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On value distribution of L-functions sharing finite sets with meromorphic functions

by

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Abstract

In [17] the authors showed the existence of subsets $S \subset \mathbb{C}$ with 7 elements such that if a non-constant meromorphic function f, having finitely many poles, and an L-function in the Selberg class share S CM, then $f = L$. In this paper, we present a class of such subsets S with 5 elements. Moreover, when avoiding the hypothesis of having finitely many poles, we show a class of such subsets S with 9 elements.

Key Words: L-function, Selberg class, meromorphic function, unique range set.
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1 Introduction

In the last few years, the value distribution and uniqueness of L-functions has been studied extensively. Let us recall some basic notations and known results on the value distribution of L-functions.
An $L-$function in the Selberg class is defined to be a Dirichlet series

$$L(s) = \sum_{n=0}^{\infty} \frac{a(n)}{n^s},$$

satisfying the following axioms:

(i) *Ramanujan hypothesis:* for all positive ϵ, $a(n) \ll n^{\epsilon}$;

(ii) *Analytic continuation:* there exists a non-negative integer m such that $(s-1)^m L(s)$ is an entire function of finite order;

(iii) *Functional equation:* there are positive real numbers Q, λ_i, and there exists a positive integer K, and there are complex numbers μ_i,ω with $Re\mu_i \geq 0$ and $|\omega| = 1$ such that $\Lambda_L(s) = \omega\overline{\Lambda_L(1-\overline{s})}$, where $\Lambda_L(s) := L(s)Q^s \prod_{i=1}^{K} \Gamma(\lambda_i s + \mu_i)$.

(iv) *Euler product hypothesis:* $L(s) = \prod_p L_p(s)$, where

$$L_p(s) = \exp\big(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\big),$$

with coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$, where the product is taken over all prime numbers p.

Note that the Riemann Zeta function is an L-function in the Selberg class.

On the other hand, an L-function can be analytically continued as a meromorphic function in the complex plane $\mathbb{C}$. Therefore, for the problem of value distribution of L-functions sharing finite sets with meromorphic functions, one of the main tools is the Nevanlinna theory on the value distribution of meromorphic functions.

In this paper, by a meromorphic function we mean a meromorphic function in the complex plane $\mathbb{C}$.

Let f be a meromorphic function in $\mathbb{C}$, $a \in \mathbb{C} \cup \infty$. Denote by $E_f(a)$ the set of a-points of f counted with its multiplicities.

For a nonempty subset $S \subset \mathbb{C} \cup \infty$, define

$$E_f(S) = \cup_{a \in S} E_f(a).$$

Two meromorphic functions f, g are said to *share* S, *counting multiplicities* (share S CM), if $E_f(S) = E_g(S)$.

In 1976 F. Gross ([4]) proved that there exist three finite sets $S_j, (j = 1, 2, 3)$, such that any two entire functions f and g, satisfying $E_f(S_j) = E_g(S_j)$, $j = 1, 2, 3$, must be identical. In the same paper, F. Gross posed the following question:

Question A. *Can one find two (or possible even one) finite sets* $S_j, (j = 1, 2)$ *such that any two entire functions* f, g, *satisfying* $E_f(S_j) = E_g(S_j)$, $(j = 1, 2)$, *must be identical?*

H. X. Yi ([14]-[16]) first gave an affirmative answer to Question A. He showed that the set $\{z \in \mathbb{C} : z^n(z^p + a) + b = 0\}$ with $a, b \neq 0$, $n \geq p + 9$, $p \geq 2$, $(n, p) = 1$ is a unique range set for meromorphic functions.

In the last few years, the value distribution and uniqueness of L-functions has been studied extensively. J. Steuding ([11]) showed that an L-function is uniquely defined by its preimage of a single point $c \in \mathbb{C}$, counted with multiplicity:

Theorem A ([11]). *If two* $L-$*functions with* $a(1) = 1$ *share a complex value* $c \neq \infty$ *CM, then they are identically equal.*

P. C. Hu and B. Q. Li ([5]) pointed out that one should add the condition $c \neq 1$.

In 2004, J. Steuding ([10], Theorem 4) showed that, two L-functions, satisfying some additional conditions, coincide if they share two values IM. In 2011 B. Q. Li ([8]) was able to remove these conditions.

Theorem B. *Let* L_1 *and* L_2 *be two* L*-functions, satisfying the same functional equation with* $a(1) = 1$, *and let* $a_1, a_2 \in \mathbb{C}$ *be two distinct values. If* $L_1^{-1}(a_j) = L_2^{-1}(a_j), j = 1, 2$, *then* $L_1 \equiv L_2$.

In 2015 P. C. Hu and A. D. Wu ([6] obtained uniqueness theorems for L-functions, sharing a finite subset of $\mathbb{C} \setminus \{1\}$, counted with multiplicities.

