Bull. Math. Soc. Sci. Math. Roumanie
Tome 65 (113), No. 3, 2022, 329–340

Domination number in the Zariski topology-graph of modules

by

Shokoufeh Habibi[(1)], Zohreh Habibi[(2)], Masoomeh Hezarjaribi[(3)]

Abstract

Let M be a module over a commutative ring and let $Spec(M)$ be the collection of all prime submodules of M. One can define a Zariski topology on $Spec(M)$, which is analogous to that on $Spec(R)$, and then for any non-empty set T of $Spec(M)$, it is possible to define a simple graph $G(\tau_T)$, called the Zariski topology-graph. In this paper, we study the domination number of $G(\tau_T)$ and some connections between the graph-theoretic properties of $G(\tau_T)$ and algebraic properties of the module M.

Key Words: Rings and modules, Zariski topology, graph, domination number.
2010 Mathematics Subject Classification: Primary 13C13, 13C99; Secondary 05C75.

1 Introduction

Throughout this paper R is a commutative ring with a non-zero identity and M is a unital R-module. By $N \leq M$ (resp. $N < M$) we mean that N is a submodule (resp. proper submodule) of M.

Define $(N :_R K)$ or simply $(N : K) = \{r \in R | \ rK \subseteq N\}$ for any $N, K \leq M$. We denote $((0) : M)$ by $Ann_R(M)$ or simply $Ann(M)$. M is said to be faithful if $Ann(M) = (0)$. Let $N, K \leq M$. Then the product of N and K, denoted by NK, is defined by $(N : M)(K : M)M$ (see [3]). Define $ann(N)$ or simply $annN = \{m \in M | \ m(N : M) = 0\}$.

The prime spectrum of M is the set of all prime submodules of M and denoted by $Spec(M)$, $Max(M)$ is the set of all maximal submodules of M, and $J(M)$, the jacobson radical of M, is the intersection of all elements of $Max(M)$, respectively [15].

If N is a submodule of M, then $V(N) = \{P \in Spec(M) | \ (P : M) \supseteq (N : M)\}$ [16].

The *Zariski topology* on $X = Spec(M)$ is the topology τ_M described by taking the set $Z(M) = \{V(N) | \ N \text{ is a submodule of } M\}$ as the set of closed sets of $Spec(M)$ [16].

There are many papers on assigning graphs to rings or modules (see, for example, [1, 4, 6, 7, 10, 11, 18, 20]). In [4], the present authors introduced and studied the graph $G(\tau_T)$ and $AG(M)$, called *the Zariski topology-graph* and *the annihilating-submodule graph*, respectively.

Let T be a non-empty subset of $Spec(M)$. *The Zariski topology-graph* $G(\tau_T)$ is an undirected graph with vertices $V(G(\tau_T))= \{N < M | \text{ there exists } K < M \text{ such that } V(N) \cup V(K) = T \text{ and } V(N), V(K) \neq T\}$ and distinct vertices N and L are adjacent if and only if $V(N) \cup V(L) = T$ (see [4, Definition 2.3]).

$AG(M)$ is an undirected graph with vertices $V(AG(M))= \{N \leq M | \text{ there exists } (0) \neq K < M \text{ with } NK = (0)\}$. In this graph, distinct vertices $N, L \in V(AG(M))$ are adjacent if and only if $NL = (0)$. Let $AG(M)^*$ be the subgraph of $AG(M)$ with vertices $V(AG(M)^*) = \{N < M$ with $(N : M) \neq Ann(M)|$ there exists a submodule $K < M$ with $(K : M) \neq Ann(M)$ and $NK = (0)\}$. By [4, Theorem 3.4], one concludes that $AG(M)^*$ is a connected subgraph.

If $Spec(M) \neq \emptyset$, the mapping $\psi : Spec(M) \to Spec(R/Ann(M))$ such that $\psi(P) = (P : M)/Ann(M)$ for every $P \in Spec(M)$, is called the *natural map* of $Spec(M)$ [16].

The prime radical $\sqrt{N}$ is defined to be the intersection of all prime submodules of M containing N, and in case N is not contained in any prime submodule, $\sqrt{N}$ is defined to be M [15].

In this paper, we study the domination number of $G(\tau_T)$ and some connections between the graph-theoretic properties of $G(\tau_T)$ and algebraic properties of the module M.

