [21], and then by S. Malik in [17]-[20].

Suppose that H is a hamiltonian cycle in $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$. The hamiltonian cycle H is determined by two paths $H_{1\to n}$ (from 1 to n) and $H_{n\to 1}$ (from n to 1), i.e., $H = H_{1\to n} \cup H_{n\to 1}$.

In [18], the hamiltonicity of the Toeplitz graphs $T_n\langle 1,3;1,t\rangle$ was investigated. In this paper, we improve upon [18]. In [18], it was shown that: For odd t, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if and only if n is even. For even $t \le 6$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian for all n. For even $t \ge 8$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 0,2,4,6,5,7,9,\ldots,t-3\, mod(t-1)$, or if $n \cong 3\, mod(t-1)$ and $t \cong 0,2\, mod\, 3$. Here we prove that, for even $t \ge 8$ and $t \cong 1\, mod\, 3$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 3\, mod(t-1)$, which together with a result in [18], says that, for even $t \ge 8$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 3\, mod(t-1)$. We also prove that, for even $t \ge 8$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 1\, mod(t-1)$. For even $t \ge 8$, we also discuss the hamiltonicity of $T_n\langle 1,3;1,t\rangle$ for $n \cong 8,10,12,\ldots,t-2\, mod(t-1)$. We see that $T_n\langle 1,3;1,t\rangle$ is hamiltonian for $n \cong s\, mod(t-1)$ if $t \cong s\, mod\, 6$, where $s \in \{8,10,12,\ldots,t-2\}$. The paper will be concluded with a conjecture that, for even $t \ge 8$, $T_n\langle 1,3;1,t\rangle$ is non-hamiltonian for $n \cong 8, 10, 12, \ldots, t-2\,mod(t-1)$ if $t \ncong s\,mod\,6$, which completes the hamiltonicity investigation in Toeplitz graphs $T_n\langle 1,3;1,t\rangle$.

For any vertex a and $b > a$, of the Toeplitz graph $T_n\langle 1,3;1,t\rangle$, we define a path $P_{a\to b}$ in $T_n\langle 1,3;1,t\rangle$ from a to b as $P_{a\to b} = (a, a+3, a+4, a+7, \ldots, a+4k, a+4k+3, \ldots, b)$, where k is a non-negative integer, see Figure 2.

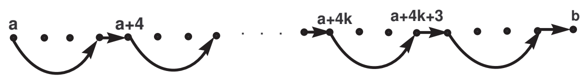

Figure 2: $P_{a\to b}$

2 Toeplitz Graphs $T_n\langle 1,3;1,t\rangle$

Lemma 1. *If $T_n\langle 1,3;1,t\rangle$ has a hamiltonian cycle containing the edge $(n-2,\,n-1)$, then $T_{n+t-1}\langle 1,3;1,t\rangle$ has the same property.*

Proof. Let $T_n\langle 1,3;1,t\rangle$ have a hamiltonian cycle containing the edge $(n-2,\,n-1)$. We transform this hamiltonian cycle to a hamiltonian cycle in $T_{n+t-1}\langle 1,3;1,t\rangle$, by replacing the edge $(n-2,\,n-1)$ with the path $(n-2, n+1, n+2, \ldots, (n+t-1)-2, (n+t-1)-1, n+t-1, n-1)$, see Figure 3. This shows that $T_{n+t-1}\langle 1,3;1,t\rangle$ has the same property. This finishes the proof.□

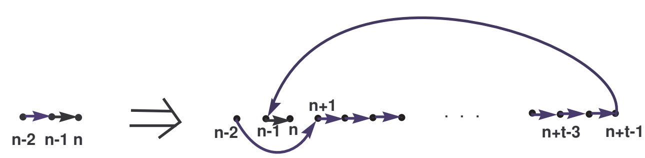

Figure 3:

In [18], it was proved that, for even $t \geq 8$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 5, 7, 9, \ldots, t-3\,mod(t-1)$, and it was also proved that, for even $t \geq 8$ and $t \cong 0, 2\,mod\,3$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 3\,mod(t-1)$. Here we prove that, for even $t \geq 8$ and $t \cong 1\,mod\,3$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 3\,mod(t-1)$. This shows that, for even $t \geq 8$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 3\,mod(t-1)$. We also prove that for even $t \geq 8$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 1\,mod(t-1)$.