Theorem C ([6]). *Fix a positive integer* n *and take a subset* $S = \{c_1, ..., c_n\} \subset \mathbb{C} \setminus \{1\}$ *of distinct complex numbers, satisfying*

$$n + (n-1)\sigma_1(c_1, ..., c_n) + \cdots + 2\sigma_{n-2}(c_1, ..., c_n) + \sigma_{n-1}(c_1, ..., c_n) \neq 0,$$

where σ_j *are the elementary symmetric polynomials, defined by*

$$\sigma_j(c_1, ..., c_n) = (-1)^j \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} c_{i_1} c_{i_2} \cdots c_{i_j}, j = 1, ..., n-1.$$

If two L*-functions with* $a(1) = 1$ *share* S *CM, then they are identically equal.*

In 2017 Q. Q. Yuan, X. M. Li, and H. X. Yi [17] posed the following question:

Question B. *What can be said about the relationship between a meromorphic function f and an L-function L, if $E_f(S) = E_L(S)$?*

In this direction, they obtained the following result:

Theorem D.[17] *Let f be a non-constant meromorphic function having finitely many poles, and let L be an L-function. Let $P(z) = z^n + az^m + b$, where m, n are positive integers, satisfying $n > 2m + 4$, and (m,n)=1, $a, b \in \mathbb{C}$ are nonzero constants. Denote by S the zero set of P. If f and L share S CM, then $f = L$.*

From Theorem D it follows the existence of a class of subsets S with 7 elements, which are zero sets of Yi's polynomials, such that if $E_f(S) = E_L(S)$, then $f = L$, where f is a non-constant meromorphic function having finitely many poles, L is an L-function.

In this paper we show the existence of a class of subsets S with 9 elements, such that for a non-constant meromorphic function f and an L-function L, if $E_f(S) = E_L(S)$, then $f = L$.

For the case of non-constant meromorphic functions having finitely many poles, we present a class of subsets $S \subset \mathbb{C}$ with 5 elements having the above property.

The obtained results improve the recent results due to Q.Q. Yuan, X.M. Li, and H.X. Yi [17], where the cardinalities of subsets S should be at least 7.

Note that the subsets S considered in this paper are not zero sets of Yi's polynomials, as in [17], and our method uses the Second Fundamental Theorem of Nevanlinna theory for moving targets.

Now let us describe main results of the paper.

Let $n, m \in \mathbb{N}^*$, $a \in \mathbb{C}$, $a \neq 0$.

Consider polynomials $P(z)$ of the following form:

$$P(z) = (n+m+1)\Big(\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i} z^{n+m+1-i} a^i\Big) + 1 = Q(z) + 1,$$

where

$$Q(z) = (n+m+1)\Big(\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i} z^{n+m+1-i} a^i\Big). \tag{1.1}$$

Suppose that

$$(n+m+1)\Big(\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i}\Big) a^{n+m+1} \neq -1, -2. \tag{1.2}$$

Then $P'(z) = (n+m+1)z^n(z-a)^m$, and P' has a zero at 0 of order n, a zero at a of order m. Note that, from the condition (1.2) it follows that P has only simple zeros.

We shall prove the following theorems.

Theorem 1. *Let f be a non-constant meromorphic function, L be an L-function, $P(z)$ be defined as in (1.1) with conditions (1.2), $S = \{z | P(z) = 0\}$. If $n \geq 2, m \geq 2, n + m \geq 8$, then the condition $E_f(S) = E_L(S)$ implies $f = L$.*

Theorem 2. *Let f be a non-constant meromorphic function, having finitely many poles, L be an L-function, $P(z)$ be defined as in (1.1) with conditions (1.2), $S = \{z | P(z) = 0\}$. If $n - m \geq 2$, then the condition $E_f(S) = E_L(S)$ implies $f = L$.*

Remark. i) From Theorem 1 it follows that there exists a class of subsets S with 9 elements such that, if $E_f(S) = E_L(S)$, then $f = L$, where f is a non-constant meromorphic function, L is an L-function.

ii) In Theorem 2, take $m = 1, n = 3$, then $\deg P = 5$, and we have a class of subsets S with 5 elements such that if $E_f(S) = E_L(S)$, then $f = L$, where f is a non-constant meromorphic function having finitely many poles.

2 Preliminaries

We recall some basic notions and known results on value distribution of meromorphic functions and L-functions. We assume that the reader is familiar with the notations in the Nevanlinna theory (see [3]).

Let $f(z)$ be a meromorphic function. The number of poles of $f(z)$ in the disc $\{|z| \le r\}$ will be denoted by $n(r, f)$, and we assume that a pole of order m contributes m to the value of $n(r, f)$. Then the *counting function* is defined as

$$N(r, f) = \int_o^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

and $\overline{N}(r, f)$ is defined in the same way with $n(t, f)$ being replaced by the number of poles of f (ignoring multiplicities) in $\{|z| < t\}$.

The *approximating function* is defined as

$$m(r, f) = \frac{1}{2\pi} \int_o^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \quad \log^+ |x| = \max(0, \log |x|).$$

The *characteristic function* is defined as

$$T(r, f) = N(r, f) + m(r, f).$$

Then we have two Fundamental Theorems of the Nevanlinna theory:

First Fundamental Theorem. *Let $f(z)$ be a non-constant meromorphic function. Then*

$$T(r, f) = T(r, \frac{1}{f}) + O(1).$$

Second Fundamental Theorem. *Let $f(z)$ be a non-constant meromorphic function, let $a_1, a_2, \cdots, a_q$ be distinct values in $\mathbb{C}$. Then we have*

$$(q-1)T(r, f) \le \overline{N}(r, f) + \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f - a_i}) - N_0(r, \frac{1}{f'}) + S(r, f),$$

where $N_0(r, \frac{1}{f'})$ is the counting function of those zeros of f', which are not zeros of function $(f - a_1)...(f - a_q)$.