$Z(R)$ and $Nil(R)$ will denote the set of all zero-divisors and the set of all nilpotent elements of R, respectively. Also, $Z_R(M)$ or simply $Z(M)$, the set of zero divisors on M, is the set $\{r \in R|\ rm = 0$ for some $0 \neq m \in M\}$. If $Z(M) = 0$, then we say that M is a domain. An ideal $I \leq R$ is said to be nil if I consists of nilpotent elements.

Now we introduce some notions. A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set of vertices, $V(G)$, a set $E(G)$ of edges, and an incident function ψ_G that associates an unordered pair of distinct vertices with each edge. The edge e joins x and y if $\psi_G(e) = \{x, y\}$, and we say x and y are adjacent. The number of edges incident at x in G is called the degree of the vertex x in G and is denoted by $d_G(x)$ or simply $d(x)$. A path in graph G is a finite sequence of vertices $\{x_0, x_1, \ldots, x_n\}$, where x_{i-1} and x_i are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for existing an edge between x_{i-1} and x_i. The distance between two vertices x and y, denoted $d(x, y)$, is the length of the shortest path from x to y. The diameter of a connected graph G is the maximum distance between two distinct vertices of G. For any vertex x of a connected graph G, the eccentricity of x, denoted $e(x)$, is the maximum of the distances from x to the other vertices of G. The set of vertices with minimum eccentricity is called the center of the graph G, and this minimum eccentricity value is the radius of G. For some $U \subseteq V(G)$, we denote by $N(U)$, the set of all vertices of $G \setminus U$ adjacent to at least one vertex of U and $N[U] = N(U) \cup \{U\}$.

A graph H is a subgraph of G, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. A subgraph H of G is a spanning subgraph of G if $V(H) = V(G)$. A spanning subgraph H of G is called a perfect matching of G if every vertex of G has degree 1. A subset S of the vertex set $V(G)$ is called independent if any two vertices of S are not adjacent in G.

A clique of a graph is a complete subgraph and the supremum of the sizes of cliques in G, denoted by $cl(G)$, is called the clique number of G. Let $\chi(G)$ denote the chromatic number of the graph G, that is, the minimal number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Obviously $\chi(G) \geq cl(G)$.

A graph G is a split graph if $V(G)$ can be partitioned into two subsets A and B such that the subgraph induced by A in G is a clique in G, and B is an independent subset of $V(G)$.

A subset D of $V(G)$ is called a dominating set if every vertex of G is either in D or adjacent to at least one vertex in D. The domination number of G, denoted by $\gamma(G)$, is the number of vertices in a smallest dominating set of G. A total dominating set of a graph G is a dominating set S such that every vertex is adjacent to a vertex in S. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. A dominating set of cardinality $\gamma(G)$ ($\gamma_t(G)$) is called a γ-set (γ_t-set). A dominating set D is a connected dominating set if the subgraph $< D >$ induced by D is a connected subgraph of G. The connected domination number of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of G. A dominating set D is a clique dominating set if the subgraph $< D >$ induced by D is complete in G. The clique domination number $\gamma_{cl}(G)$ of G equals the minimum cardinality of a clique dominating set of G. A dominating set D is a paired-dominating set if the subgraph $< D >$ induced by D has a perfect matching. The paired-domination number $\gamma_{pr}(G)$ of G equals the minimum cardinality of a paired-dominating set of G.

A vertex u is a neighbor of v in G, if uv is an edge of G, and $u \neq v$. The set of all neighbors of v is the open neighborhood of v or the neighbor set of v, and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v in G.

Let S be a dominating set of a graph G, and $u \in S$. The private neighborhood of u relative to S in G is the set of vertices which are in the closed neighborhood of u, but not in the closed neighborhood of any vertex in $S \backslash \{u\}$. Thus the private neighborhood $P_N(u, S)$ of u with respect to S is given by $P_N(u, S) = N[u] \setminus (\cup_{v \in S \backslash \{u\}} N[v])$. A set $S \subseteq V(G)$ is called irredundant if every vertex v of S has at least one private neighbor. An irredundant set S is a maximal irredundant set if for every vertex $u \in V \setminus S$, the set $S \cup \{u\}$ is not irredundant. The irredundance number $ir(G)$ is the minimum cardinality of maximal irredundant sets. There are so many domination parameters in the literature and for more details we refer to [13].