Theorem 1. *For even $t \geq 8$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 1\,mod(t-1)$.*

Proof. Let $n \cong 1\,mod(t-1)$, then the smallest possible value for n is t which we can not consider as $n > t$. So the next value for n is $t + (t-1)$, i.e., $n = 2t-1$.

Case 1. If $t \cong 0\,mod\,4$, then a hamiltonian cycle in $T_{n=2t-1}\langle 1,3;1,t\rangle$ is $(P_{1\to n-t-2}, n-t+1, n-t+4, n-t+5, \ldots, n-2, n-1, n, n-t, n-t+3 = t+2, 2, P_{3\to n-t-4}, n-t-1, n-t+2 = t+1, 1)$, see Figure 4.

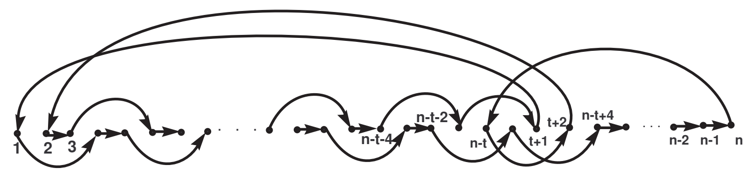

Figure 4: A hamiltonian cycle in $T_{n=2t-1}\langle 1,3;1,t\rangle$, where $t \cong 0\,mod\,4$

Case 2. If $t \cong 2\,mod\,4$, then a hamiltonian cycle in $T_{n=2t-1}\langle 1,3;1,t\rangle$ is $(P_{1\to n-t-8}, n-t-5, n-t-2, n-t+1, n-t+4, n-t+5, \ldots, n-2, n-1, n, n-t, n-t+3 = t+2, 2, P_{3\to n-t-6}, n-t-3, n-t-4, n-t-1, n-t+2 = t+1, 1)$, see Figure 5.

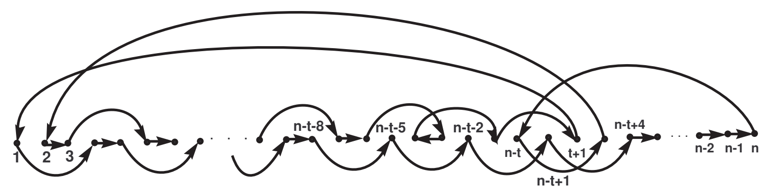

Figure 5: A hamiltonian cycle in $T_{n=2t-1}\langle 1,3;1,t\rangle$, where $t \cong 2\,mod\,4$

Note that $(n-2,\, n-1)$ is an edge in both of the above hamiltonian cycles. Suppose $T_n\langle 1,3;1,t\rangle$, with $n = (2t-1) + r(t-1)$, has a hamiltonian cycle containing the edge $(n-2,\, n-1)$, for some non-negative integer r. By Lemma 1, $T_{n+t-1}\langle 1,3;1,t\rangle$ enjoys the same property. This finishes the proof.□

Theorem 2. *For even $t \geq 8$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 3\,mod(t-1)$.*

Proof. By Theorem 6 in [18], for even $t \geq 8$ and $t \cong 0, 2\,mod\,3$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 3\,mod(t-1)$. Here we show that, for even $t \geq 8$ and $t \cong 1\,mod\,3$, it is also hamiltonian if $n \cong 3\,mod(t-1)$.

Let $t \geq 8$ (even) and $t \cong 1\,mod\,3$. Assume $n \cong 3\,mod(t-1)$; then the smallest possible value for n is $t+2$, which is an even number.

Case 1. If $n \cong 0\,mod\,12$, then a hamiltonian cycle in $T_{n=t+2}\langle 1,3;1,t\rangle$ is $(P_{1\to n-3}, n, n-t =$

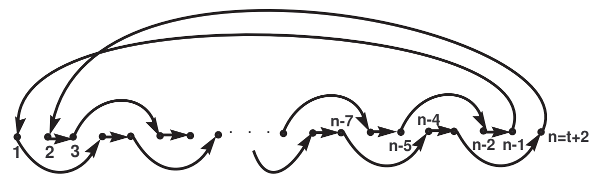

Figure 6: A hamiltonian cycle in $T_{n=t+2}\langle 1,3;1,t\rangle$; $n \cong 0 \, mod \, 12$

$2, P_{3\to n-5}, n-2, n-1, n-1-t=1)$, see Figure 6.
Case 2. If $n \not\cong 0 \, mod \, 12$, then a hamiltonian cycle in $T_{n=t+2}\langle 1,3;1,t\rangle$ is $(P_{1\to n-9}, n-6, n-3, n, n-t=2, P_{3\to n-7}, n-4, n-5, n-2, n-1, n-1-t=1)$, see Figure 7.

