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Convexity at a point

by

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Dedicated to Professors Toma Albu and Constantin Năstăsescu on the occasion of their 80th birthdays

Abstract

There are several notions of convexity at a point in the literature, with applications to inequalities, the mechanics and thermodynamics of continuous media, and nonlinear programming. We came upon these notions in the process of constructing examples of nonconvex minima, and we ended up introducing another one, inspired by a definition of convexity with difference quotients. We study the interrelationships of these notions for real functions of one variable under various smoothness assumptions. To argue that a "generic minimum" is nonconvex, we demonstrate how to build one from any discontinuity of a second derivative.

Key Words: Point of convexity of a function, function convex at a point, punctual convexity.

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1 Introduction

This article was started as an attempt to address a common misconception among beginners in analysis — a function must be convex near a local minimum. In the course of producing examples of nonconvex minima, we were naturally led to question whether these examples are at least "convex" in some sense at the minimum point in question. We ended up finding several ways of defining convexity of a function at a point in the literature. The notion of a *point of convexity* was defined by Šilhavý in [19] and by Niculescu and Rovenţa in [12]. As Niculescu and Rovenţa explain, this notion is present, in an equivalent form, in a paper of Dragomirescu and Ivan [5], and the technique of convex minorants, described by Steele in [22, pp. 96–99], is also close to the concept of point of convexity. They also indicate that another sufficient condition for convexity at a point, formulated in terms of secant line slopes, can be found in the papers of Niculescu and Stephan [14, 15], and that related work was done by Niculescu and Spiridon in [13]. Bazaraa and Shetty in [3] define the notion of a function *convex at a point*. This was studied by Minuţă in [10]. A third notion of *punctual convexity* was given by Florea and Păltănea in [6]. We introduce the notion of *total convexity*, inspired by Galvani's definition of convexity in terms of difference quotients, which appears in E. Artin's monograph on the Gamma function [1].

In Section 2, we study the relationships between these notions of pointwise convexity. In the context of relating these notions with convexity near a point, we give a few examples in

Section 3 clearing the misconception mentioned at the beginning of this article. Convexity of a function can be expressed as its epigraph being a convex set. In the last section we prove that the epigraph of a function being "convex at a point" is equivalent to that point being a point of convexity of the function (also see [12, Lemma 2.1] and [19, Propositions 16.2.3 and 16.2.4]). To argue that the examples in Section 3 are rather generic, we demonstrate how to construct a nonconvex minimum from any discontinuity of a second derivative. We conclude the article by questioning whether smooth functions with certain convexity requirements exist.

Throughout this article I stands for an open interval on the real line. Let f be a real-valued function defined on I. We denote the *left limit* and the *right limit* of f at x_0 by $f(x_0^-)$ and $f(x_0^+)$, respectively:

$$f(x_0^-) := \lim_{x\to x_0^-} f(x) \quad \text{and} \quad f(x_0^+) := \lim_{x\to x_0^+} f(x).$$

For distinct x_0 and x_1, let $\varphi_f(x_0, x_1)$ be the quotient

$$\varphi_f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}.$$

Note that $\varphi_f(x_0, x_1) = \varphi_f(x_1, x_0)$. We write φ for φ_f if f is understood. The *left derivative* of f at x_0, denoted by $f'_-(x_0)$, is the left limit at x_0 of $\varphi(x, x_0)$. Analogously, the *right derivative* of f at x_0, denoted by $f'_+(x_0)$, is the right limit of $\varphi(x, x_0)$ at x_0. Either of them is called a *one-sided derivative of f at x_0*. We say that *the one-sided derivatives of f are increasing on I* if both one-sided derivatives of f exist at every point in I and for all $x_1 < x_2$ in I the following two conditions hold:

$$f'_-(x_1) \le f'_+(x_1) \quad \text{and} \quad f'_+(x_1) \le f'_-(x_2).$$

We say that f is *weakly convex* (i.e., *Jensen convex*) on I if for any $x_1, x_2 \in I$,

$$f\left(\frac{x_1 + x_2}{2}\right) \le \frac{1}{2}(f(x_1) + f(x_2)).$$

Definition 1. *A function f is* convex *on I if for any distinct $x_1, x_2 \in I$ and $0 < t < 1$,*

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2). \tag{1.1}$$

It is clear that (1.1) is also true when $x_1 = x_2$, or $t = 0$, or $t = 1$, and requiring $x_1 < x_2$ makes no difference to the concept being defined. Geometrically, the definition means that the graph of f is below the line segment between any two points on the graph.

The following definition of convexity, closer in spirit to calculus, appears in [1]. It seems it was first given by Galvani in [7] (see also [11, Exercise 3, p. 31]).