A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V; that is, U and V are each independent sets and is denoted by $B_{n,m}$, where V and U are of size n and m, respectively. A complete bipartite graph on n and m vertices, denoted by $K_{n,m}$, where V and U are of size n and m, respectively, and $E(G)$ connects every vertex in V with all vertices in U. Note that a graph $K_{1,m}$ is called a star graph and the vertex in the singleton partition is called the center of the graph. We denote by C_n and P_n a cycle and a path of order n, respectively (see [12]).

In section 2, a dominating set of $G(\tau_T)$ is constructed using elements of the center when M is an Artinian module. Also we prove that the domination number of $G(\tau_T)$ is equal to the number of factors in the Artinian decomposition of M and we also find several domination parameters of $G(\tau_T)$. In section 3, some relations between the domination numbers and the total domination numbers of Zariski topology-graphs are studied. Also, we study the domination number of the Zariski topology-graphs for reduced rings with finitely many minimal primes and faithful modules.

Throughout the rest of this paper, we denote by T a non-empty subset of $Spec(M)$, $F := \cap_{P \in T} P$, $Q := (F : M)M$, $\bar{M} := M/Q$, $\bar{N} := N/Q$, $\bar{m} := m + Q$, and $\bar{I} := I/(Q : M)$, where N is a submodule of M containing Q, $m \in M$, and I is an ideal of R containing $(Q : M)$. **Also, throughout this paper $\bar{M}$ is a module which does not have a non-zero submodule $\bar{F} \neq \bar{N}$ with $V(N) = T$.**

The following results are useful for further reference in this paper.

Remark 1. *Let N be a submodule of M. Set $V^*(N) := \{P \in Spec(M) | \ P \supseteq N\}$. By [4, Remark 2.2], for submodules N and K of M, we have*

$$V(N) \cup V(K) = V(N \cap K) = V(NK) = V^*(NK).$$

By [4, Remark 2.5], we have T is a closed subset of $Spec(M)$ if and only if $T = V(F)$ and $G(\tau_T) \neq \emptyset$ if and only if $T = V(F)$ and T is not irreducible. So if N and K are adjacent in $G(\tau_T)$, then $V^*(NK) = V^*(Q)$ and hence $\sqrt{NK} = F$. Therefore $F \subseteq \sqrt{(N : M)M}$ and $F \subseteq \sqrt{(K : M)M}$.

The following is well known.

Proposition 1. *Suppose that e is an idempotent element of R. We have the following statements.*

- *(a) $R = R_1 \times R_2$, where $R_1 = eR$ and $R_2 = (1 - e)R$.*
- *(b) $M = M_1 \times M_2$, where $M_1 = eM$ and $M_2 = (1 - e)M$.*
- *(c) For every submodule N of M, $N = N_1 \times N_2$ such that N_1 is an R_1-submodule M_1, N_2 is an R_2-submodule M_2, and $(N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$.*
- *(d) For submodules N and K of M, $NK = N_1K_1 \times N_2K_2$ such that $N = N_1 \times N_2$ and $K = K_1 \times K_2$.*
- *(e) Prime submodules of M are $P \times M_2$ and $M_1 \times Q$, where P and Q are prime submodules of M_1 and M_2, respectively.*

We need the following results.

Lemma 1. *(See [2, Proposition 7.6].) Let $R_1, R_2, \ldots, R_n$ be non-zero ideals of R. Then the following statements are equivalent:*

- *(a) $R = R_1 \times \ldots \times R_n$;*
- *(b) As an abelian group R is the direct sum of $R_1, \ldots, R_n$;*
- *(c) There exist pairwise orthogonal idempotents $e_1, \ldots, e_n$ with $1 = e_1 + \ldots + e_n$, and $R_i = Re_i$, $i = 1, \ldots, n$.*

Lemma 2. *(See [14, Theorem 21.28].) Let I be a nil ideal in R and $u \in R$ be such that $u + I$ is an idempotent in R/I. Then there exists an idempotent e in uR such that $e - u \in I$.*

Lemma 3. *(See [7, Lemma 2.4].) Let N be a minimal submodule of M and let $Ann(M)$ be a nil ideal. Then we have $N^2 = (0)$ or $N = eM$ for some idempotent $e \in R$.*

Lemma 4. *(See [4, Lemma 4.10].) Let R be an Artinian ring and suppose $\bar{M}$ is a finitely generated module which is not a vertex in $AG(\bar{M})$. Then for every non-zero proper submodule $\bar{N}$ of $\bar{M}$, $\bar{N}$ and N are vertices in $AG(\bar{M})$ and $G(\tau_T)$, respectively.*

