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Flurries of Ducci waves
by
Cristian Cobeli[(1)], Alexandru Zaharescu[(2)]

Dedicated to Professors Constantin Năstăsescu and Toma Albu
on the occasion of their 80th birthdays

Abstract

We introduce the class of trees with integer nodes generated by the Ducci process applied on the digit representation of the nodes in some number system. There are two possible options: one is to let the size of the torus go down freely during the process and the other is to keep it fixed. We give revealing examples and discuss general characteristics in the two cases.

Key Words: Ducci games, cycle, precycle, period, discrete dynamics.
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1 Introduction

Started probably in the late 19th century [15, 18] as a simple challenging problem with numbers, the Ducci game developed into a provocative subject with features linking combined aspects from algebra [4, 5, 11], combinatorics, geometry of numbers, number theory [2, 3, 6–9, 12, 21, 24] cellular automata, discrete dynamical systems and/or theoretical computer science [16, 17, 20, 22, 23, 25–27].

In the original settings, a tuple of $d \geq 2$ integers are placed around a circle (a torus of dimension one) and then, recursively, these numbers are replaced in between by the absolute value of the difference of the adjacent numbers. As the size of the tuple, called also the size of the torus, is finite and the set of natural numbers has a first element, the process eventually enters into a cycle. Notable is the fact that whatever the starting d-tuple, the final cycle consists only of zeros if and only if d is a power of two. This has been demonstrated several times independently by different authors approaching the subject from different angles (see [1, 4, 13–15, 17, 27, 30] and the references therein). A distinguished motivation for the outgrowth was the entry of the subject into mathematical folklore through its occasional tempting popularization in educational mathematics journals [1, 10, 13, 15, 17, 27–30].

In a related variant, the process can be described as a sequence of numbers, such as

$$1001 \to 1012 \to 1133 \to 2464 \to 6006 \to 6062 \to 6688 \to 2464 \to 6006, \dots \tag{1.1}$$

Here, the subsequence $[2464, 6006, 6002, 6688]$ repeats and is called *cycle*. The *length of the cycle* is called *period*, a term that is also used for multiples of the length of a cycle. The subsequence of numbers before the cycle, $[1001, 1012, 1133]$ in (1.1), is called *precycle*.

Characterizing the length of cycles and precycles is one of the interesting and complex problems of Ducci processes in their various variants. If the components of the game are seen as the digits of numbers written in a numeration base, the maximum length of the cycles is studied by Breuer [4] and Dular [19].

To explain the drive behind the action, let $b, d \geq 2$ be the *base* and the *size* (also called *level* or *dimension*), and let $1 \leq h \leq d-1$ be a *displacement*, which are supposed to be fixed integers. Consider the operator $T_{b,h} : \mathbb{Z}_b^d \to \mathbb{Z}_b^d$ defined by

$$T_{b,h}(n_1, \ldots, n_d) := \big((n_1 + n_{1+h}) \,(\mathrm{mod}\, b)\,, (n_2 + n_{2+h}) \,(\mathrm{mod}\, b)\,, \ldots, (n_d + n_{d+h}) \,(\mathrm{mod}\, b)\,\big)\,,$$

where the notation of the subscripts is circular mod d, that is, n_j with j larger than d is the same as n_{j-d}. Adjoining the operations of representation of an integer in base b before and after the application of $T_{b,h}$, we associate $T_{b,h}$ with an operator $\tilde{T}_{b,h}(n)$ that acts on positive integers. To preserve the size during the iterations, the convention is to see the smaller numbers as having in front zero digits for all necessary large orders. Thus, in size $d \geq 2$, if $n_1, \ldots, n_d$ are the digits of the representation of n, in base b, that is, $n = \overline{n_1 n_2 \cdots n_d} = n_1 b^{d-1} + \cdots + n_{d-1} b + n_d$, and $T_{b,h}(n_1, \ldots, n_d) = (m_1, \ldots, m_d)$, then $\tilde{T}_{b,h}(n) = m$, where $m = \overline{m_1 m_2 \cdots m_d} = m_1 b^{d-1} + \cdots + m_{d-1} b + m_d$. On torus of size d, this holds for integers $n \in [0, b^d - 1]$ and some of the digits $n_1, n_2, \ldots$ are allowed to be equal to zero. The case of collapsing levels, where the cluster of zeros from the left of the tuples are left aside, is discussed in Section 2. In any case, $[0]$ is always a cycle of length 1. But there are other non-trivial cycles, which by chance are very long, as may happen in fortuitous arithmetic concurences, such as those in level $d = 31$ from Table 2, or if 2 is a primitive root mod d (see the remarks from [17, Section 2] and the levels $d = 19, 29$ in Table 2). With these definitions, the details left in the shade in the generation of sequence (1.1) are shown in Table 1. Accordingly, the numbers from 0 to $b^d - 1$ are organized in a well-favored hierarchy of trees with roots in cycles.