Definition 2. *A function f is* convex *on I if for any $x_0 \in I$, $\varphi(x, x_0)$ is an increasing function in x on $I \setminus \{x_0\}$.*

It is easy to see that f is convex on I (in the sense of Definition 2) if and only if for any distinct x_0, x_1, x_2 in I with $x_1 < x_2$, the second order difference quotient,

$$\begin{aligned} \Psi_f(x_0, x_1, x_2) &:= \frac{\varphi_f(x_2, x_0) - \varphi_f(x_1, x_0)}{x_2 - x_1} \\ &= \frac{(x_2 - x_1)f(x_0) + (x_0 - x_2)f(x_1) + (x_1 - x_0)f(x_2)}{(x_0 - x_1)(x_1 - x_2)(x_2 - x_0)} \ge 0. \end{aligned} \tag{1.2}$$

Since the value $\Psi_f(x_0, x_1, x_2)$ does not change by permuting the arguments, convexity of f on I is equivalent to $\Psi_f(x_0, x_1, x_2)$ being nonnegative for all distinct triple x_0, x_1, x_2 in I. From now on, we simply write Ψ for Ψ_f if f is understood.

Proposition 1. *Definition 1 and Definition 2 are equivalent.*

Proof. Identifying the numbers strictly between x_1 and x_2 with the values strictly between 0 and 1 via

$$x_0 \mapsto t := \frac{x_0 - x_2}{x_1 - x_2} \quad \text{and} \quad t \mapsto x_0 := tx_1 + (1-t)x_2 \tag{1.3}$$

shows that the inequality in (1.1) holds if and only if

$$f(x_0) \leq \frac{x_0 - x_2}{x_1 - x_2} f(x_1) + \frac{x_1 - x_0}{x_1 - x_2} f(x_2) \tag{1.4}$$

holds. A rearrangement of terms after dividing $(x_0 - x_1)(x_2 - x_0)$ (which is positive since x_0 is strictly between x_1 and x_2) on both sides of (1.4) shows that it is equivalent to $\Psi(x_0, x_1, x_2) \geq 0$; letting x_1 and x_2 range through all distinct pairs of points in I establishes the proposition. □

A number of properties of convex functions follow readily from the definition. For example, since $\Psi_{cf+g} = c\Psi_f + \Psi_g$, if f and g are convex on I and $c \geq 0$, then so is $cf + g$. Also, under the coordinate transformation $X = ax + b, Y = cy + d$ $(a, c \neq 0)$, the value of $\Psi(x_0, x_1, x_2)$ is multiplied by c. Thus, its sign will not change if $c > 0$. In particular, translation of axes and reflection about the y-axis will not affect convexity.

We assume the reader is familiar with the following characterization of convexity [1, Theorem 1.4]: A function is convex if, and only if, it has increasing one-sided derivatives. Consequently, for a twice differentiable f, convexity of f on I is equivalent to f' being increasing on I and that is equivalent to $f'' \geq 0$ on I.

2 Pointwise Convexity

We list below the notions of pointwise convexity mentioned in the introduction. Inspired by Definition 2, we add one of our own at the end of this list.

Definition 3. *Let I be an open interval and f be a real-valued function defined on I. Then, with repect to I,*

1. *[19, 12] a point $x_0 \in I$ is a point of convexity of f if*

$$f(x_0) \leq tf(x_1) + (1-t)f(x_2) \tag{2.1}$$

for any $x_1, x_2 \in I$ and $0 < t < 1$ such that $x_0 = tx_1 + (1-t)x_2$.

2. *[3] f is convex at $x_0 \in I$ if for any $x_1 \in I$ other than x_0, and $0 < t < 1$,*

$$f(tx_0 + (1-t)x_1) \leq tf(x_0) + (1-t)f(x_1). \tag{2.2}$$

3. *[6] f is punctually convex (or p-convex, for short) at $x_0 \in I$ if*

$$f(x_0) + f(x_1 + x_2 - x_0) \le f(x_1) + f(x_2) \tag{2.3}$$

whenever x_0 is strictly between $x_1, x_2 \in I$.

4. *f is totally convex at $x_0 \in I$ if $\varphi(x, x_0)$ is an increasing function of x on $I \setminus \{x_0\}$. Equivalently, $\Psi(x_0, x_1, x_2) \ge 0$ for any distinct x_1, x_2 in $I \setminus \{x_0\}$.*

Note that in [6] p-convexity at a point was also called convexity at a point, so we had to use the new name in order to distinguish the two notions. For readability, we often skip mentioning the interval I if it is understood. For distinct points x_0, x_1, x_2, we say that x_1, x_2 are *on opposite sides of* x_0 if x_0 is strictly between x_1 and x_2, i.e. $(x_0 - x_1)(x_2 - x_0) > 0$. Analogously, x_1, x_2 are on the *same side of* x_0 if they are not on opposite sides of x_0, i.e. $(x_0 - x_1)(x_2 - x_0) < 0$. Also, since we deal with convexity, we assume neighborhoods are open and convex, so, on the real line, when we say a neighborhood of a point x_0 we mean an open interval containing x_0.

Our first move to understand the relationships between these notions is to also characterize the first three of them in terms of difference quotients.