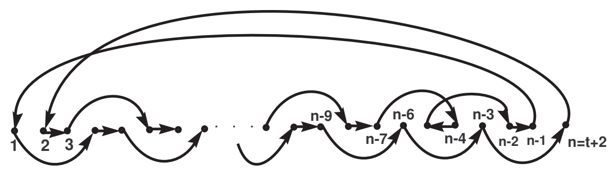

Figure 7: A hamiltonian cycle in $T_{n=t+2}\langle 1,3;1,t\rangle$; $n \not\cong 0 \, mod \, 12$

Note that $(n-2,\, n-1)$ is an edge in both of the above hamiltonian cycles. Suppose $T_n\langle 1,3;1,t\rangle$, with $n=(t+2)+r(t-1)$, has a hamiltonian cycle containing the edge $(n-2,\, n-1)$, for some non-negative integer r. By Lemma 1, $T_{n+t-1}\langle 1,3;1,t\rangle$ enjoys the same property. This finishes the proof.□

In [18], it was proved that, for even $t \geq 8$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $n \cong 0,2,4,6 \, mod(t-1)$. Now, for even $t \geq 8$, we will discuss the hamiltonicity of $T_n\langle 1,3;1,t\rangle$, if $n \cong 8,10,12,\ldots,t-2 \, mod(t-1)$. Clearly, here $t \geq 10$.

Theorem 3. *For even $t \geq 10$, and $n \cong s \, mod(t-1)$ where $s \in \{8,10,12,\ldots,t-2\}$, $T_n\langle 1,3;1,t\rangle$ is hamiltonian if $t-s \cong 0 \, mod \, 6$ or ($t-s \cong 4 \, mod \, 6$ and $s \neq 8$) or ($t-s \cong 2 \, mod \, 6$ and $n \neq s+t-1$).*

Proof. For even $t \geq 10$, let $n \cong s \, mod(t-1)$, where $s \in \{8,10,12,\ldots,t-2\}$. The smallest possible value for n is $s+t-1$, i.e., $n=s+t-1$, which is an odd number.
Case 1. Let $t-s \cong 0 \, mod \, 6$.
(i) If $s \cong 0 \, mod \, 4$, then a hamiltonian cycle in $T_{n=s+t-1}\langle 1,3;1,t\rangle$ is $(P_{1\to n-t-2}, n-t+1, n-t+4, \ldots, t+3, t+4, \ldots, n-2, n-1, n, n-t, n-t+3, \ldots, t+2, 2, P_{3\to n-t-4}, n-t-1, n-t+2, \ldots, t+1, 1)$, see Figure 8.
(ii) If $s \cong 2 \, mod \, 4$, then a hamiltonian cycle in $T_{n=s+t-1}\langle 1,3;1,t\rangle$ is $(P_{1\to n-t-8}, n-t-$

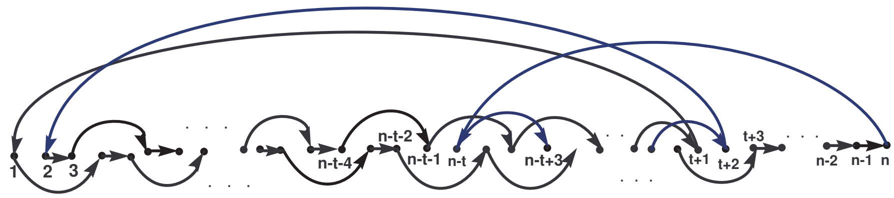

Figure 8: A hamiltonian cycle in $T_{n=s+t-1}\langle 1,3;1,t\rangle$, where $s \cong 0\, mod\, 4$

$5, n-t-2, \ldots, t+3, t+4, \ldots, n-2, n-1, n, n-t, n-t+3, \ldots, t+2, 2, P_{3\to n-t-6}, n-t-3, n-t-4, n-t-1, n-t+2, \ldots, t+1, 1)$, see Figure 9.