TABLE 1. *The sequence of iterations $\tilde{T}_{b,h}^{(k)}(n)$ for $k \geq 0$, with $n = 1001$, $b = 10$, $h = 1$, and $d = 4$.*

j	n	digits of n in base b	$T_{b,h}(n)$	$\tilde{T}_{b,h}(n)$
1	1001	$(1,0,0,1)$	$(1,0,1,2)$	1012
2	1012	$(1,0,1,2)$	$(1,1,3,3)$	1133
3	1133	$(1,1,3,3)$	$(2,4,6,4)$	2464
4	2464	$(2,4,6,4)$	$(6,0,0,6)$	6006
5	6006	$(6,0,0,6)$	$(6,0,6,2)$	6062
6	6062	$(6,0,6,2)$	$(6,6,8,8)$	6688
7	6688	$(6,6,8,8)$	$(2,4,6,4)$	2464

The representation of such a structure for base $b = 10$, displacement $h = 1$ and size $d = 2$ is shown in Figure 1. In general, the whole system depends on the decomposition in prime factors of b and d, and, to a considerable degree, on the 'parity', more precisely particularly on the number of factors of 2 they have.

In the case $b = 10$, $h = 1$, and $d = 4$, from which the sequence (1.1) was extracted, the numbers from 0 to 9999 are arranged as trees rooted into the cycles $[0]$, $[4268, 6842]$, $[2684, 8426]$ and 30 other cycles of length four each. Together, these disjoint cycles take $1 + 2 \cdot 2 + 30 \cdot 4 = 125$ values. We remark that there are no odd digits in the representation of the numbers in the cycles, but not all numbers with all digits even belong to the cycles.

The remaining $10000 - 125 = 9875$ numbers belong to precycles, which may have lengths $1, 2, 3$ or 4.

The shape is quite different on the neighbor torus of size $d = 3$, and the same $b = 10$ and $h = 1$. In this case, there are three singular cycles $[0]$, $[55, 505, 550]$ and $[222, 444, 888, 666]$, 16 cycles of length six and 33 cycles of length twelve. All together these disjoint cycles take $1+3+4+16\cdot 6+33\cdot 12 = 500$ values. The other $1000-500 = 500$ nonegative integers ≤ 999 belong to precycles of length one each.

We remark that if $b = 2$ and $h = 1$, in all tori of size $d \geq 2$, the operator $T_{b,h}$ coincides with the original Ducci transformation. The reason is that iterating the operation of taking the absolute value of the difference between neighbor integers placed around a torus in dimension one decreases the numbers and, eventually, it leads to a tuple of numbers having at most two distinct components of which one is necessarily zero (see [17]).

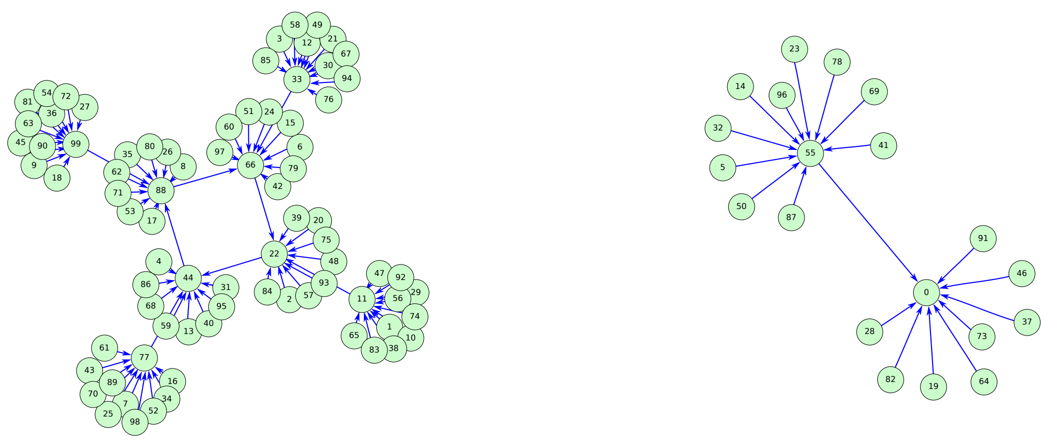

FIGURE 1. *The crown–rooted tree hierarchy of integers* $n \in \{0, 1, \ldots, 99\}$ *generated by the Ducci operator* $\tilde{T}_{b,h}(n)$ *for* $b = 10$, $h = 1$, $d = 2$. *There are two cycles* $[0]$ *and* $[22, 44, 88, 66]$, *and* 95 *numbers in the precycles of length* 1 *or* 2.

For each base $b \geq 2$, and size $d \geq 2$, the integers $0 \leq n \leq b^d - 1$ are organized by the Ducci operator in a hierarchy expressed as series of '*crown-rooted trees*' bond on the cycles. There are two types in this class of structures, according to whether the level is kept fixed or it is dropping during the iterations. In addition to Figures 1 and 2, in Section 4, we present and compare in Figures 3–6 the graphic representations of several relevant examples of the two types of members of this class.

2 The game with dropping levels

In our original game defined by the interations of the $\tilde{T}_{b,h}(n)$ operator, the size is preserved during the evolution by adding zeros in front of the representations in base b of the numbers that have fewer digits than the apriori fixed size. In this section we discuss the case where no more zeros are added. Adapting to context, we will call *level* instead of size or dimension the size of the torus at a given moment. Since the level will no longer be fixed, any number can be part of the game. Thus, if $n \geq 1$, the Ducci operator will be applied for the level $d = \lfloor \log_b(n) \rfloor + 1$ to the tuple of digits $(n_1, \ldots, n_d)$, $0 \leq n_j < b$, $n_1 > 0$. By convention, if $d = 1$, at the next step, (n) becomes $\big(2n \,(\mathrm{mod}\, b)\big)$, since geometrically, on the one dimentional torus, n is next only to itself. Also, by definition, the level of $n = 0$ is 1.