Proposition 2. *If f is defined on an open interval containing x_0,*

- *x_0 is a point of convexity of f if, and only if, whenever x_1, x_2 are on opposite sides of x_0*
$$\Psi(x_0, x_1, x_2) = \frac{\varphi(x_2, x_0) - \varphi(x_1, x_0)}{x_2 - x_1} \ge 0.$$
- *f is convex at x_0 if, and only if, whenever x_1, x_2 are distinct points on the same side of x_0*
$$\Psi(x_0, x_1, x_2) = \frac{\varphi(x_2, x_0) - \varphi(x_1, x_0)}{x_2 - x_1} \ge 0.$$
- *f is p-convex at x_0 if, and only if, whenever x_1, x_2 are on opposite sides of x_0 with $x_0 + x_0' = x_1 + x_2$,*
$$\frac{\varphi(x_2, x_0') - \varphi(x_1, x_0)}{x_2 - x_1} \ge 0$$
and this inequality is also equivalent to
$$\begin{cases} \Psi(x_1, x_0, x_0') + \Psi(x_2, x_0, x_0') \ge 0 & \text{if } x_0 \ne x_0'; \\ \Psi(x_0, x_1, x_2) \ge 0 & \text{if } x_0 = x_0'. \end{cases}$$

Proof. The points x_1, x_2 are on opposite sides of x_0 if, and only if, there exists $0 < t < 1$, such that $x_0 = tx_1 + (1-t)x_2$. Thus, the proof of Proposition 1, shows that x_0 being a point of convexity of f is equivalent to $\Psi(x_0, x_1, x_2) \ge 0$ for any x_1, x_2 on opposite sides of x_0. The same proof also shows that the inequality in (2.2) holds if and only if $\Psi(x_2, x_0, x_1) \ge 0$ with $x_2 = tx_0 + (1 - t)x_1$. For distinct x_1, x_2 on the same side of x_0, by switching x_1 and x_2, if necessary, we can assume x_2 is between x_0 and x_1. Since the value $\Psi(x_0, x_1, x_2)$ is unchanged by permutations of x_0, x_1, x_2, we conclude that f being convex at x_0 is equivalent to $\Psi(x_0, x_1, x_2) \geq 0$ for any distinct x_1, x_2 on the same side of x_0.

A function f being p-convex at x_0 means for any x_1, x_2 on opposite sides of x_0,

$$f(x_2) - f(x_0') + f(x_1) - f(x_0) \geq 0 \tag{2.4}$$

where $x_0' := x_1 + x_2 - x_0$. Dividing this inequality by the positive quantity (both x_0 and x_0' are strictly between x_1 and x_2)

$$(x_2 - x_1)(x_2 - x_0') = (x_2 - x_1)(x_0 - x_1)$$

establishes the equivalence of (2.4) with

$$\frac{\varphi(x_2, x_0') - \varphi(x_1, x_0)}{x_2 - x_1} \geq 0. \tag{2.5}$$

When $x_0 = x_0'$, then (2.5) becomes $\Psi(x_0, x_1, x_2) \geq 0$ and if $x_0 \neq x_0'$, then $\varphi(x_0', x_0)$ is defined. As the quantity

$$r := \frac{x_2 - x_1}{x_2 - x_0} = \frac{x_2 - x_1}{x_0' - x_1}$$

is also positive, the inequality in (2.5) is equivalent to

$$\begin{aligned} & r\left(\frac{\varphi(x_2, x_0') - \varphi(x_0, x_0')}{x_2 - x_1} - \frac{\varphi(x_1, x_0) - \varphi(x_0', x_0)}{x_2 - x_1}\right) \\ &= \frac{x_2 - x_1}{x_2 - x_0}\frac{\varphi(x_2, x_0') - \varphi(x_0, x_0')}{x_2 - x_1} - \frac{x_2 - x_1}{x_0' - x_1}\frac{\varphi(x_1, x_0) - \varphi(x_0', x_0)}{x_2 - x_1} \\ &= \Psi(x_2, x_0, x_0') + \Psi(x_1, x_0, x_0') \geq 0. \end{aligned}$$

This finishes the proof. □

With these characterizations the following proposition is immediate.

Proposition 3. *Total convexity implies the other notions of pointwise convexity in Definition 3.*

The next few examples show that for each of the first three notions in Definition 3 there exist functions that possess that property but not the other two. Hence, in view of Proposition 3, all three of them are strictly weaker than total convexity.

Example 1. *The point* 0 *is an absolute minimum of the function* $f(x) = x^{2/3}$*. Hence,* 0 *is a point of convexity of* f *and* f *is also p-convex at* 0*. However,* f *is not convex at* 0*. The opposite is true about* 0 *for* $-f$*; that is,* $-f$ *is convex at* 0 *but* 0 *is neither a point of convexity of* $-f$ *nor* $-f$ *is p-convex at* 0*.*