Theorem 1. *(See [5, Theorem 4.2].) Assume that $\bar{M}$ is a faithful module. Then the following statements are equivalent.*

- *(a) $\chi(G(\tau_{Spec(M)})) = 2$.*
- *(b) $G(\tau_{Spec(M)})$ is a bipartite graph with two non-empty parts.*

(c) $G(\tau_{Spec(M)})$ *is a complete bipartite graph with two non-empty parts.*

(d) *Either R is a reduced ring with exactly two minimal prime ideals or* $G(\tau_{Spec(M)})$ *is a star graph with more than one vertex.*

Proposition 2. *(See [13, Proposition 3.9].) Every minimal dominating set in a graph G is a maximal irredundant set of G.*

2 Domination number in Zariski topology-graph for Artinian modules

The main goal in this section, is to obtain the value certain domination parameters of the Zariski topology-graph for Artinian modules.

Lemma 5. *Let* $\bar{M}$ *be a faithful module. Then the following statements are equivalent.*

(a) *There is a vertex of* $G(\tau_{Spec(M)})$ *which is adjacent to every other vertex of* $G(\tau_{Spec(M)})$.

(b) $G(\tau_{Spec(M)})$ *is a star graph.*

(c) $M = F \oplus D$, *where F is a simple module and D is a prime module.*

(d) $\gamma(G(\tau_T)) = 1$.

Proof. Trivial from [5, Corollary 3.2]. □

Theorem 2. *Let* $\bar{M}$ *be a finitely generated Artinian local module and* $G(\tau_T) \neq \emptyset$. *Assume that* $\bar{N}$ *is the unique maximal submodule of* $\bar{M}$. *Then the radius of* $G(\tau_T)$ *is* 0 *or* 1 *and the center of* $G(\tau_T)$ *is* $\{K|\ \bar{K} \subseteq ann(\bar{N}), \bar{0} \neq \bar{K} \leq \bar{M}\}$.

Proof. Suppose that $G(\tau_T) \neq \emptyset$. Then the number of non-zero proper submodules of $\bar{M}$ is greater than 1. Since $\bar{M}$ is finitely generated Artinian module, there exists $m \in \mathbb{N}$, $m > 1$ such that $\bar{N}^m = (\bar{0})$ and $\bar{N}^{m-1} \neq (\bar{0})$. For any non-zero submodule $\bar{K}$ of $\bar{M}$, $\bar{K}\bar{N}^{m-1} \subseteq \bar{N}\bar{N}^{m-1} = (\bar{0})$ and so $d(N^{m-1}, K) = 1$. Hence $e(N^{m-1}) = 1$ and so the radius of $G(\tau_T)$ is 1. Suppose $\bar{K}$ and $\bar{L}$ are arbitrary non-zero submodules of $\bar{M}$ and $\bar{K} \subseteq ann(\bar{N})$. Then $\bar{K}\bar{L} \subseteq \bar{K}\bar{N} = (\bar{0})$ and hence $e(K) = 1$. Suppose $(\bar{0}) \neq \bar{K}' \nsubseteq ann(\bar{N})$. Then $\bar{K}'\bar{N} \neq (\bar{0})$ and so $e(K') > 1$. Hence the center of $G(\tau_T)$ is $\{K|\ \bar{K} \subseteq ann(\bar{N}), 0 \neq \bar{K} \leq \bar{M}\}$. □

Corollary 1. *Let* $\bar{M}$ *be a finitely generated Artinian local module and* $\bar{N}$ *is the unique maximal submodule of* $\bar{M}$. *Then the following hold good.*

(a) $\gamma(G(\tau_T)) = 1$.

(b) *D is a* γ*-set of* $G(\tau_T)$ *if and only if* $\bar{D} \subseteq ann(\bar{N})$.

Proof. (a) It follows directly from Theorem 2.
(b) Let $D = \{K\}$ be a γ-set of $G(\tau_T)$. Suppose $\bar{K} \nsubseteq ann(\bar{N})$. Then $\bar{K}\bar{N} \neq (\bar{0})$ and so N is not dominated by K, a contradiction. Conversely, suppose $\bar{D} \subseteq ann(\bar{N})$. Let K be an arbitrary vertex in $G(\tau_T)$. Then $\bar{K}\bar{L} \subseteq \bar{N}\bar{L} = (\bar{0})$ for every $\bar{L} \in D$, i.e., every vertex K is adjacent to every $L \in D$. If $|D| > 1$, then $D \setminus \{L'\}$ is also a dominating set of $G(\tau_T)$ for some $L' \in D$ and so D is not minimal. Thus $|D| = 1$ and so D is a γ-set by (a). □