Formally, to define the process, let $b \geq 2$ and $h \geq 1$ be fixed. Then define $\tilde{U}_{b,h} : \mathbb{N} \to \mathbb{N}$ by the same formula from the Introduction, $\tilde{U}_{b,h}(n) = \tilde{T}_{b,h}(n)$, where $\tilde{U}_{b,h}(0) = 0$, and if $n \geq 1$ the level is $d = d(n) = \lfloor \log_b(n) \rfloor + 1$ and $(n_1, \dots, n_d)$ are the digits of n in base b, with $0 \leq b_1, \dots, b_d < b$ and $b_1 > 0$. Then the sequence of iterations $\tilde{U}_{b,h}^{(k)}(n)$ is composed by pieces of the sequences $\tilde{T}_{b,h}^{(k)}(n)$ calculated for various fixed levels that are merged together. The result is a sequence of integers that is not necessarily increasing or decreasing, but the levels can only be stationary or occasionally drop. Eventually, the sequence $\tilde{U}_{b,h}^{(k)}(n)$ for $k \geq 1$ enters into a cycle, also.

For example, if $b = 7$, $h = 1$, and $k \geq 0$, the sequence $\tilde{U}_{b,h}^{(k)}(440)$, is

$$440 \to 721 \to 968 \to 151 \to 175 \to 31 \to 0 \to 0 \to, \dots$$

because on the digits side the Ducci process mod 7 is

$$(1,1,6,6) \to (2,0,5,0) \to (2,5,5,2) \to (\cancel{0},3,0,4) \to (3,4,0) \to (\cancel{0},4,3) \to (\cancel{0},0) \to, \dots$$

For other examples compare the hierarchy generated by the iterations of $\tilde{U}_{b,h}$ in Figure 2 with that of the $\tilde{T}_{b,h}$ operator in Figure 1 and also see Figure 4.

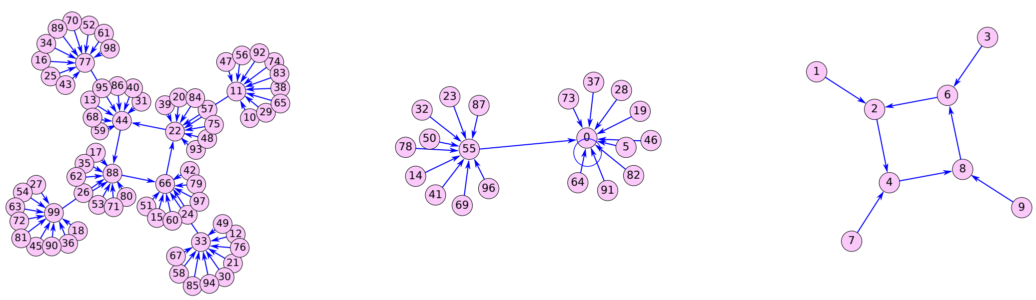

FIGURE 2. *The crown-rooted tree hierarchy of integers $n \in \{0, 1, \dots, 99\}$ generated by the Ducci operator with dropping levels allowed $\tilde{U}_{b,h}(n)$ for $b = 10$ and $h = 1$. Here are three cycles $[0]$, $[2,4,8,6]$ and $[22,44,88,66]$, and 90 numbers in the precycles of length 1 or 2.*

Notice that on even levels the largest gap between consecutive numbers obtained during the iterations of $\tilde{U}_{b,1}^{(k)}$ or $\tilde{T}_{b,1}^{(k)}$ are produced by the 'leap-up' from n with digits $(1, b-2, 1, b-2, \dots)$ to $b^d - 1$, and the 'leap-down' from n with digits $(b-1, 1, b-1, 1, \dots)$ to 0.

Cycles can be longer than 1, but notice that all its terms are always at the same level. If $b = 3$ and $h = 1$, an example of a cycle of length 6 is $[9, 10, 14, 18, 20, 25]$, whose Ducci action on the digits side is $(1,0,0) \to (1,0,1) \to (1,1,2) \to (2,0,0) \to (2,0,2) \to (2,2,1)$. Into this cycle enters the branch with the leaf 120 in four steps because: $(1,1,1,1,0) \to (2,2,2,1,1) \to (1,1,0,2,0) \to (2,1,2,2,1) \to (\cancel{0},\cancel{0},1,0,0)$.

A clear difference from the case of the Ducci game on a fixed level is that in base $b = 2$ the only possible cycle is $[0]$, a fact that is proved in the next theorem.

Theorem 1. *Let $b = 2$ and $h \geq 1$. Then, for any integer $n \geq 0$, there exists $k_0 \geq 0$ such that $\tilde{U}_{b,h}^{(k)}(n) = 0$ for all $k \geq k_0$.*

Proof. If $n=1$, by the definition, $\tilde{U}_{b,h}^{(k)}(1)=0$ for all $k\geq 2$.