Example 2. *The point* $x = 0$*, being an absolute minimum of the function* $f(x) = 1 - \cos(x)$*, is a point of convexity of* f*. As*

$$f(7\pi/4 - \pi/2) + f(0) > f(7\pi/4) + f(-\pi/2),$$

and $\varphi(3\pi/4, 0) > \varphi(7\pi/8, 0)$ *(or, alternatively,* $\varphi'(3\pi/4, 0) < 0$*), we have that* f *is neither p-convex nor convex at* 0*. The restriction of* f *to* $I = (-\pi, \pi)$ *becomes p-convex at* 0*. This can be seen from Theorem 2 of [6], as* $f'(x) = \sin(x)$ *and* x *have the same sign on* I*. Yet, this restriction of* f *to* I *is still not convex at* 0*, because both* $3\pi/4$ *and* $7\pi/8$ *are in* I *(or, alternatively,* $3\pi/4 \in I$*). However, if we move to a small enough neighborhood of* 0*, say* $(-\pi/2, \pi/2)$*, then* f *becomes outright convex on that interval.*

The above example highlights the role of the interval I in all notions of pointwise convexity in Definition 3. For a function that is neither convex nor p-convex at a point of convexity with respect to any neighborhood of that point, see Example 7.

Example 3. *Take a Hamel basis of* $\mathbb{R}$ *over* $\mathbb{Q}$ *containing a positive and a negative number, say* -1 *and* $\sqrt{2}$*. Map the basis elements to* -1 *and extend this assignment to a function* f *on* $\mathbb{R}$ *by linearity. First, let us note that* f *is p-convex at every point* x_0*, in particular at* 0*. This is because by* $\mathbb{Q}$*-linearity of* f*,*

$$f(x_0) + f(x_1 + x_2 - x_0) = f(x_0) + f(x_1) + f(x_2) - f(x_0) = f(x_1) + f(x_2).$$

Next, let us also note that for any $0 \neq q \in \mathbb{Q}$*,* $\varphi(q, 0) = 1$ *and* $\varphi(q\sqrt{2}, 0) = -1/\sqrt{2}$*. In particular, for each* $n \geq 1$*,* $\varphi(1/n, 0) = \varphi(-1/n, 0) = 1$ *and* $\varphi(\sqrt{2}/n, 0) = -1/\sqrt{2}$*. So,* f *is neither convex at* 0 *nor* 0 *is a point of convexity of* f *with respect to any neighborhood of* 0*.*

It is clear from Definition 3 and the characterizations in Proposition 2 that a function convex on I must be pointwise convex at each point in the interval with respect to any of the notions in Definition 3. The converse is also true except for punctual convexity. This anomaly is demonstrated by the function f in Example 3. We have already argued that it is p-convex everywhere. By $\mathbb{Q}$-linearity, f only takes rational values and is not constant on any open interval. Thus, f is not continuous and hence not convex on any open interval. What punctual convexity of f at every point in an open interval I does imply is that f is weakly convex on I: for distinct $x_1, x_2 \in I$, by p-convexity of f at $(x_1 + x_2)/2$,

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1)}{2} + \frac{f(x_2)}{2}$$

and the inequality also holds when $x_1 = x_2$ for trivial reasons. So, the anomaly of punctual convexity goes away under continuity as weakly convex continuous functions are convex [1, Theorem 1.5]. So, we have just proved:

Proposition 4. *If a function is convex on an open interval* I*, then it is pointwise convex at every point in* I *with respect to any notion of pointwise convexity in Definition 3. The converse is also true, except continuity of* f *on* I *is required in the case of punctual convexity.*

Actually, there is a finer statement to be made. If f is continuous on I and is p-convex at x_0, then x_0 must be a point of convexity of f by Lemma 3 (case $n = 2$) in [6]. Also, note that the functions in Example 1 and Example 2 are continuous. So, no additional implications among these notions of pointwise convexity are true under continuity. However, if a function is convex at a differentiable point, then the point must also be a point of convexity. In fact, something slightly more general is true.

Proposition 5. *Suppose f is convex at x_0 and $f'_-(x_0) \leq f'_+(x_0)$. Then x_0 is a point of convexity of f.*

Proof. Since f is convex at x_0, so $\varphi(x, x_0)$ is increasing on both sides of x_0. So, for any $x_1 < x_0 < x_2$,

$$\varphi(x_1, x_0) \leq \varphi(x_0^-, x_0) = f'_-(x_0) \leq f'_+(x_0) = \varphi(x_0^+, x_0) \leq \varphi(x_2, x_0).$$

Thus, x_0 is a point of convexity of f. □

We conclude this section with two brief remarks. First, monotonicity of $\varphi(x, x_0)$ on the left (resp. right) side of x_0 implies $f'_-(x_0) \in \mathbb{R} \cup \{+\infty\}$ (resp. $f'_+(x_0) \in \mathbb{R} \cup \{-\infty\}$). Thus, the assumption $f'_-(x_0) \leq f'_+(x_0)$, treated as an inequality of extended reals, already implies both limits are real numbers. Second, a weakly convex function needs not be p-convex at every point, as the following example demonstrates.