Theorem 3. *Let $\bar{M} = \oplus_{i=1}^{n} \bar{M}_i$, where $\bar{M}_i$ is a finitely generated Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Then the radius of $G(\tau_T)$ is 2 and the center of $G(\tau_T)$ is $\{K|\bar{K} \subseteq J(\bar{M}), \bar{0} \neq \bar{K} \leq \bar{M}\}$.*

Proof. Assume that $\bar{M} = \oplus_{i=1}^{n} \bar{M}_i$, where $\bar{M}_i$ is a finitely generated Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Let $\bar{J}_i$ be the unique maximal submodule in $\bar{M}_i$ with nilpotency n_i. Note that $Max(\bar{M}) = \{\bar{N}_1, \ldots, \bar{N}_n |\ \bar{N}_i = \bar{M}_1 \oplus \ldots \oplus \bar{M}_{i-1} \oplus \bar{J}_i \oplus \bar{M}_{i+1} \oplus \ldots \oplus \bar{M}_n, 1 \leq i \leq n\}$ is the set of all maximal submodules in $\bar{M}$. Consider $\bar{D}_i = (\bar{0}) \oplus \ldots \oplus (\bar{0}) \oplus \bar{J}_i^{\,n_i - 1} \oplus (\bar{0}) \oplus \ldots \oplus (\bar{0})$ for $1 \leq i \leq n$. Note that $J(\bar{M}) = \bar{J}_1 \oplus \ldots \oplus \bar{J}_n$ is the Jacobson radical of $\bar{M}$ and any non-zero submodule in $\bar{M}$ is adjacent to $\bar{D}_i$ for some i. Let $\bar{K}$ be any non-zero submodule of $\bar{M}$. Then $\bar{K} = \oplus_{i=1}^{n} \bar{K}_i$, where $\bar{K}_i$ is a submodule of $\bar{M}_i$.
Case 1. If $\bar{K} = \bar{N}_i$ for some i, then $\bar{K}\bar{D}_j \neq (\bar{0})$ and $\bar{K}\bar{N}_j \neq (\bar{0})$ for all $j \neq i$. Note that $N(K) = \{(0) \oplus \ldots \oplus (0) \oplus L_i \oplus (0) \oplus \ldots \oplus (0)|\ \bar{J}_i\bar{L}_i = (\bar{0}), \bar{L}_i$ is a nonzero submodule in $\bar{M}_i\}$. Clearly $N(K) \cap N(N_j) = (0)$, $d(K, N_j) \neq 2$ for all $j \neq i$, and so $K - D_i - D_j - N_j$ is a path in $G(\tau_T)$. Therefore $e(K) = 3$ and so $e(N) = 3$ for all $\bar{N} \in Max(\bar{M})$.
Case 2. If $\bar{K} \neq \bar{D}_i$ and $\bar{K}_i \subseteq \bar{J}_i$ for all i. Then $\bar{K}\bar{D}_i = (\bar{0})$ for all i. Let $\bar{L}$ be any non-zero submodule of $\bar{M}$ with $\bar{K}\bar{L} \neq (\bar{0})$. Then $\bar{L}\bar{D}_j = (\bar{0})$ for some j, $K - D_j - L$ is a path in $G(\tau_T)$ and so $e(K) = 2$.
Case 3. If $\bar{K}_i = \bar{M}_i$ for some i, then $\bar{K}\bar{D}_i \neq (\bar{0})$, $\bar{K}\bar{N}_i \neq (\bar{0})$ and $\bar{K}\bar{D}_j = (\bar{0})$ for some $j \neq i$. Thus $K - D_j - D_i - N_i$ is a path in $G(\tau_T)$, $d(K, N_i) = 3$ and so $e(K) = 3$. Thus $e(K) = 2$ for all $\bar{K} \subseteq J(\bar{M})$. Further note that in all the cases center of $G(\tau_T)$ is $\{K|\bar{K} \subseteq J(\bar{M}), \bar{0} \neq \bar{K} \leq \bar{M}\}$. □