Suppose that $n\geq 2$. On the first step the level is $d=\lfloor\log_b(n)\rfloor+1$ and suppose the digits of n are $(n_1,\dots,n_d)$, where $n_1=1$ and $0\leq n_2,\dots,n_d\leq 1$. If $n_{h+1}=1$, then $n_1+n_{h+1}\equiv 0\,(\mathrm{mod}\ 2)$, so that the level of $\tilde{U}_{b,h}^{(2)}(n)$ is strictly less than d, being dropped by at least 1. If $n_{h+1}=0$, then the repeated application of $\tilde{U}_{b,h}$ to the successive outcomes, after at most $d-1$ steps, will make the $(h+1)$th component equal to 1 and, consequently, the level will drop at the next iteration. Then the theorem follows by inverse induction. □

As observed before, if $b=2$, it may happen that the evolution of the game at level d may arrive at the integer 2^d-1, from which the whole process will end in 0 in just one step, because $\tilde{U}_{b,h}(2^d-1)=0$. In any case, starting from n, the repeated application of $\tilde{U}_{b,h}$ will drop from its level, possibly by skipping some levels, and ending in 0 in a maximum number of $d+(d-1)+\cdots+1=d(d+1)/2$ steps, where d is the level of n.

If $b>2$, there are integers n for which $\{\tilde{U}_{b,h}^{(k)}(n)\}_{k\geq 0}$ ends in longer cycles.

Question (L)**.** *Given $L\geq 2$, does there always exist integers $b\geq 3$ and $n>0$ for which the sequence $\{\tilde{U}_{b,h}^{(k)}(n)\}_{k\geq 0}$ ends in a cycle of length L?*

Table 2. *Long cycles generated by the iterations $\tilde{U}_{b,h}^{(k)}(n)$ for $k\geq 0$, with $b=9$ and $h=1$. The initial leafs are the numbers $n=4,13,94,823,\dots$, whose representations in base 9 are 4_9, 14_9, 114_9, 1114_9,... The precycles are empty if $d=1,3,5,7,9$, and* [13] *if $d=2$,* [823] *if $d=4$,* [66433, 132890, 265939] *if $d=6$,* [5380843] *if $d=8$, and* [435848053] *if $d=10$. Here, by periods we mean the lengths of the cycles.*

d	n	digits of n in base b	cycle	period
1	4	(4)	[4, 8, 7, 5, 1, 2]	6
2	13	$(1,4)$	[50, 10, 20, 40, 80, 70]	6
3	94	$(1,1,4)$	[94, 212, 583, 725, 607, 236]	6
4	823	$(1,1,1,4)$	[1670, 3499, 2180,...,3520]	24
5	7384	$(1,1,1,1,4)$	[7384, 14792, 29743,...,19190]	120
6	66433	$(1,1,1,1,1,4)$	[527060, 421210, 492020, 70810, 176660, 105850]	6
7	597874	$(1,1,1,1,1,1,4)$	[597874, 1195772,...,1554464]	546
8	5380843	$(1,1,1,1,1,1,1,4)$	[10761710, 21523579,...,36051631]	24
9	48427564	$(1,1,1,1,1,1,1,1,4)$	[48427564, 96855152,...,125911658]	18
10	435848053	$(1,1,1,1,1,1,1,1,1,4)$	[871696130, 1743392419,...,1874146609]	240

d	length of precycle	period	d	length of precycle	period
11	0	726	24	3	24
12	3	24	25	0	4 428 600
13	0	78	26	1	78
14	1	1 092	27	0	54
15	0	120	28	1	2184
16	1	240	29	0	416 118 216
17	0	83 640	30	3	240
18	9	18	31	0	102 649 866
19	0	1 121 874	32	1	19680
20	1	240	33	0	726
21	0	546	34	1	669120
22	1	726	35	0	797160
23	0	531 438	36	9	72

Table 2 includes a list of long cycles in which enters fast the sequence started by the integers n whose representation in base 9 are $(1,1,\dots,1,4)$. Notice the especial d-digits numbers with $d=19,25,29,31$ that generate cycles longer than one million.

For some small lengths L the answer to Question L is positive, in abundance, with some infinite 'threads' of cycles. For instance, if $b = 3$, a thread of cycles of length two is: $[4, 8]$; $[13, 26]$; $[40, 80]$; $[121, 242]$; $[364, 728]$; $[1093, 2186]$;..., with the general formula: $\big[(3^k - 1)/2, 3^k - 1\big]$ for $k \geq 2$.

If $b = 5$, a thread composed by cycles of length four is: $[6, 12, 24, 18]$; $[31, 62, 124, 93]$; $[156, 312, 624, 468]$;... whose general formula is: $\big[(5^k - 1)/4, (5^k - 1)/2, (5^k - 1), 3(5^k - 1)/4\big]$ for $k \geq 2$.

3 Ducci process on fixed levels

The Ducci game works by repeatedly transforming a finite string of numbers arranged around a one-dimensional torus. These strings are the intermediate steps that account for the evolution of the process. We express the different Ducci operators as matrices that are applied on row-numbers written as tuples or vectors. The entries of vectors circulate around during the process and, while denoted, by convention, with the same bold letters, depending on the context, they can be thought of and written with the same meaning both horizontally or vertically.