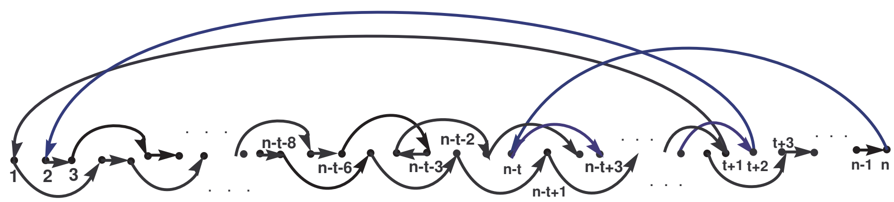

Figure 9: A hamiltonian cycle in $T_{n=s+t-1}\langle 1,3;1,t\rangle$, where $s \cong 2\, mod\, 4$

Note that $(n-2,\, n-1)$ is an edge in both of the hamiltonian cycles in Case 1. Suppose $T_n\langle 1,3;1,t\rangle$, with $n=(s+t-1)+r(t-1)$, has a hamiltonian cycle containing the edge $(n-2,\, n-1)$, for some non-negative integer r. By Lemma 1, $T_{n+t-1}\langle 1,3;1,t\rangle$ enjoys the same property.

Case 2. Let $t-s \cong 4\, mod\, 6$ and $s \neq 8$.
(i) If $s \cong 0\, mod\, 4$ and $s \neq 8$, then a hamiltonian cycle in $T_{n=s+t-1}\langle 1,3;1,t\rangle$ is $(P_{1\to s-11}, s-8, s-5, \ldots, t+3, t+4, \ldots, s+t-4, s+t-1, s+t-2, s+t-3, s-3, s, \ldots, t+2, 2, P_{3\to s-9}, s-6, s-7, s-4, \ldots, t+1, 1)$, see Figure 10.
(ii) If $s \cong 2\, mod\, 4$, then a hamiltonian cycle in $T_{n=s+t-1}\langle 1,3;1,t\rangle$ is $(P_{1\to s-5}, s-2, s+1, \ldots, t+3, t+4, \ldots, s+t-4, s+t-1, s+t-2, s+t-3, s-3, s, \ldots, t+2, 2, P_{3\to s-7}, s-4, s-1, \ldots, t+1, 1)$, see Figure 11.

Since $(s+t-1, s+t-2)$ is an edge in both of the hamiltonian cycles in Case 2, in $T_{s+t-1}\langle 1,3;1,t\rangle$, we transform each of this hamiltonian cycle to a hamiltonian cycle in $T_{(s+t-1)+t-1=s+2t-2}\langle 1,3;1,t\rangle$, by replacing the edge $(s+t-1, s+t-2)$ with the path $(s+t-1, s+t, \ldots, s+2t-4, s+2t-3, s+2t-2, s+t-2)$, which contains the edge $(s+2t-4, s+2t-3)$, see Figure 12. Suppose $T_n\langle 1,3;1,t\rangle$, with $n=(s+t-1)+r(t-1)$, has a hamiltonian cycle containing the edge $(n-2,\, n-1)$, for some non-negative integer

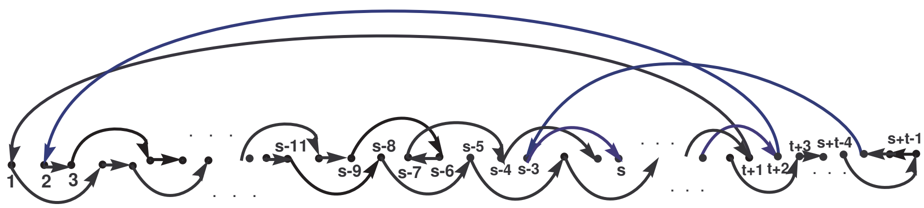

Figure 10: A hamiltonian cycle in $T_{s+t-1}\langle 1,3;1,t\rangle$, where $s \cong 0\, mod\, 4$, $s \neq 8$