Recall that $S(r, f)$ denotes a quantity satisfying $S(r, f) = O\{\log(rT(r, f))\}$ for all r outside possibly a set of finite Lebesgue measure.

A meromorphic function f is said to be a *small function* with respect to a meromorphic function g if $T(r, f) = o(T(r, g))$ when $r \to +\infty$. For the convenience of the reader, we recall Second Fundamental Theorem of the Nevanlinna theory for moving targets (see, for example, [9]).

Lemma 1. (Second Fundamental Theorem for moving targets) *Let f be a non-constant meromorphic function and let $a_1, a_2, \cdots, a_q$ be distinct meromorphic functions on $\mathbb{C} \cup \{\infty\}$. Assume that a_i are small functions with respect to f for all $i = 1, \cdots, q$. Then, the inequality*

$$(q-2)T(r,f) \leq \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f-a_i}) + S(r,f)$$

holds for all r, except for a set of finite Lebesgue measure.

Lemma 2. ([1]) $\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i}$ *is not an integer, where $n, m \geq 1$ are integers.*

In ([1], Lemma 2.2), Banerjee proved the Lemma for $n, m \geq 3$, but it is clear that the Lemma is valid for $n, m \geq 1$.

For a discrete subset $S = \{a_1, a_2, ..., a_q\} \subset \mathbb{C}$, we consider its generated polynomial of the following form

$$R(z) = (z-a_1)(z-a_2)...(z-a_q). \tag{1.3}$$

Assume that the derivative of $R(z)$ has mutually distinct k zeros $d_1, d_2, \cdots, d_k$ with multiplicities $q_1, q_2, \cdots, q_k$, respectively. We often consider polynomials satisfying the following condition, introduced by Fujimoto ([2]):

$$R(d_i) \neq R(d_j), 1 \leq i < j \leq q. \tag{1.4}$$

A polynomial $P(z)$ is called a *uniqueness polynomial for meromorphic (entire) functions* if for arbitrary two non-constant meromorphic (entire) functions f and g, the condition $P(f) = P(g)$ implies $f = g$.

Lemma 3. ([2]) *Let $R(z)$ be a polynomial of the form (1.3), satisfying the condition (1.4). Then $R(z)$ is a uniqueness polynomial if and only if*

$$\sum_{1 \leq l < j \leq k} q_l q_j > \sum_{i=1}^{k} q_l.$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $\max\{q_1, q_2, q_3\} \geq 2$*, or when* $k = 2$, $\min\{q_1, q_2\} \geq 2$*, and* $q_1 + q_2 \geq 5$.

Lemma 4. ([3]). *Let f be an entire function of finite order ρ. If f has no zeros, then $f(z) = e^{h(z)}$, where $h(z)$ is a polynomial of degree less than ρ.*

Lemma 5. ([3]) *For any non-constant meromorphic function f, we have*
i) $T(r, f^{(k)}) \leq (k+1)T(r,f) + S(r,f)$;
ii) $S(r, f^{(k)}) = S(r,f)$.

Now let k be a positive integer. As usually, denote by $\overline{N}_{(k}(r,f)$ the counting function of the poles of order $\geq k$ of f, where each pole is counted only once, and by $\overline{N}(r, \frac{1}{f'}; f \neq 0)$ the counting function of the zeros z of f' satisfying $f(z) \neq 0$, where each zero is counted only once. We also denote by $N_{1)}(r,f)$ the counting function of the simple poles of f. If z is a zero of f, denote by $\nu_f(z)$ its multiplicity.

Lemma 6. *Let f, g be two non-constant meromorphic functions. Set*

$$F = \frac{1}{f},\ G = \frac{1}{g},\ H = \frac{F''}{F'} - \frac{G''}{G'}.$$

Suppose $H \not\equiv 0$, and $E_f(0) = E_g(0)$. Then

$$N(r, H) \le \overline{N}_{(2}(r, f) + \overline{N}_{(2}(r, g) + \overline{N}(r, \frac{1}{f'}; f \neq 0) + \overline{N}(r, \frac{1}{g'}; g \neq 0).$$

For the proof, see [7] (Lemma 2.3). Moreover, from the proof of Lemma 2.3 in [7] it follows that if a is a common simple zero of f and g, then $H(a) = 0$.

We shall use the following lemma on L-functions.

Lemma 7. **([11]**. *Let L be a non-constant L-function. Then*

i) $T(r, L) = \frac{d_L}{\pi} r \log r + O(r)$, where $d_L = 2\sum_{i=1}^{K} \lambda_i$ is the degree of L, and K, λ_i are respectively the positive integer and positive real number in the functional equation of the definition of L-functions;

ii) $N(r, \frac{1}{L}) = \frac{d_L}{\pi} r \log r + O(r)$, $N(r, L) = S(r, L)$.