Consider the near identical $d \times d$ matrices

$$A_d = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \quad \text{and} \quad B_d = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \tag{3.1}$$

Let us note the transformations produced on a matrix X of suitable size by the multiplication by A_d and B_d:

Remark 1. *a. The lines of X are circularly shifted in A_dX by one position from bottom to top and the columns of X are circularly shifted in XA_d by one position from left to right.*

b. The columns of X are circularly shifted in XB_d by one position from right to left and the lines of X are circularly shifted in B_dX by one position from top to bottom.

c. In particular, we see that if $d \geq 1$, then $A_d^d = A_d$ and $B_d^d = B_d$.

Then, the Ducci operations are done by applying on vectors the operators

$$M_d := I + A_d \qquad \text{and} \qquad N_d := I + B_d, \tag{3.2}$$

where I is the identity matrix. Both M_d and N_d or combinations of their powers can be used to generate Ducci-type processes. Here we stick to M_d, which is the closest the original Ducci game.

Let p be a prime number and let $d \geq 2$ be the size (or level) of a discrete torus. The *integer Ducci operation* (mod p) is the result of the operator $D_d : \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ defined by

$$D_d(\boldsymbol{x}) := (x_0 + x_1, x_1 + x_2, \dots, x_{d-1} + x_0)$$

for any $\boldsymbol{x} = (x_0, \dots, x_{d-1})$. The Ducci process is the sequence produced by the iterations $D_d^{(k)}(\boldsymbol{x})$, $k \geq 0$, applied on the initial positions $\boldsymbol{x} \in \mathbb{Z}_p^d$. With the elements of $\mathbb{Z}_p^d$ viewed as column vectors, we see that

$$D_d(\boldsymbol{x}) = (I + A_d)\boldsymbol{x} = M_d\boldsymbol{x}\,. \tag{3.3}$$

An immediate calculation finds that the characteristic polynomial of M_d is

$$\psi_d(t) = (t-1)^d - 1. \tag{3.4}$$

It follows that 1 is not an eigenvalue of the matrix M_d, thus there is no nonzero vector $\boldsymbol{v}$ such that $M_d\boldsymbol{v} = \boldsymbol{v}$. Therefore, the only cycle of length one is $[\mathbf{0}]$, where $\mathbf{0} = (0, \dots, 0)$.

Example 1. *Three examples with small bases $b = p$ and sizes $d = p^k$. Here the vectors that are steps in the Ducci process are expressed, as in Introduction, by the numbers whose digits in base b are their components. Note that if $p = 3$ and 5, then 2 is a primitive root mod p and the order of 2 mod 9 is 6. Checking the evolution of all integers $n \in [0.b^d - 1]$, one finds:*

1. If $p = 3$ and $d = 3$, then $[0]$ and $[13, 26]$ are the only cycles of length one and two, respectively, and there are four other cycles of length 6.

2. If $p = 3$ and $d = 3^2$, then $[0]$ and $[9841, 19682]$ are the only cycles of length one and two, respectively. Also, there are four cycles of length 6 and 1092 cycles of length 18. Two of the longer ones are $[757, 3028, 12112, 1514, 6056, 17411]$ and $[1, 4, 16, 28, 112, 448, 784, 3136, 12301, 2, 8, 23, 56, 224, 644, 1487, 5948, 17222]$.

We remark that in base 3 the digits of the essential generators of the cycles are: $757 = (0, 0, 1, 0, 0, 1, 0, 0, 1)$ and $9841 = (1, 1, 1, 1, 1, 1, 1, 1, 1)$.

3. If $p = 5$ and $d = 5$, then $[0]$ and $[781, 1562, 2343, 3124]$ are the only cycles of length 1 and 4, respectively, and there are other 156 cycles of length 20.

Theorem 2. *Let the base $b = p$ be an odd prime and let $d = p^k$, where $k \geq 1$, be the size of the torus. Denote by $o_p = \operatorname{ord}_p(2)$ the order of 2 modulo p. Then:*

1. Any cycle length of the Ducci operation on $\mathbb{Z}_p^d$ must be a divisor of do_p.

2. The order o_p is a cycle length and any cycle length greater than 1 is divisible by o_p.

3. The order o_p is not the largest cycle length.

4. If $d = p$, then 1, o_p and do_p are the only possible cycle lengths.

5. Let $b, d \geq 2$ be integers. Then, all possible cycle lengths of Ducci games on $\mathbb{Z}_b^d$ are also possible lengths of cycles on $\mathbb{Z}_b^e$, for any e that is divisible by d.

Proof. 1. Note first that since 1 is not an eigenvalue of M_d, the only cycle of length one is $[\mathbf{0}]$. Then, since

$$\begin{aligned} M_d^d = (I + A_d)^{p^k} &\equiv I + A_d^{p^k} \equiv 2I \pmod{p}\,, \text{ which implies} \\ M_d^{do_p} = (M_d^d)^{o_p} &\equiv (2I)^{o_p} \equiv I \pmod{p}\,, \end{aligned} \tag{3.5}$$

it follows that any cycle length must be a divisor of do_p.