Corollary 2. *Let $\bar{M} = \oplus_{i=1}^{n} \bar{M}_i$, where $\bar{M}_i$ is a simple module for all $1 \leq i \leq n$ and $n \geq 2$. Then the radius of $G(\tau_T)$ is 1 or 2 and the center of $G(\tau_T)$ is $\cup_{i=1}^{n} D_i$, where $\bar{D}_i = (\bar{0}) \oplus \ldots \oplus (\bar{0}) \oplus \bar{M}_i \oplus (\bar{0}) \oplus \ldots \oplus (\bar{0})$ for $1 \leq i \leq n$.*

Proposition 3. *Let $\bar{M} = \oplus_{i=1}^{n} \bar{M}_i$, where $\bar{M}_i$ is a finitely generated Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$ ($\bar{M} \neq \bar{M}_1 \oplus \bar{M}_2$, where $\bar{M}_1$ and $\bar{M}_2$ are simple modules). Then*

(a) $\gamma(G(\tau_T)) = n$.

(b) $ir(G(\tau_T)) = n$.

(c) $\gamma_c(G(\tau_T)) = n$.

(d) $\gamma_t(G(\tau_T)) = n$.

(e) $\gamma_{cl}(G(\tau_T)) = n$.

(f) $\gamma_{pr}(G(\tau_T)) = n$, *if n is even and* $\gamma_{pr}(G(\tau_T)) = n + 1$, *if n is odd.*

Proof. Let $\bar{J}_i$ be the unique maximal submodule in $\bar{M}_i$ with nilpotency n_i. Let $\Omega = \{D_1, D_2, \ldots, D_n\}$, where $\bar{D}_i = (\bar{0}) \oplus \ldots \oplus (\bar{0}) \oplus \bar{J}_i^{n_i - 1} \oplus (\bar{0}) \oplus \ldots \oplus (\bar{0})$ for $1 \leq i \leq n$. Note that any non-zero submodule in $\bar{M}$ is adjacent to D_i for some i. Therefore $N[\Omega] = V(G(\tau_T))$, Ω is a dominating set of $G(\tau_T)$ and so $\gamma(G(\tau_T)) \leq n$. Suppose S is a dominating set of $G(\tau_T)$ with $|S| < n$. Then there exists $\bar{N} \in Max(\bar{M})$ such that $\bar{N}\bar{K} \neq (\bar{0})$ for all $K \in S$, a contradiction. Hence $\gamma(G(\tau_T)) = n$. By Proposition 2, Ω is a maximal irredundant set with minimum cardinality and so $ir(G(\tau_T)) = n$. Clearly $< \Omega >$ is a complete subgraph of $G(\tau_T)$. Hence $\gamma_c(G(\tau_T)) = \gamma_t(G(\tau_T)) = \gamma_{cl}(G(\tau_T)) = n$. If n is even, then $< \Omega >$ has a perfect matching and so Ω is a paired-dominating set of $G(\tau_T)$. Thus $\gamma_{pr}(G(\tau_T)) = n$. If n is odd, then $< \Omega \cup K >$ has a perfect matching for some $K \in V(G(\tau_T)) \setminus \Omega$. and so $\Omega \cup K$ is a paired-dominating set of $G(\tau_T)$. Thus $\gamma_{pr}(G(\tau_T)) = n$ if n even and $\gamma_{pr}(G(\tau_T)) = n + 1$ if n is odd.

□

Note that when $\bar{M} = \oplus_{i=1}^{n} \bar{M}_i$, where $\bar{M}_i$ is a finitely generated Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Then by Theorem 3, radius of $G(\tau_T)$ is 2. Further, by Proposition 3, the domination number of $G(\tau_T)$ is equal to n, where n is the number of distinct maximal submodules of $\bar{M}$. However, this need not be true if the radius of $G(\tau_T)$ is 1. For, consider $\bar{M} = \bar{M}_1 \oplus \bar{M}_2$, where $\bar{M}_1$ and $\bar{M}_2$ are simple modules. Then $G(\tau_T)$ is a star graph and so has radius 1, whereas $\bar{M}$ has two distinct maximal submodules. The following corollary shows that a more precise relationship between the domination number of $G(\tau_T)$ and the number of maximal submodules in $\bar{M}$, when $\bar{M}$ is finite.