Example 4. *Let f be the function in Example 3. Then for any x_1, x_2,*

$$f^2\left(\frac{x_1 + x_2}{2}\right) = \left(\frac{1}{2}(f(x_1) + f(x_2))\right)^2 \leq \frac{1}{2}f^2(x_1) + \frac{1}{2}f^2(x_2).$$

So, f^2 is weakly convex on $\mathbb{R}$. However, f^2 is not p-convex at 0 as

$$f^2(0) + f^2(\sqrt{2} - 1) = 4 > 2 = f^2(\sqrt{2}) + f^2(-1).$$

3 Convexity near a point

In this section, we consider convexity locally. We examine the relationship between local convexity and the notions of pointwise convexity discussed in the previous section. For convenience, we say that something happens *near a point* if it happens in all sufficiently small neighborhoods of that point. Since convexity on an open interval implies total convexity at any point in that interval, convexity near x_0 implies total convexity at x_0 with respect to all sufficiently small neighborhoods of x_0. The reverse implication, however, is invalidated by the following function.

Example 5. *The function*

$$f(x) = \begin{cases} x^2\left(\frac{2}{5} + x^2 \cos\left(\frac{1}{x}\right)\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

is totally convex at 0 on $\mathbb{R}$ but not convex near 0. This can be seen by checking that the derivative of $\varphi(x, 0)$ ($x \neq 0$) is strictly positive and that $f''(1/(2k\pi)) < 0 < f''(1/(2k+1)\pi)$ for all $k \geq 2$.

In fact, even smooth (i.e., $\mathcal{C}^\infty$) examples exist.

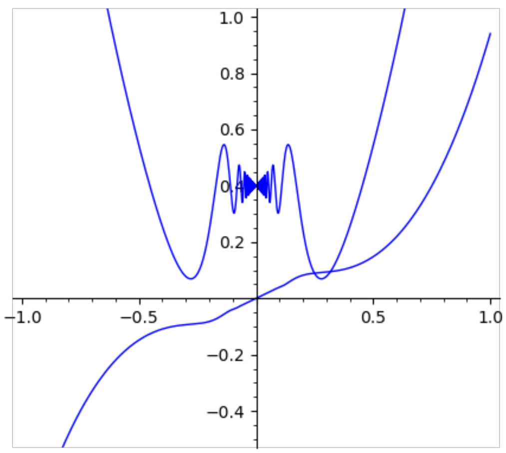

Figure 1: $\varphi_f(x,0)$ and $\varphi'_f(x,0)$ for the function f in Example 5

Example 6. *Consider the smooth function*

$$h(x) = \begin{cases} e^{-1/x^2}(\cos(1/x)+2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Let $g(x) = \int_0^x h(t)dt$ and $f(x) = xg(x)$. Then $f(x)$ is also smooth and $\varphi(x,0) = g(x)$, and so its derivative is $h(x)$ which is positive for all nonzero x. Hence, $f(x)$ is totally convex at 0. One checks readily that the dominating term of $f''(x) = xh'(x) + 2h(x)$ for x near 0 is

$$e^{-1/x^2}(3\sin(1/x^3) + 2\cos(1/x^3) + 2).$$

This term is positive when both $\sin(1/x^3)$ and $\cos(1/x^3)$ are $\sqrt{2}/2$ and is negative when both are $-\sqrt{2}/2$. Consequently, $f''(x)$ changes signs near 0 and hence f is not convex near 0.

This phenomenon, however, cannot happen for real analytic functions.

Proposition 6. *Suppose f is a twice differentiable function on I and f'' is continuous at a point of convexity $x_0 \in I$ of f. If x_0 is not a limit point of the zeros of f'', then f is convex near x_0.*

Proof. Since x_0 is not a limit point of the zeros of f'', by passing to an open subinterval of I, we can assume f'' is zero-free on $I_0 := I \setminus \{x_0\}$. If $f'' > 0$ on I_0, then $f'' \geq 0$ on I by continuity, and so f is convex near x_0. If $f'' < 0$ on I_0, then $-f$ is convex near x_0. Consequently, $-\varphi(x_1,x_0) \leq -\varphi(x_2,x_0)$ for all $x_1 < x_2$. But, as x_0 is a point of convexity of f, $\varphi(x_1,x_0) \leq \varphi(x_2,x_0)$ for all $x_1 < x_0 < x_2$. Therefore, $\varphi(x,x_0)$ must be constant near x_0 (and the constant is $f'(x_0)$). Thus, $f(x)$ is linear, contradicting the fact that f'' is zero-free on I_0. The remaining case is that f'' changes sign across x_0. Without loss of generality, $f''(x) > 0$ for $x > x_0$ and $f''(x) < 0$ for $x < x_0$. Since x_0 is a point of convexity of f, for $x_1 < x_0 < x_2$,

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1} \leq \frac{f(x_2) - f(x_0)}{x_2 - x_0}$$

Letting $x_2 \to x_0$, we conclude that for all $x_1 < x_0$,

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1} \leq f'(x_0).$$

So $f'(u) \leq f'(x_0)$ for some $x_1 < u < x_0$. But as $f'' < 0$ on (x_1, x_0), $f'(u) \geq f'(x_0)$. This means $f'(u) = f'(x_0)$, and so by the mean value theorem $f''(v) = 0$ for some $v \in (u, x_0) \subseteq I_0$, contradicting the fact that f is zero-free on I_0. ∎

Theorem 1. *A function is convex near an analytic point of convexity.*

Proof. Suppose a function f is analytic at a point of convexity x_0. Then f satisfies the assumption of Proposition 6 on some open neighborhood I of x_0. So, f is convex near x_0 if x_0 is not a limit point of the zeros of f''. Now, if x_0 is a limit point of the zeros of f'' then, as f'' is also analytic at x_0, it follows from the identity theorem of power series [17, Theorem 8.5] that f'' must be identically zero near x_0. Therefore, f is convex near x_0 as well. ∎

Our next example shows that a smooth function does not need to be convex or p-convex at a point of convexity.