From this Lemma it follows that $N(r, L) = S(r, L) = O(\log r)$.

3 Proof of main results

3.1 More Lemmas

First we establish some lemmas.

Lemma 8. *Let f be a non-constant meromorphic function. Then*

$$\overline{N}(r, \frac{1}{f}) - \frac{1}{2} N_{1)}(r, \frac{1}{f}) \le \frac{1}{2} N(r, \frac{1}{f}).$$

Proof. We have

$$\begin{aligned}
\overline{N}(r, \frac{1}{f}) - \frac{1}{2} N_{1)}(r, \frac{1}{f}) &= \frac{1}{2}(2\overline{N}(r, \frac{1}{f}) - N_{1)}(r, \frac{1}{f})) \\
&= \frac{1}{2}(\overline{N}(r, \frac{1}{f}) + N_{1)}(r, \frac{1}{f}) + \overline{N}_{(2}(r, \frac{1}{f}) - N_{1)}(r, \frac{1}{f})) \\
&= \frac{1}{2}(\overline{N}(r, \frac{1}{f}) + \overline{N}_{(2}(r, \frac{1}{f})) \le \frac{1}{2} N(r, \frac{1}{f}).
\end{aligned}$$

∎

Lemma 9. *Let f be a non-constant meromorphic function and L be an L-function, $P(z)$ be defined as in (1.1). If either $n \ge 3$, $m = 1$ or $n, m \ge 2$, $m + n \ge 5$, and $P(f) = P(L)$, then $f = L$.*

Proof. Recall that $P(z)$ is a polynomial in $\mathbb{C}[z]$, having no multiple zeros, and of degree $n+m+1$. Write

$$P(z) = (n+m+1)\Big(\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i} z^{n+m+1-i} a^i\Big) + 1 = Q(z) + 1,$$

where

$$Q(z) = (n+m+1)\Big(\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i} z^{n+m+1-i} a^i\Big).$$

We have $P'(z) = (n+m+1)z^n(z-a)^m$. Consider the following possible cases:

Case 1. $n \geq 3, \ m = 1$. Then

$$P(z) = z^{n+2} - \frac{(n+2)a}{n+1} z^{n+1} + 1, \ P'(z) = (n+2)z^n(z-a).$$

Set $b = \dfrac{(n+2)a}{n+1}$, $h = \dfrac{f}{L}$. Since $P(f) = P(L)$ we obtain

$$f^{n+2} - bf^{n+1} = L^{n+2} - bL^{n+1}. \tag{3.1}$$

It implies

$$L = b\frac{h^{n+1} - 1}{h^{n+2} - 1}. \tag{3.2}$$

Suppose that h is not a constant. Let $r_1, r_2, ..., r_{n+1}$, $(r_j \neq 1, \ j = 1, 2, \cdots, n+1)$ be the roots of unity of degree $n+2$. Since $n+1 \geq 4$, by the Picard Theorem we always find two distinct numbers r_i, r_j such that $h - r_i$, $h - r_j$, have zeros. Because $r_j^{n+1} \neq 0$, $(r_j \neq 1, \ j = 1, 2, \cdots, n+1)$ from (3.2) we see that L has at least two distinct poles, a contradiction, since L has only one possible pole at $s = 1$.

So h is a constant. Then (3.2) implies $h^{n+2} = 1$ and $h^{n+1} = 1$, because L is not a constant. Therefore $h = 1$ and $f = L$.

Case 2. $n, m \geq 2, \ n+m \geq 5$.

By Lemma 2.2, we see that $\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i}$ is not an integer. For polynomial $P(z)$, $P'(z) = (n+m+1)z^n(z-a)^m$ has two distinct zeros $z = 0, z = a$. Set

$$A = \sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i},$$

then $A \neq 0$. We have $P(0) = Q(0) + 1 = 1$, $P(a) = Q(a) + 1 = (n+m+1)Aa^{n+m+1} + 1$. Since $a \neq 0$, $P(a) \neq P(0)$. On the other hand, $\min\{n, m\} \geq 2$, $n+m \geq 5$, from Lemma 2.3 it follows that $P(z)$ is a uniqueness polynomial, and from $P(f) = P(L)$ we get $f = L$. □

Lemma 10. *Let f, g be two non-constant meromorphic functions, and $P(z)$ be defined as in (1.1). If $\min\{n, m\} \geq 2$ and*

$$\frac{1}{P(f)} = \frac{c}{P(g)} + c_1.$$

Then $c_1 = 0$.

Proof. From the proof of Lemma 3.2, we see that $P(a) \neq P(0)$, where $0, a$ are two distinct zeros of $P^{'}(z)$. Applying Proposition 7.1 in [2] we get $c_1 = 0$. □

Lemma 11. *Let f be a non-constant meromorphic function, L be an L-function, $P(z)$ be defined as in (1.1) with the condition (1.2). If either $n \geq 3, m = 1$, or $n, m \geq 2$, then the condition $P(f) = cP(L)$ for a constant $c \neq 0$ implies $c = 1$ and $f = L$.*

Proof. From Case 2 of Lemma 3.2 we have $P(a) \neq P(0)$. Set $F = P(f), G = P(L)$. From $P(f) = cP(L), c \neq 0$, it implies

$$F = cG,\ T(r, f) = T(r, L) + O(1), \quad S(r, f) = S(r, L). \tag{3.3}$$

First, assume that $c \neq 1$.