2. For $d = p^k$, in $\mathbb{Z}_p[t]$ the characteristic polynomial (3.4) of M_d becomes $t^d - 2 = (t-2)^d$. Thus 2 is the only eigenvalue of M_d. Let $\boldsymbol{v} \in \mathbb{Z}_p^d$ be an eigenvector, that is, $M_d\boldsymbol{v} = 2\boldsymbol{v}$.

Then it follows that $M_d^{o_p}\boldsymbol{v} = 2^{o_p}\boldsymbol{v} \equiv \boldsymbol{v} \pmod p$, by the definition of o_p. This means that $\boldsymbol{v}$ is in a cycle of length o_p and also that o_p is the shortest possible cycle length. Furthermore, also by the definition of the order o_p, it follows that any longer cycle lengths are multiples of o_p.

3. Let us note that $M_d^{\lambda} = I$ if λ is the length of the longest cycle. But $M_d^{o_p}$ cannot be the identity, since, otherwise, $M_d^{p-1} = I$, because o_p divides $p-1$. Then this would imply $M_d^p = M_d$ and, recursively, $M_d^d = M_d$. But this contradicts (3.5), from which we know that $M_d^d \equiv 2I \pmod p$. Consequently, o_p is not the largest length of a cycle.

4. This part is the case $k = 1$ and follows directly from parts 1-3.

5. If $\boldsymbol{x} \in \mathbb{Z}_b^d$, then denote by $\overline{\boldsymbol{x}}$ a vector that contains a number of copies of the components of $\boldsymbol{x}$. Then, if $\boldsymbol{x} \in \mathbb{Z}_b^d$ is part of a cycle, it follows that $\overline{\boldsymbol{x}} \in \mathbb{Z}_b^e$ is also part of a cycle of the same length, for any e that is divisible by d, since, by the definition of the Ducci operator, the game evolves mirrored identically on each of the e/d copies of $\boldsymbol{x}$ that compose $\overline{\boldsymbol{x}}$. □

Notice that in part 2 of Example 1, $p = 3$, $d = 3^2$, $o_3 = 2$ and the lengths of cycles are $1, 2, 6, 18$. These values do not include 3 as a length of a cycle, although 3 divides $9 \cdot 2$, in agreement with the first three parts of Theorem 2.

Remark 2 (Number of Cycles for $d = p^k$)**.** *In the proof of Theorem 2, let $\nu(k)$ be the number of eigenvectors of $M_d^{o_p}$ corresponding to the eigenvalue 1, so that, $1 \le \nu(k) \le d-1$. Then the number of distinct cycles of length o_p is $(p^{\nu(k)} - 1)/o_p$ and the number of distinct cycles of length $o_p p$ is $\left(p^d - p^{\nu(k)}\right)/o_p p$.*

If 2 is a primitive root modulo p, then Theorem 2 ensures the possibility of the existence of only the following cycle lengths.

Corollary 1. *Let p be an odd prime and let $d = p^k$, with $k \ge 1$, be the size of the torus. Suppose 2 is a primitive root mod p. Then the only possible cycle lengths that divide $p(p-1)$ are 1, $p-1$ and $p(p-1)$.*

There is a distinction between the aspect of the trees generated by the Ducci process on fixed size $d \ge 2$, a distinction which is determined by the parity of d (see Breuer [4, Theorem 3.2]). The remarkable fact is that if p is an odd prime, the base is p^k, $k \ge 1$, and the size d is odd, then all precycles are empty, while this never happens if the size d is even. We can use this fact to see that any vector $\boldsymbol{x}$ can canonically be embeded as $\boldsymbol{x}^{\lceil e \rceil}$ into infinity Ducci many processes of sizes e so that $\boldsymbol{x}^{\lceil e \rceil}$ is part of some cycle.

Theorem 3. *Let $d \ge 1$ be odd, let p be an odd prime, and let $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{Z}_p^d$. Then there exist infinitely many sizes $e > d$ such that by embedding $\boldsymbol{x}$ into $\mathbb{Z}_p^e$ as the vector $\boldsymbol{x}^{\lceil e \rceil} = (0, \ldots, 0, x_1, \ldots, x_d) \in \mathbb{Z}_p^e$, $\boldsymbol{x}^{\lceil e \rceil}$ lies in a cycle of the Ducci process in $\mathbb{Z}_p^e$.*

Proof. Let $e = p^k d$ for some $k \ge 1$, and let $s = \mathrm{ord}_d(p)$ be the order of $p \mod d$, that is, $p^s \equiv 1 \pmod d$. Then $M_d^{p^{k+s}} = (I + A_d)^{p^{k+s}} = (I + A_d)^{p^k} = M_d^{p^k}$. Thus, applying Ducci operation p^k times, we see that any vector is in a cycle of period $p^k(p^s - 1)$.

Note that since e is odd, 0 is not a root of the characteristic polynomial of M_e. Thus $\mathrm{Det}\, M_e \ne 0$ which implies $\mathrm{Det}\, M_e^{p^{k+s}} \ne 0$. So for any $\boldsymbol{v} \in \mathbb{Z}_p^e$, there is $\boldsymbol{w} \in \mathbb{Z}_p^e$ such that $M_e^{p^{k+s}} \boldsymbol{w} = \boldsymbol{v}$. As $M_e^{p^k} \boldsymbol{w}$ is in a cycle, $\boldsymbol{v}$ is in a cycle.