Corollary 3. *Let $\bar{M}$ be a finitely generated Artinian module, $\bar{M}$ is a faithful module, and $\gamma(G(\tau_T)) = n$. Then either $\bar{M} = \bar{M}_1 \oplus \bar{M}_2$, where $\bar{M}_1$ and $\bar{M}_2$ are simple modules or $\bar{M}$ has n maximal submodules.*

Proof. When $\gamma(G(\tau_T)) = 1$, proof follows from [7, Corollary 2.12]. If $\gamma(G(\tau_T)) = n$, where $n \geq 2$, then $\bar{M}$ can not be $\bar{M} = \bar{M}_1 \oplus \bar{M}_2$, where $\bar{M}_1$ and $\bar{M}_2$ are simple modules. Hence $\bar{M} = \oplus_{i=1}^{n} \bar{M}_i$, where $\bar{M}_i$ is a finitely generated Artinian local module for all $1 \leq i \leq m$ and $m \geq 2$. By Proposition 3, $\gamma(G(\tau_T)) = m$. Hence by assumption $m = n$, i.e., $\bar{M} = \oplus_{i=1}^{n} \bar{M}_i$, where $\bar{M}_i$ is a finitely generated Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. One can see now that $\bar{M}$ has n maximal submodules. □

Theorem 4. *Let $\bar{M}$ be a faithful module and let S be the set of all maximal elements of the set $V(G(\tau_{Spec(M)}))$. If $|S| > 1$, then $\gamma_t(G(\tau_{Spec(M)})) = |S|$.*

Proof. Suppose that S is the set of all maximal elements of the set $V(G(\tau_{Spec(M)}))$. Let $K \in S$. First we show that $K = ann(annK)$ and there exists $m \in M$ such that $K = ann(Rm)$. Since $annK \neq 0$, there exists $0 \neq m \in annK$. Hence $K \subseteq ann(annK) \subseteq ann(Rm)$. Thus by the maximality of K, we have $K = ann(annK) = ann(Rm)$. For any $K \in S$, choose $m_K \in M$ such that $K = ann(Rm_K)$. We assert that $D = \{Rm_K | \ K \in S\}$ is a total dominating set of $G(\tau_{Spec(M)})$. Since for every $L \in V(G(\tau_{Spec(M)}))$ there exists $K \in S$ such that $L \subseteq K = ann(Rm_K)$, L and Rm_K are adjacent. Also for each pair $K, K' \in S$, we have $(Rm_K)(Rm_{K'}) = 0$. Namely, if there exists $m \in (Rm_K)(Rm_{K'}) \setminus \{0\}$, then $K = K' = ann(Rm)$. Thus $\gamma_t(G(\tau_{Spec(M)})) \leq |S|$. To complete the proof, we show that each element of an arbitrary γ_t-set of $G(\tau_{Spec(M)})$ is adjacent to exactly one element of S. Assume to the contrary, that a vertex L' of a γ_t-set of $G(\tau_{Spec(M)})$ is adjacent to K and K', for $K, K' \in S$. Thus $K = K' = annL'$, which is impossible. Therefore $\gamma_t(G(\tau_{Spec(M)})) = |S|$. □

Corollary 4. *Let $\bar{M} = \oplus_{i=1}^{n} \bar{M}_i$, where $\bar{M}_i$ is a finitely generated Artinian local module for all $1 \leq i \leq n$, $n \geq 2$ ($\bar{M} \neq \bar{M}_1 \oplus \bar{M}_2$, where $\bar{M}_1$ and $\bar{M}_2$ are simple modules). Then $\gamma_t(G(\tau_T)) = \gamma(G(\tau_T)) = |Max(\bar{M})|$.*

Proof. Let $\bar{M} = \oplus_{i=1}^{n} \bar{M}_i$, where $(\bar{M}_i, \bar{J}_i)$ is a finitely generated Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Recall that $Max(\bar{M}) = \{\bar{N}_1, \ldots, \bar{N}_n | \ \bar{N}_i = \bar{M}_1 \oplus \ldots \oplus \bar{M}_{i-1} \oplus \bar{J}_i \oplus \bar{M}_{i+1} \oplus \ldots \oplus \bar{M}_n, 1 \leq i \leq n\}$. By Lemma 4, every nonzero proper submodule of M which is contain F, is a vertex in $G(\tau_T)$. So the set of maximal elements of $V(G(\tau_T))$ and $Max(\bar{M})$ are equal and hence by Theorem 4, $\gamma_t(G(\tau_T)) = |Max(\bar{M})|$. Finally, the result follows from Proposition 3. □

Example 1. *Let $\mathbb{Z}_3 \times \mathbb{Z}_4$ as $\mathbb{Z}_{24}$-module and $T = Spec(M)$. $S = \{(0) \times \mathbb{Z}_4, \mathbb{Z}_3 \times \bar{2}\mathbb{Z}_4\}$ is the set of all maximal elements of $G(\tau_T)$ and $\gamma_t(G(\tau_T)) = \gamma_t(P_4) = 2 = |S|$.*

3 The relationship between $\gamma_t(G(\tau_T))$ and $\gamma(G(\tau_T))$

The main goal in this section is to study the relation between $\gamma_t(G(\tau_T))$ and $\gamma(G(\tau_T))$.