If $c = P(a)$, from (3.3) and $P(a) \neq 0$, we have

$$F - 1 = P(a)(G - \frac{1}{P(a)}). \tag{3.4}$$

We consider $P(z) - \frac{1}{P(a)}$. By $P(0) = 1$ and $P(a) = c \neq 1$ we obtain $P(0) - \frac{1}{P(a)} \neq 0$. Moreover, since $P(a) \neq -1$ and $P(a) = c \neq 1$ we obtain $P(a) - \frac{1}{P(a)} \neq 0$. Therefore $P(z) - \frac{1}{P(a)}$ has only simple zeros, let they be given by $b_i^{'}, i = 1, 2, ..., n + m + 1$.

Note that $P(z) - 1$ has a zero at 0 of order $n + 1$, and m distinct simple zeros. Let $c_i^{'}, i = 1, 2, ..., m$, be distinct simple zeros of $P(z) - 1$. Applying Second Fundamental Theorem to the function L and the values $b_1^{'}, b_2^{'}, ..., b_{n+m+1}^{'}$, by (3.3), (3.4) and noting that $N(r, L) = S(r, L)$ we get

$$\begin{aligned}
(n + m)T(r, L) &\leq \overline{N}(r, L) + \sum_{i=1}^{n+m+1} \overline{N}(r, \frac{1}{L - b_i^{'}}) + S(r, L),\\
&\leq \overline{N}(r, \frac{1}{f}) + \sum_{i=1}^{m} \overline{N}(r, \frac{1}{f - c_i^{'}}) + S(r, L),\\
&\leq T(r, f) + mT(r, f) + S(r, L)\\
&= (m + 1)T(r, L) + S(r, L).
\end{aligned}$$

This is a contradiction to the assumption that $n \geq 2$.

Therefore, $c \neq P(a)$. Then from (3.3) we have

$$F - c = c(G - 1). \tag{3.5}$$

From $P(f) = cP(L), c \neq 0$, it implies $T(r, f) = T(r, L) + O(1)$ and $\overline{N}(r, f) = \overline{N}(r, L)$, and therefore $S(r, f) = S(r, L)$.

Now consider $P(z) - c$. By $P(0) = 1$ and $c \neq 1$ we have $P(0) - c = 1 - c \neq 0$. Moreover $c \neq P(a)$. So $P(z) - c$ has only simple zeros, let they be given by $e_i, i = 1, 2, ..., n + m + 1$. Now we consider $P(z) - 1$. We see that $P(0) = 1, P(z) - P(0) = P(z) - 1$ has a zero at 0 of order $n + 1$, and m distinct simple zeros. Let $t_i, i = 1, 2, ..., m$, be distinct simple zeros of $P(z)-1$. Applying Second Fundamental Theorem to the function f and the values $e_1, e_2, \cdots, e_{n+m+1}$, by (3.5) we get

$$\begin{aligned}(n+m)T(r,f) &\le \overline{N}(r,f) + \sum_{i=1}^{n+m+1} \overline{N}(r, \frac{1}{f-e_i}) + S(r,f)\\ &\le \overline{N}(r, \frac{1}{L}) + \sum_{i=1}^{m} \overline{N}(r, \frac{1}{L-t_i}) + S(r,f)\\ &\le T(r,L) + mT(r,L) + S(r,f)\\ &= (m+1)T(r,f) + S(r,f).\end{aligned}$$

This is a contradiction to the assumption that $n \ge 2$.

Therefore, we have $c = 1$. Then

$$P(f) = P(L) \tag{3.6}$$

From Lemma 3.2 we obtain $f = L$. ∎

3.2 Proof of Theorem 1

Proof. 1. Let $n, m \ge 2,\ n+m \ge 9,\ P(z) = (z-a_1)...(z-a_{n+m+1})$. Set

$$F = \frac{1}{P(f)},\ G = \frac{1}{P(L)}, H = \frac{F^{''}}{F'} - \frac{G^{''}}{G'}.$$

We first prove that $H \equiv 0$.

Suppose that $H \not\equiv 0$.

Claim 1. *We have*

1/ $(n+m)T(r,L) \le \overline{N}(r, \frac{1}{P(L)}) - N_o(r, \frac{1}{L'}) + S(r,L)$,

where $N_o(r, \frac{1}{L'})$ *is the counting function of those zeros of* $L^{'}$, *which are not zeros of function* $L(L-a)(L-a_i),\quad i = 1, ..., n+m+1$,

and

$(n+m-1)T(r,f) \le \overline{N}(r, \frac{1}{P(f)}) - N_o(r, \frac{1}{f'}) + S(r,f)$,

where $N_o(r, \frac{1}{f'})$ *is the counting function of those zeros of* $f^{'}$, *which are not zeros of function* $f(f-a)(f-a_i),\quad i = 1, ..., n+m+1$.