To conclude the proof of the theorem, we notice that the embedding of $\boldsymbol{x}$ into $\boldsymbol{x}^{\lceil e \rceil}$ makes sense for infinitely many odd e larger than d. □

4 Crown-rooted trees generated by Ducci processes

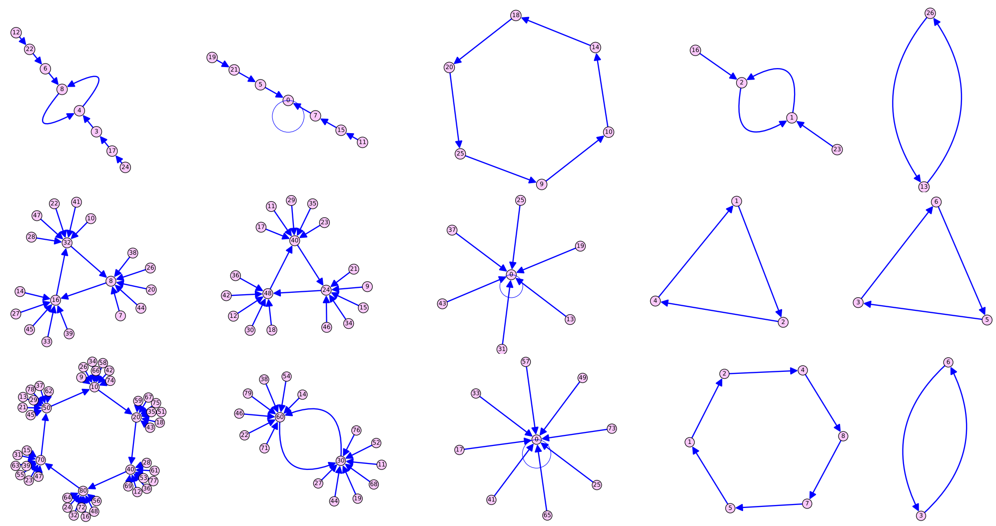

FIGURE 3. *Droping levels. The crown–rooted tree hierarchy of integers $n \in \{0, 1, \ldots, b^d - 1\}$ generated by the Ducci operator $\tilde{U}_{b,1}(n)$ for $b = 3$, $d = 3$ (top); $b = 7$, $d = 2$ (middle); $b = 9$, $d = 2$ (bottom).*

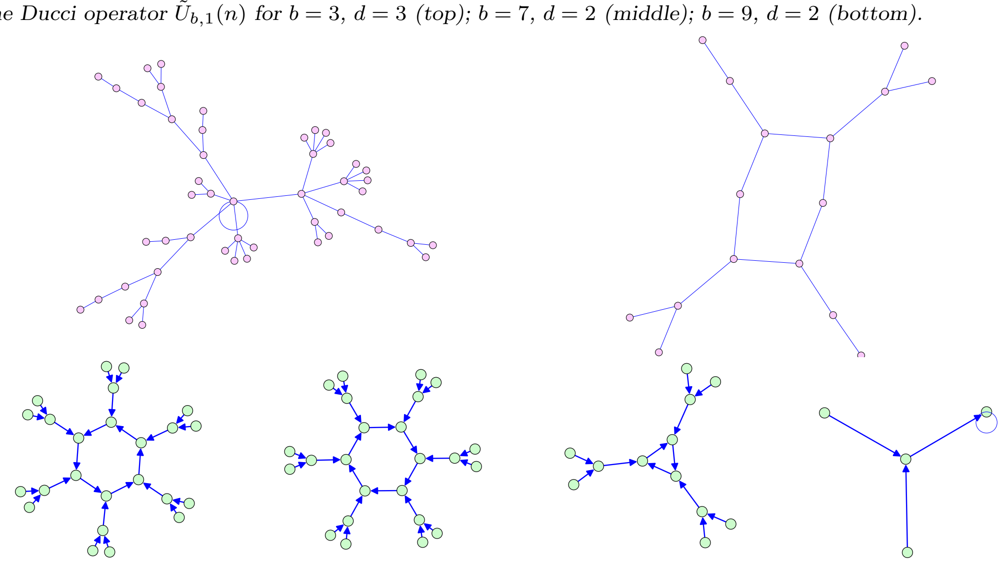

FIGURE 4. *Comparison between dropping level (top) and fixed level (bottom) cases for $b = 4$ and $d = 3$. The tree hierarchy of integers $n \in \{0, 1, \ldots, 4^3 - 1\}$ generated by $\tilde{U}_{b,1}(n)$ (top) and $\tilde{T}_{b,1}(n)$ (bottom).*