Example 7. *The function*

$$f(x) = \begin{cases} e^{-1/x^2}(\cos(1/x^3) + 2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

is positive for nonzero x. Hence, 0 is a point of convexity of the function on $\mathbb{R}$. One checks that, near 0, the dominating term of $\varphi'(x, 0)$ and $f'(x)$, respectively, are

$$\frac{3e^{-1/x^2}}{x^5} \sin\left(\frac{1}{x^3}\right) \quad \text{and} \quad \frac{3e^{-1/x^2}}{x^4} \sin\left(\frac{1}{x^3}\right).$$

So, near 0, $\varphi'(x, 0)$ takes opposite signs on either side of 0, and therefore f is not convex at 0 with respect to any neighborhood of 0. Moreover, for all n sufficiently large,

$$f'(a_n) > f'(0) = 0 > f'(b_n)$$

where $a_n = -((2n + 1/2)\pi)^{-1/3} < 0 < b_n = ((2n - 1/2)\pi)^{-1/3}$. So, according to [6, Theorem 2], f is not p-convex at 0 with respect to any neighborhood of 0.

Next we focus on the pairwise implications of the notions in Definition 3. Since total convexity implies the others, we only need to focus on the remaining ones. From their characterizations in terms of Ψ, the following proposition is immediate.

Proposition 7. *If a function is convex at a point of convexity, then it is totally convex, and hence p-convex, at that point.*

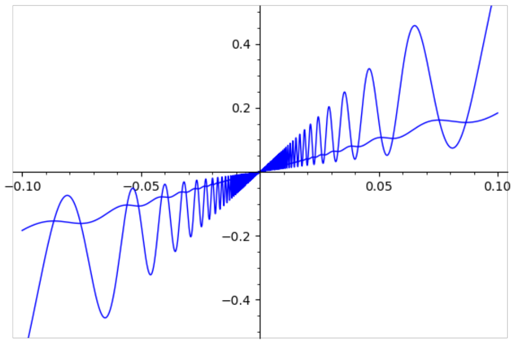

Figure 2: $f'(x)$ and $\varphi_f(x,0)$ of the function f in Example 8

Example 2 shows that a function f, with respect to some neighborhood of x_0, does not need to be convex at x_0 even if x_0 is a point of convexity of f and f is p-convex at x_0. The following example shows that this can happen even with respect to an arbitrarily small neighborhood of x_0.

Example 8. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$f(x) = \begin{cases} 2x^2 + 3x^3 \sin(1/x) & x \neq 0; \\ 0 & x = 0. \end{cases}$$

Then, $f(x) > 0$ for all nonzero x that is sufficiently close to 0. Thus, 0 is a point of convexity of f. Also, $f'(0) = 0$ and for $x \neq 0$,

$$\begin{aligned} f'(x) &= x(4 - 3\cos(1/x)) + 9x^2 \sin(1/x), \\ \varphi'(x,0) &= 2 - 3\cos(1/x) + 6x\sin(1/x). \end{aligned}$$

So, $f'(x)$ and x have the same sign near 0 (in fact, this is true for all x). Therefore, according to Theorem 2 of [6], f is p-convex at 0 with respect to any sufficiently small neighborhood of 0. On the other hand, $\varphi'(x,0)$ changes sign near 0. Therefore, f is not convex at 0 with respect to any neighborhood of 0.

The same kind of analysis shows that, near 0, the function defined by

$$\begin{cases} ax^{2n} + bx^{2n+1} \sin(1/x) & x \neq 0; \\ 0 & x = 0 \end{cases}$$

has 0 as a point of convexity and is p-convex but not convex at 0 as long as $0 < (2n-1)a < b < 2na$. In particular, taking $a = 2$ and $b = 4n - 1$ gives a family of C^n examples.

We now prove that convexity and punctual convexity at a point together imply the point in question must be a point of convexity.

Proposition 8. *If f is both convex and p-convex at a point, then that point must be a point of convexity of f.*

Proof. Suppose f is both convex and p-convex at a point x_0. By a translation of axes, we can assume $x_0 = 0$. Take any $x_1 < 0 < x_2$. Since either $0 < -x_1 \leq x_2$ or $x_1 \leq -x_2 < 0$, arguing with $f(-x)$ if necessary, we can assume $-x_1 \leq x_2$. The inequalities $\varphi(x_1, 0) \leq \varphi(-x_1, 0)$ and $\varphi(-x_1, 0) \leq \varphi(x_2, 0)$ follow, respectively, from f being p-convex and convex at 0. Consequently, $\varphi(x_1, 0) \leq \varphi(x_2, 0)$ establishing that $x_0 = 0$ is a point of convexity of f. □

4 Final Remarks

In this last section we include the results that we believe are worth mentioning but do not quite fit into our earlier discourse.