2/ $\overline{N}(r, \frac{1}{P(L)}) \le \frac{n+m+1}{2}T(r,L) + \frac{1}{2}N_{1)}(r, \frac{1}{P(L)}) + S(r,L)$,

and

$\overline{N}(r, \frac{1}{P(f)}) \le \frac{n+m+1}{2}T(r,f) + \frac{1}{2}N_{1)}(r, \frac{1}{P(f)}) + S(r,f)$.

Proof. 1/ Applying Second Fundamental Theorem to L and the values $a_1, a_2, \cdots, a_{n+m+1}$, and $0, a$, we obtain

$$(n+m+2)T(r,L) \leq \overline{N}(r,L) + \overline{N}(r,\frac{1}{L}) + \overline{N}(r,\frac{1}{L-a}) + \\ + \sum_{i=1}^{n+m+1} \overline{N}(r,\frac{1}{L-a_i}) - N_o(r,\frac{1}{L'}) + S(r,L).$$

On the other hand

$$\overline{N}(r,L) = S(r,L),\ \overline{N}(r,\frac{1}{L}) \leq T(r,L) + S(r,L),$$

and

$$\overline{N}(r,\frac{1}{L-a}) \leq T(r,L) + S(r,L),\quad \sum_{i=1}^{n+m+1} \overline{N}(r,\frac{1}{L-a_i}) = \overline{N}(r,\frac{1}{P(L)}).$$

Then we have

$$(n+m)T(r,L) \leq \overline{N}(r,\frac{1}{P(L)}) - N_o(r,\frac{1}{L'}) + S(r,L).$$

The inequality for f is proved by a similar argument.

2/ Applying Lemma 3.1 we get

$$\overline{N}(r,\frac{1}{P(L)}) \leq \frac{1}{2}[N(r,\frac{1}{P(L)}) + N_{1)}(r,\frac{1}{P(L)})].$$

On the other hand

$$N(r,\frac{1}{P(L)}) \leq T(r,P(L)) + S(r,L) = (n+m+1)T(r,L) + S(r,L).$$

Therefore,

$$\overline{N}(r,\frac{1}{P(L)}) \leq \frac{n+m+1}{2}T(r,L) + \frac{1}{2}N_{1)}(r,\frac{1}{P(L)}) + S(r,L).$$

Similarly, we have the inequality for f.

Claim 1 is proved.

Claim 2. *We have*

1/ $(n+m)T(r,L) + S(r,L) \leq (n+m+1)T(r,f) + S(r,f)$,

$(n+m-1)T(r,f) + S(r,f) \leq (n+m+1)T(r,L) + S(r,L)$.

In particular, $S(r,f) = S(r,L)$.

2/ $N(r,H) \leq 3T(r,f) + 2T(r,L) + N_o(r,\frac{1}{f'}) + N_o(r,\frac{1}{L'}) + S(r)$, *where we denote* $S(r) = S(r,f) = S(r,L)$.

Proof of Claim 2.

1/ Applying Second Fundamental Theorem to the functions L and the values $a_1, a_2, \cdots, a_{n+m+1}$, we have

$$(n+m)T(r,L) \leq \overline{N}(r,L) + \sum_{i=1}^{n+m+1} \overline{N}(r,\frac{1}{L-a_i}) + S(r,L).$$

Noting that $\overline{N}(r,L)=S(r,L),\ \ E_L(S)=E_f(S)$, we obtain

$$\begin{aligned}(n+m)T(r,L)+S(r,L) &\leq \sum_{i=1}^{n+m+1}\overline{N}(r,\frac{1}{f-a_i})+S(r,f)\\ &\leq (n+m+1)T(r,f)+S(r,f).\end{aligned}$$

Similarly,

$$(n+m)T(r,f)\leq \overline{N}(r,f)+\sum_{i=1}^{n+m+1}\overline{N}(r,\frac{1}{f-a_i})+S(r,f),$$

it implies

$$(n+m)T(r,f)\leq T(r,f)+\sum_{i=1}^{n+m+1}\overline{N}(r,\frac{1}{L-a_i})+S(r,f).$$

Therefore

$$(n+m-1)T(r,f)+S(r,f)\leq (n+m+1)T(r,L)+S(r,L).$$

Part 1 is proved.

2/ Noting that H has only simple poles, from Lemma 2.6 we obtain

$$\begin{aligned}N(r,H) &\leq \overline{N}_{(2}(r,P(f))+\overline{N}_{(2}(r,P(L))\\ &\quad +\overline{N}(r,\frac{1}{P'(f)};P(f)\neq 0)+\overline{N}(r,\frac{1}{P'(L)};P(L)\neq 0)+S(r).\end{aligned}\tag{3.7}$$

On the other hand,

$$\begin{aligned}\overline{N}(r,P(L)) &=\overline{N}(r,L)=S(r),\\ \overline{N}_{(2}(r,P(f)) &=\overline{N}(r,f)\leq T(r,f)+S(r).\end{aligned}$$

Then

$$\begin{aligned}N(r,H) &\leq T(r,f)+\overline{N}(r,\frac{1}{P'(f)};P(f)\neq 0)\\ &\quad +\overline{N}(r,\frac{1}{P'(L)};P(L)\neq 0)+S(r).\end{aligned}\tag{3.8}$$