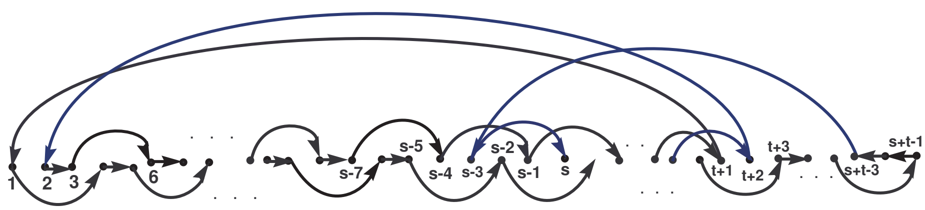

Figure 11: A hamiltonian cycle in $T_{s+t-1}\langle 1,3;1,t\rangle$, where $s \cong 2\, mod\, 4$

r. By Lemma 1, $T_{n+t-1}\langle 1,3;1,t\rangle$ enjoys the same property.

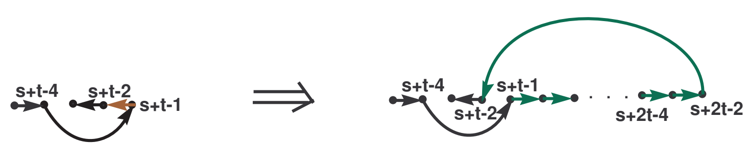

Figure 12: Transformation of the edge $(s+t-1, s+t-2)$ to the path $(s+t-1, s+t, \ldots, s+2t-4, s+2t-3, s+2t-2, s+t-2)$

Case 3. Let $t-s \cong 2\, mod\, 6$ and $n \neq s+t-1$.

In this case, the smallest possible value for n different from $s+t-1$, will be $(s+t-1)+(t-1)$, i.e., $n = s+2t-2$, which is an even number.

(i) If $s \cong 0\, mod\, 4$.

For $s = 8$, a hamiltonian cycle in $T_{s+2t-2=2t+6}\langle 1,3;1,t\rangle$ is $(2t+6, 2t+5, 2t+4, t+4, t+3, 3, 2, 1, 4, 5, \ldots, t+2, t+5, t+6, \ldots, 2t+3, 2t+6)$, see Figure 13.

For $s \neq 8$, a hamiltonian cycle in $T_{s+2t-2}\langle 1,3;1,t\rangle$ is $(P_{1\rightarrow s-7}, s-3, s, \ldots, t+3, t+$

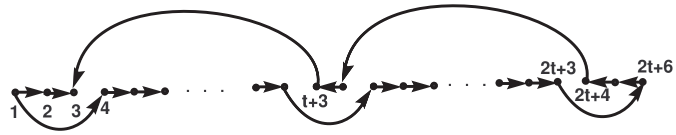

Figure 13: A hamiltonian cycle in $T_{2t+6}\langle 1,3;1,t\rangle$

$4,\dots,s+t-6,s+t-3,s+t-2,\dots,s+2t-5,s+2t-2,s+2t-3,s+2t-4,s+t-4,s+t-5,s-5,s-2,\dots,t+2,2,P_{3\to s-9},s-6,s-3,\dots,t+1,1)$, see Figure 14.

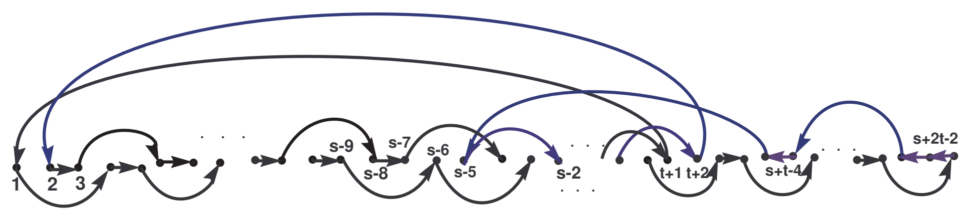

Figure 14: A hamiltonian cycle in $T_{s+2t-2}\langle 1,3;1,t\rangle$, where $s\cong 0\,mod\,4$ and $s\neq 8$

(ii) If $s\cong 2\,mod\,4$.