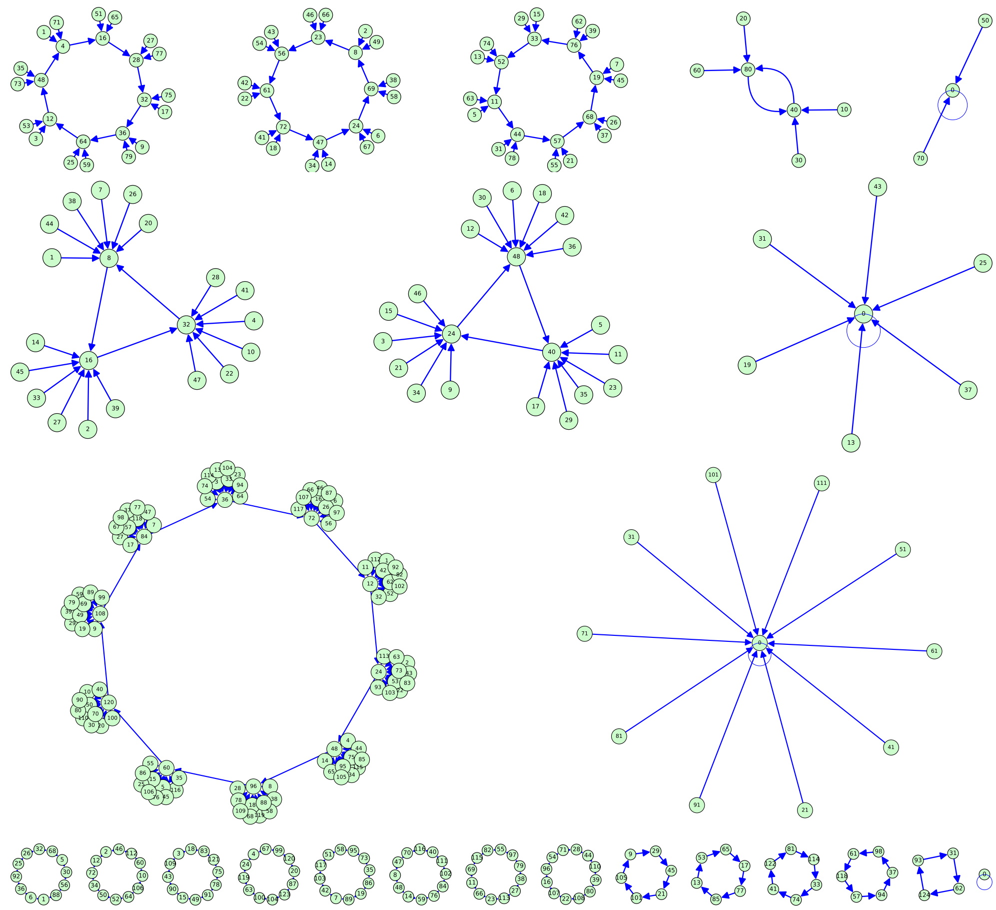

FIGURE 5. *Fixed levels. The crown–rooted tree hierarchy of integers $n \in \{0, 1, \ldots, b^d - 1\}$ generated by the Ducci operator $\tilde{T}_{b,1}(n)$ for $b = 3$, $d = 4$ (top); $b = 7$, $d = 2$ (middle up); $b = 11$, $d = 2$ (middle down); $b = 5$, $d = 3$ (bottom).*

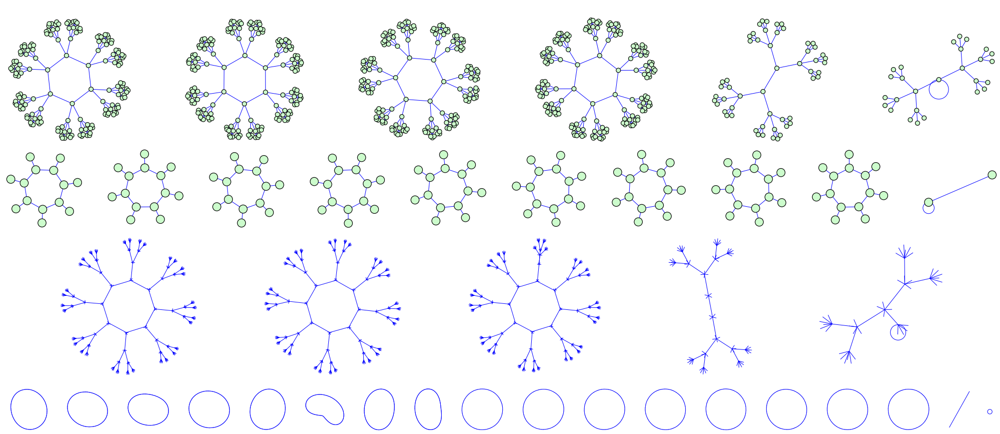

FIGURE 6. *Fixed levels. The crown–rooted tree hierarchy of integers $n \in \{0, 1, \ldots, b^d - 1\}$ generated by the Ducci operator $\tilde{T}_{b,1}(n)$ for $b = 3$, $d = 6$ (top); $b = 2$, $d = 7$ (middle up); $b = 6$, $d = 4$ (middle down); $b = 3$, $d = 7$ (bottom).*

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[1] Simion Stoilow Institute of Mathematics of the Romanian Academy,
P. O. Box 1-764, RO-014700 Bucharest, Romania
email: `cristian.cobeli@imar.ro`

[2] Simion Stoilow Institute of Mathematics of the Romanian Academy,
P. O. Box 1-764, RO-014700 Bucharest, Romania
and
Department of Mathematics, University of Illinois at Urbana-Champaign,
Altgeld Hall, 1409 W. Green Street, Urbana, IL, 61801, USA
email: `zaharesc@illinois.edu`