Moreover, we have

$$\begin{aligned}\overline{N}(r,\frac{1}{[P(f)]'};P(f)\neq 0) &= \overline{N}(r,\frac{1}{f^n})\\ &\quad +\overline{N}(r,\frac{1}{(f-a)^m};(f-a_1)\cdots(f-a_{n+m+1})\neq 0)\\ &\leq \overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f-a})+N_o(r,\frac{1}{f'})\\ &\leq 2T(r,f)+N_o(r,\frac{1}{f'})+S(r).\end{aligned}\tag{3.9}$$

Similarly,

$$\overline{N}(r, \frac{1}{[P(L)]'}; P(L) \neq 0) \leq 2T(r, L) + N_o(r, \frac{1}{L'}) + S(r). \tag{3.10}$$

Claim 2 follows from inequalities (3.8), (3.9), (3.10).

Claim 3. *We have*

1/ $(n+m-3)T(r, L) \leq 3T(r, f) - N_o(r, \frac{1}{L'}) + N_o(r, \frac{1}{f'}) + S(r).$

2/ $(n+m-6)T(r, f) \leq 2T(r, L) - N_o(r, \frac{1}{f'}) + N_o(r, \frac{1}{L'}) + S(r).$

Proof. Note that from Lemma 2.6, if a is a common simple zero of $P(f)$ and $P(L)$, then $H(a) = 0$. Therefore, from this and by First Fundamental Theorem we get:

$$\begin{aligned} N_{1)}(r, \frac{1}{P(L)}) = N_{1)}(r, \frac{1}{P(f)}) &\leq N(r, \frac{1}{H}) \leq T(r, H) + 0(1) \\ &= N(r, H) + m(r, H) + O(1). \end{aligned} \tag{3.11}$$

By the logarithmic derivative lemma, we have

$$\begin{aligned} m(r, H) &= m(r, \frac{F''}{F'} - \frac{G''}{G'}) \\ &\leq m(r, \frac{F''}{F'}) + m(r, \frac{G''}{G'}) + O(1) \\ &= S(r, F') + S(r, G') + O(1). \end{aligned} \tag{3.12}$$

On the other hand, from Lemma 2.5 we get

$$S(r, F') = S(r, F), \; S(r, G') = S(r, G).$$

Moreover,

$$T(r, F) = T(r, P(f)) + O(1) = (m+n+1)T(r, f) + O(1),$$

$$T(r, G) = T(r, P(L)) + O(1) = (m+n+1)T(r, L) + O(1).$$

Therefore,

$$S(r, f) = S(r, F) = S(r, F'), \; S(r, L) = S(r, G) = S(r, G').$$

Combining (3.11) and (3.12) we get

$$N_{1)}(r, \frac{1}{P(L)}) = N_{1)}(r, \frac{1}{P(f)}) \leq N(r, H) + S(r). \tag{3.13}$$

1/ From Claim 1 we have

$$\begin{aligned} (n+m)T(r, L) &\leq \overline{N}(r, \frac{1}{P(L)}) - N_o(r, \frac{1}{L'}) + S(r), \\ (n+m)T(r, L) &\leq \frac{n+m+1}{2} T(r, L) + \frac{1}{2} N_{1)}(r, \frac{1}{P(L)}) - N_o(r, \frac{1}{L'}) + S(r), \end{aligned}$$

and then

$$(n+m-1)T(r,L) \leq N_{1)}(r,\frac{1}{P(L)}) - 2N_o(r,\frac{1}{L'}) + S(r). \tag{3.14}$$

From this and (3.13) and noting that

$$N(r,H) \leq 3T(r,f) + 2T(r,L) + N_o(r,\frac{1}{f'}) + N_o(r,\frac{1}{L'}) + S(r),$$

we obtain

$$(n+m-3)T(r,L) \leq 3T(r,f) - N_o(r,\frac{1}{L'}) + N_o(r,\frac{1}{f'}) + S(r).$$

Part 1 is proved.

2/ From Claim 1 and Part 2 of Claim 2, by using similar arguments as in Part 1, we obtain Part 2.

Now we use Claims 1, 2, 3 to obtain a contradiction, and complete the proof of $H \equiv 0$.

Claim 2 and Claim 3 give us

$$(n+m-3)T(r,L) \leq 3.\frac{n+m+1}{n+m-1}T(r,L) - N_o(r,\frac{1}{L'}) + N_o(r,\frac{1}{f'}) + S(r),$$

$$(n+m-6)\frac{n+m}{n+m+1}T(r,L) \leq 2T(r,L) - N_o(r,\frac{1}{f'}) + N_o(r,\frac{1}{L'}) + S(r).$$

Adding two inequalities and using straight calculations, we obtain:

$$(2(n+m) + \frac{7}{n+m+1} - \frac{6}{n+m-1} - 15)T(r,L) \leq S(r).$$

This contradicts $n+m \geq 8$.

We have proved $H \equiv 0$. Therefore,

$$\frac{1}{P(f)} = \frac{c}{P(L)} + c_1$$

for some constants c ($\neq 0$) and c_1. By Lemma 3.3 we obtain $c_1 = 0$.