For $s\neq 10$, a hamiltonian cycle in $T_{s+2t-2}\langle 1,3;1,t\rangle$ is $(P_{1\to s-13},s-10,s-7,\dots,t+3,t+4,\dots,s+t-6,s+t-3,s+t-2,\dots,s+2t-5,s+2t-2,s+2t-3,s+2t-4,s+t-4,s+t-5,s-5,s-2,\dots,t+2,2,P_{3\to s-11},s-8,s-9,s-6,s-3,\dots,t+1,1)$, see Figure 15.

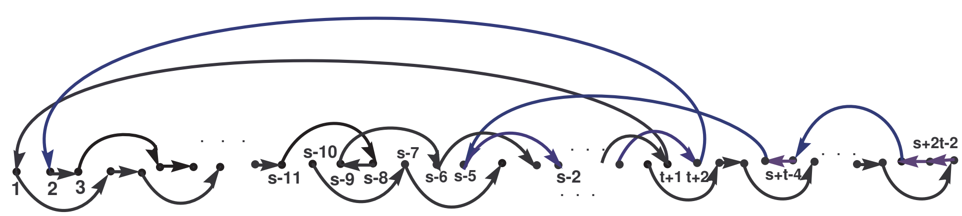

Figure 15: A hamiltonian cycle in $T_{s+2t-2}\langle 1,3;1,t\rangle$, where $s\cong 2\,mod\,4$ and $s\neq 8$

For $s=10$. If $t\cong 0\,mod\,4$, then a hamiltonian cycle in $T_{s+2t-2=2t+8}\langle 1,3;1,t\rangle$ is $(1,2,5,8,\dots,t+2,P_{t+5\to 2t+1},2t+4,2t+5,2t+8,2t+7,2t+6,t+6,P_{t+7\to 2t+3},t+3,t+$

$4, 4, P_{3\to t-5}, t-2, t-3, t, t+1, 1)$, see Figure 16. And if $t \cong 2\,mod\,4$, then a hamiltonian cycle in $T_{2t+8}\langle 1, 3; 1, t\rangle$ is $(1, 2, P_{5\to t-1}, t+2, P_{t+5\to 2t-5}, 2t-2, 2t+1, 2t+4, 2t+5, 2t+8, 2t+7, 2t+6, t+6, P_{t+7\to 2t-3}, 2t, 2t-1, 2t+2, 2t+3, t+3, t+4, 4, P_{3\to t+1}, 1)$, see Figure 17.

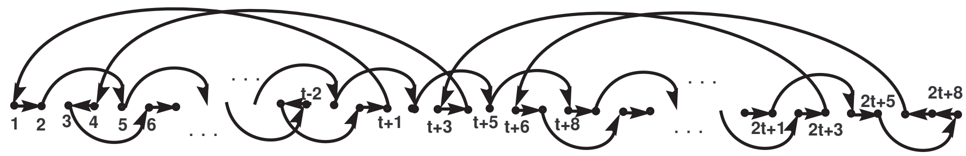

Figure 16: A hamiltonian cycle in $T_{2t+8}\langle 1, 3; 1, t\rangle$, where $t \cong 0\,mod\,4$

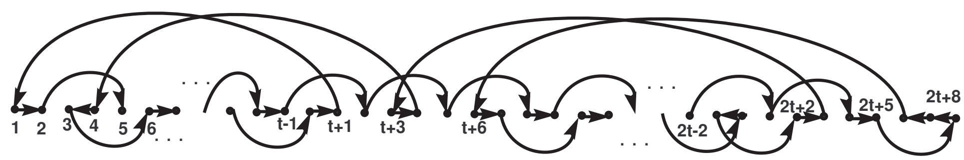

Figure 17: A hamiltonian cycle in $T_{2t+8}\langle 1, 3; 1, t\rangle$, where $t \cong 2\,mod\,4$

Since $(s+2t-2, s+2t-3)$ is an edge in all the hamiltonian cycles, in Case 3, in $T_{s+2t-2}\langle 1, 3; 1, t\rangle$, we transform each of this hamiltonian cycle to a hamiltonian cycle in $T_{(s+2t-2)+t-1=s+3t-3}\langle 1, 3; 1, t\rangle$, by replacing the edge $(s+2t-2, s+2t-3)$ with the path $(s+2t-2, s+2t-1, \ldots, s+3t-5, s+3t-4, s+3t-3, s+2t-3)$, which contains the edge $(s+3t-4, s+3t-3)$. Suppose $T_n\langle 1, 3; 1, t\rangle$, with $n = (s+3t-3) + r(t-1)$, has a hamiltonian cycle containing the edge $(n-2, n-1)$, for some non-negative integer r. By Lemma 1, $T_{n+t-1}\langle 1, 3; 1, t\rangle$ enjoys the same property.