Another common way of defining convexity of a function f on an open interval I is via its *epigraph*:

$$\operatorname{epi}(f) = \{(x, y) \colon x \in I, y \geq f(x)\}.$$

Convexity of f on I is equivalent to $\operatorname{epi}(f)$ being a convex set [11, p. 115]. It is also equivalent to $\operatorname{epi}(f)$ having a *supporting line* at each point of I. By that we mean for each $x_0 \in I$, there is an affine function $\ell(x) = f(x_0) + m(x - x_0)$ such that $f(x) - \ell(x) \geq 0$ for any $x \in I$. A slight modification of a proof of this latter equivalence [16, Theorems D and E, p. 12] actually shows the following result (also see [12, Lemma 2.1] and [19, Propositions 16.2.3 and 16.2.4]).

Proposition 9. *A point x_0 is a point of convexity of f if, and only if,* $\operatorname{epi}(f)$ *has a supporting line at x_0.*

Proof. Suppose $\ell(x) = f(x_0) + m(x - x_0)$ defines a supporting line of $\operatorname{epi}(f)$ at x_0. Then being a minimum of the function $g := f - \ell$, x_0 is a point of convexity of g. Note that Ψ_ℓ is constantly zero for $\ell(x)$ being affine, so $\Psi_f = \Psi_{g+\ell} = \Psi_g + \Psi_\ell = \Psi_g$. Thus, x_0 is a point of convexity of f as well. Conversely, if x_0 is a point of convexity of f, then $\varphi(x_1, x_0) \leq \varphi(x_2, x_0)$ for any $x_1 < x_0 < x_2$. Therefore, both

$$s := \sup_{x_1 < x_0} \varphi(x_1, x_0) \quad \text{and} \quad u := \inf_{x_2 > x_0} \varphi(x_2, x_0)$$

exist and $s \leq u$. It is straightforward to verify that $\ell(x) := f(x_0) + m(x - x_0)$ defines a supporting line of $\operatorname{epi}(f)$ for any $s \leq m \leq u$. □

Corollary 1. *If f has one-sided derivatives at a point of convexity x_0 and $f'_-(x_0) \geq f'_+(x_0)$ then f is differentiable at x_0 and the tangent of f at x_0 is the unique supporting line of* $\operatorname{epi}(f)$ *at x_0.*

Proof. It is immediate from their definitions that

$$f'_-(x_0) \leq s := \sup_{x_1 < x_0} \varphi(x_1, x_0) \quad \text{and} \quad f'_+(x_0) \geq u := \inf_{x_2 > x_0} \varphi(x_2, x_0).$$

When x_0 is a point of convexity of f, the proof of Proposition 9 shows that $s \leq u$. If, in addition, $f'_-(x_0) \geq f'_+(x_0)$, then these four quantities must be the same. Hence, f is differentiable at x_0 and $f'(x_0)$ is their common value. In that case, as shown in the proof of Proposition 9, the tangent line of f at x_0 is a supporting line of the epigraph of f at x_0. For uniqueness, let $\ell(x) = f(x_0) + m(x - x_0)$ define a supporting line of $\operatorname{epi}(f)$ at x_0. Then $f - \ell$ has a minimum at x_0 and since f is differentiable at x_0, $(f - \ell)'(x_0) = f'(x_0) - m = 0$. So, $\ell(x)$ is defining the tangent line of f at x_0. ∎

Example 5 shows that a function does not have to be convex near a local minimum. We now go a step further showing how to construct a nonconvex minimum from a discontinuity point of a second derivative. Let g be a twice differentiable function on an open interval I with g'' discontinuous at some $x_0 \in I$. Let $\ell(x)$ be the linear function $g(x_0) + g'(x_0)(x - x_0)$ defining the tangent of $g(x)$ at x_0. Since $g(x)$ and $g(x) - \ell(x)$ have the same second derivative, replacing g by $g - \ell$, we can assume $g'(x_0) = 0$. By applying the following result (Theorem 2) to $h = g''$, we conclude either $L_{g''}(0)$ or $R_{g''}(0)$ contains a nonempty open interval (m, M). Pick c in (m, M) other than $g''(x_0)$. Then $g''(x) - c$ does not vanish at x_0, and on at least one side of x_0, $g''(x) - c$ takes both positive and negative values near x_0. The function $f(x) := g(x) - c(x - x_0)^2/2$ is twice differentiable with $f'(x_0) = g'(x_0) = 0$ and $f''(x_0) = g''(x_0) - c \neq 0$. Thus, arguing with $-f$ if needed, we can assume $f''(x_0) > 0$. And so, x_0 is a local minimum of f by the second derivative test [20, 3.65.a]. However, f cannot be convex near x_0, as $f''(x) = g''(x) - c$ changes sign in every neighborhood of x_0.