Thus, there is a constant $C \neq 0$ such that $P(f) = CP(L)$. From Lemma 3.4, we obtain $f \equiv L$. Theorem 1 is proved. □

3.3 Proof of Theorem 2

We denote the order of a meromorphic function f by $\rho(f)$. Write

$$P(z) = (n+m+1)\Big(\sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n+m+1-i} z^{n+m+1-i} a^i\Big) + 1 = z^{n+1}R(z) + 1,$$

where $R(z)$ is a polynomial of degree m. Recall that $n-m \geq 2,\ m \geq 1$, and then either $n \geq 3,\ m = 1$, or $n \geq 4,\ m \geq 2,\ n+m \geq 6$.

From Part 1 of Claim 2 of Theorem 1 we get:

$$\begin{aligned}(n+m)T(r,L)+S(r,L) \leq& (n+m+1)T(r,f)+S(r,f),\\(n+m-1)T(r,f)+S(r,f) \leq& (n+m+1)T(r,L)+S(r,L).\end{aligned} \tag{3.15}$$

In particular, $S(r,f)=S(r,L)$.

From this and because f has finitely many poles, by Lemma 2.7 we obtain

$$N(r,L)=S(r,L), N(r,L)=S(r)=N(r,f),\ \rho(f)=\rho(L)=1. \tag{3.16}$$

Since f and L share S CM, we have

$$\frac{P(f)}{P(L)}=\frac{f^{n+1}R(f)+1}{L^{n+1}R(L)+1}=R_1e^{\varphi(z)}, \tag{3.17}$$

where $R_1\not\equiv 0$ is a rational function and $\varphi(z)$ is an entire function. Then $\rho(R_1)=0$, by ([3], Theorem 1.4) and (3.16) we get

$$\rho(P(f))=\rho(f)=1,\ \rho(P(L))=\rho(L)=1. \tag{3.18}$$

From (3.17), (3.18), and Lemma 2.4 we have

$$\rho(e^{\varphi(z)})=\rho(\frac{P(f)}{R_1P(L)})\leq \max\{\rho(R_1),\rho(P(f)),\rho(P(L))\}=\rho(L)=1,$$

$$\varphi(z)=Az+B,$$

where A,B are constants. Set

$$h(z)=R_1e^{\varphi(z)}.$$

Then we have

$$T(r,h)\leq T(r,R_1)+T(r,e^{\varphi})=\frac{|A|}{\pi}r+O(\log r).$$

Therefore, $T(r,h)=O(r)$. By Lemma 2.7 we have

$$T(r,L)=\frac{d_L}{\pi}r\log r+O(r),$$

where d_L is the degree of L and $d_L>0$ if $L\not\equiv 1$ ([11]). So we have $T(r,h)=o(T(r,L))$. Because $S(r,L)=O(\log r)$ (Lemma 2.7), and $S(r,f)=O(\log r)$ as f is of finite oder ([12]), from (3.15) we obtain:

$$\begin{aligned}(n+m)T(r,L)+S(r,L) \leq& (n+m+1)T(r,f)+S(r,f),\\(n+m-1)T(r,f)+S(r,f) \leq& (n+m+1)T(r,L)+S(r,L),\end{aligned}$$

where the inequalities hold except for a set of finite Lebesgue measure.

Therefore

$$\frac{n+m}{n+m+2}T(r,L)\leq T(r,f)\leq \frac{n+m+2}{n+m-1}T(r,L),$$

except for a set of finite Lebesgue measure.

It follows $T(r,h) = o(T(r,f)$. This means that h is a small function for the functions f and L.

From (3.17) it implies

$$f^{n+1}R(f) - (h-1) = hL^{n+1}R(L).$$

We shall prove that $h \equiv 1$.

Suppose $h \not\equiv 1$. Applying Second Fundamental Theorem for $f^{n+1}R(f)$ and the small functions $\infty, 0, h-1$, and noting that $T(r,L) = T(r,f) + S(r)$, $N(r,L) = S(r) = N(r,f)$, and the degree of R is m, we get

$$\begin{aligned}
&(n+m+1)T(r,f) + O(1) = T(r, f^{n+1}R(f)) \\
&\leq \overline{N}(r, f^{n+1}R(f)) + \overline{N}(r, \frac{1}{f^{n+1}R(f)}) + \overline{N}(r, \frac{1}{f^{n+1}R(f) - (h-1)}) + S(r) \\
&\leq \overline{N}(r,f) + \overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{R(f)}) + \overline{N}(r, \frac{1}{L^{n+1}R(L)}) + S(r) \\
&\leq (1+m)T(r,f) + \overline{N}(r,L) + \overline{N}(r, \frac{1}{L}) + N(r, \frac{1}{R(L)}) + S(r) \\
&\leq (1+m)(T(r,f) + T(r,L)) + S(r) \\
&\leq (2m+2)T(r,f) + S(r).
\end{aligned}$$

Therefore

$$(n-m-1)T(r,f) \leq S(r).$$

This contradicts to $n - m \geq 2$. Thus $h \equiv 1$, and

$$f^{n+1}R(f) + 1 = L^{n+1}R(L) + 1,$$

i.e. $P(f) = P(L)$. From Lemma 3.2, we obtain $f \equiv L$. Theorem 2 is proved.

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