This finishes the proof.□

In Theorem 3, it was proved that, for even $t \geq 10$, and $n \cong s\,mod(t-1)$ where $s \in \{8, 10, 12, \ldots, t-2\}$, $T_n\langle 1, 3; 1, t\rangle$ is hamiltonian if $t - s \cong 4\,mod\,6$ and $s \neq 8$. Here we will discuss the case with $s = 8$.

Theorem 4. *For even $t \geq 10$, $n \cong 8\,mod(t-1)$, and $t - 8 \cong 4\,mod\,6$. $T_n\langle 1, 3; 1, t\rangle$ is hamiltonian for all n different from $t+7$.*

Proof. For even $t \geq 10$, let $n \cong 8\,mod(t-1)$ and $t - 8 \cong 4\,mod\,6 \Rightarrow t \cong 0\,mod\,6$.

Assume $n \neq t+7$. Then the smallest possible value for n is $t + 7 + (t-1)$, i.e., $n = 2t+6$. A hamiltonian cycle in $T_{2t+6}\langle 1, 3; 1, t\rangle$ is $(2t+6, 2t+5, 2t+4, t+4, t+3, 3, 2, 1, 4, 5, \ldots, t+2, t+5, t+6, \ldots, 2t+3, 2t+6)$. Since $(2t+6, 2t+5)$ is an edge in this hamiltonian cycle in $T_{2t+6}\langle 1, 3; 1, t\rangle$, we transform this hamiltonian cycle to a hamiltonian cycle in $T_{n=(2t+6)+t-1=3t+5}\langle 1, 3; 1, t\rangle$, by replacing the edge $(2t+6, t+5)$ with the path $(2t+6, 2t+7, \ldots, 3t+3, 3t+4, n = 3t+5, 2t+5)$, which contains the edge $(n-2, n-1) =$

$(3t+3,\ 3t+4)$, see Figure 18. Suppose $T_n\langle 1,3;1,t\rangle$, with $n=(3t+5)+r(t-1)$, has a hamiltonian cycle containing the edge $(n-2,\ n-1)$, for some non-negative integer r. By Lemma 1, $T_{n+t-1}\langle 1,3;1,t\rangle$ enjoys the same property. This finishes the proof.□

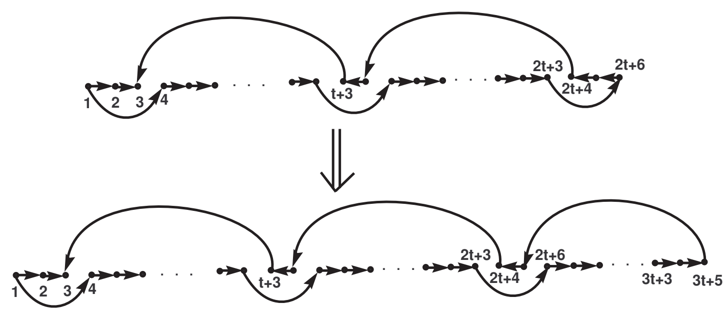

Figure 18: A hamiltonian cycle in $T_{2t+6}\langle 1,3;1,t\rangle$ and then its transformation to a hamiltonian cycle in $T_{3t+5}\langle 1,3;1,t\rangle$

Conjectures:

1. Let $t\geq 10$ and $t\cong 0\,mod\,6$. Then $T_{t+7}\langle 1,3;1,t\rangle$ is non-hamiltonian.
2. Let $t\geq 10$ and $t-s\cong 2\,mod\,6$, where $s\in\{8,10,12,\ldots,t-2\}$. Then $T_n\langle 1,3;1,t\rangle$ is non-hamiltonian if $n=s+t-1$.

Concluding Remark: An affirmative resolution of the conjecture above for $T_n\langle 1,3;1,t\rangle$ would complete the study of hamiltonicity of $T_n\langle 1,3;1,t\rangle$.

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