Theorem 2. *If a derivative $h\colon I \to \mathbb{R}$ is discontinuous at a point $x_0 \in I$, then at least one of the following two sets contains a nonempty open interval*

$$R_h(x_0) := \bigcap_{\delta > 0} h\left((x_0, x_0 + \delta)\right), \qquad L_h(x_0) := \bigcap_{\delta > 0} h\left((x_0 - \delta, x_0)\right).$$

Proof. Since h, a derivative, has a discontinuity point x_0, according to the result [8, Theorem 2.1] of Klippert, either $h(x_0^+)$ or $h(x_0^-)$ does not exist and neither of them is $+\infty$ or $-\infty$. Arguing with $h(-x)$ instead of $h(x)$, if necessary, we can assume $h(x_0^+)$ does not exist (and $h(x_0^+) \neq \pm\infty$). It follows from the intermediate value property of derivatives (Darboux's Theorem [2, 6.2.12], [4, Theorem 2.1]) that for each sufficiently small $\delta > 0$, $h\left((x_0, x_0 + \delta)\right)$ contains the open interval (m_δ, M_δ) where m_δ, M_δ are, as extended reals, the infimum and supremum of h on $(x_0, x_0 + \delta)$, respectively. As a result,

$$\bigcap_{\delta > 0} h\left((x_0, x_0 + \delta)\right) \supseteq \bigcap_{\delta > 0} (m_\delta, M_\delta) \supseteq (m, M)$$

where $m = \lim_{\delta \to 0^+} \inf h\left((x_0, x_0 + \delta)\right)$ and $M = \lim_{\delta \to 0^+} \sup h\left((x_0, x_0 + \delta)\right)$. It remains to argue that m is strictly less than M. Suppose on the contrary that $m = M$. Since for each sufficiently small $\delta > 0$,

$$-\infty \leq \inf h\left((x_0, x_0 + \delta)\right) \leq h(x_0 + \delta/2) \leq \sup h\left((x_0, x_0 + \delta)\right) \leq +\infty,$$

letting $\delta \to 0^+$, we conclude that $h(x_0^+) = m = M$. That contradicts the requirements being put on $h(x_0^+)$. ∎

Let us also note that in our construction of nonconvex minimum $g''(x_0)$ cannot fall outside the interval $[m, M]$ because of Darboux's Theorem. However, $g''(x_0)$ needs not be in (m, M). For instance, there is a differentiable function F on $\mathbb{R}$, given by Sahoo in [18], whose derivative f is nonnegative, discontinuous at 0 with $f(0) = 0$. In addition, f is nonconstant near 0, but vanishing at some point on either side of 0. So, for the function $g(x) := \int_0^x F(t)dt$, we have that $g''(x) = f(x)$ and

$$m := \lim_{\delta\to 0^+} \inf g''\left((x_0 - \delta, x_0)\right) = \lim_{\delta\to 0^+} \inf g''\left((x_0, x_0 + \delta)\right) = g''(0) = 0.$$

In Example 8, we give, for each n, a C^n function that is p-convex but not convex at 0 with respect to all sufficient small neighborhood of 0. However, no analytic example is possible as the p-convex point must be a point of convexity [6, Lemma 3] and so according to Theorem 1 the function must be convex at that point with respect to some neighborhood. This leaves us with the natural question: *Is there a smooth function f that is p-convex, but not convex, at* 0*, with respect to any neighborhood of* 0*, no matter how small?*

To give a better overview of the relationships between different notions of pointwise convexity established in this article, we display them in the following diagram and table.

$$\begin{tikzcd}
 & \text{convex near } x_0 \arrow[d, Rightarrow] & \\
 & \text{totally convex at } x_0 \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{p-convex at } x_0 \arrow[r, Rightarrow, "\text{continuity}"'] & \begin{array}{l} x_0 \text{ is a point of} \\ \text{convexity} \end{array} \arrow[d, Rightarrow, "\text{analytic}"', "\text{at } x_0"] & \text{convex at } x_0 \arrow[l, Rightarrow, "f'_-(x_0)\leq f'_+(x_0)"] \\
 & \text{convex near } x_0 &
\end{tikzcd}$$

Figure 3: Relationships between different notions of pointwise convexity

Statement			Reference
convex at x_0	$\not\Rightarrow$	p-convex at x_0 or x_0 is a point of convexity	Example 1
p-convex at x_0 and x_0 is a point of convexity	$\not\Rightarrow$	convex at x_0	Example 2, Example 8
p-convex at x_0	$\not\Rightarrow$	convex at x_0 or x_0 is a point of convexity	Example 3
convex at x_0 and x_0 is a point of convexity	$\Longrightarrow$	totally convex (and hence p-convex) at x_0	Proposition 7, Proposition 3
x_0 is a point of convexity	$\not\Rightarrow$	convex at x_0 or p-convex at x_0	Example 2, Example 7
convex at x_0 and p-convex at x_0	$\Longrightarrow$	x_0 is a point of convexity (and hence the function is totally convex at x_0)	Proposition 8, Proposition 7

Table 1: A summary of results: joint implications and missing arrows in Figure 3

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