The first result of this section provides the domination number of the Zariski topology-graph of a finite direct product of modules.

Theorem 5. *For a module M, which is a product of two (nonzero) modules, one of the following holds.*

(a) If $M \cong F \times D$, where F is a simple module and D is a prime module, then $\gamma(G(\tau_T)) = 1$.

(b) If $M \cong D_1 \times D_2$, where D_1 and D_2 are prime modules which are not simple, then $\gamma(G(\tau_T)) = 2$.

(c) If $M \cong M_1 \times D$, where M_1 is a module which is not prime and D is a prime module, then $\gamma(G(\tau_T)) = \gamma(G(\tau_{T_1})) + 1$.

(d) If $M \cong M_1 \times M_2$, where M_1 and M_2 are two modules which are not prime, then $\gamma(G(\tau_T)) = \gamma(G(\tau_{T_1})) + \gamma(G(\tau_{T_2}))$.

Proof. Parts (a) and (b) are trivial.

(c) Without loss of generality, one can assume that $\gamma(G(\tau_{T_1})) < \infty$. Suppose that $\gamma(G(\tau_{T_1})) = n$ and $\{K_1, \ldots, K_n\}$ is a minimal dominating set of $G(\tau_{T_1})$. It is not hard to see that $\{K_1 \times F_2, \ldots, K_n \times F_2, F_1 \times D\}$ is the smallest dominating set of $G(\tau_T)$.

(d) We may assume that $\gamma(G(\tau_{T_1})) = m$ and $\gamma(G(\tau_{T_2})) = n$, for some positive integers m and n. Let $\{K_1, \ldots, K_m\}$ and $\{L_1, \ldots, L_n\}$ be two minimal dominating sets in $G(\tau_{T_1})$ and $G(\tau_{T_2})$, respectively. It is easy to see that $\{K_1 \times F_2, \ldots, K_m \times F_2, F_1 \times L_1 \ldots F_1 \times L_n\}$ is the smallest dominating set in $G(\tau_T)$. □

Theorem 6. *Let $\bar{M}$ be a module. Then*
$\gamma_t(G(\tau_T)) = \gamma(G(\tau_T))$ *or* $\gamma_t(G(\tau_T)) = \gamma(G(\tau_T)) + 1$.

Proof. Assume that $\gamma_t(G(\tau_T)) \neq \gamma(G(\tau_T))$ and D is a γ-set of $G(\tau_T)$. If $\gamma(G(\tau_T)) = 1$, then it is clear that $\gamma_t(G(\tau_T)) = 2$. So let $\gamma(G(\tau_T)) > 1$ and put $k = Max\{n|$ there exist $L_1, \ldots, L_n \in D$ such that $\sqcap_{i=1}^{n} L_i \neq F\}$. Since $\gamma_t(G(\tau_T)) \neq \gamma(G(\tau_T))$, we have $k \geq 2$. Let $L_1, \ldots, L_k \in D$ be such that $\sqcap_{i=1}^{k} L_i \neq F$. Then $S = \{\sqcap_{i=1}^{k} L_i, ann\bar{L}_1, \ldots, ann\bar{L}_k\} \cup D \setminus \{L_1, \ldots, L_k\}$ is a γ_t-set. Hence $\gamma_t(G(\tau_T)) = \gamma(G(\tau_T)) + 1$. □

Example 2. *Let C_n and P_n be a cycle and a path with n vertices, respectively.*

(a) Clearly, $\gamma(C_n) = \gamma(P_n) = [n/3]$ (see [17, Example 1]).

(b) Let $\mathbb{Z}_2 \times \mathbb{Z}_3$ as $\mathbb{Z}_{12}$-module and $T = Spec(M)$. It is easy to see that $G(\tau_T) = P_2$ and $\gamma_t(P_2) = 2 = \gamma(P_2) + 1$.

(c) By [9, Lemma 10.9.5], for any split graph G, $\gamma_t(G) = \gamma(G)$. Let $\mathbb{Z}_3 \times \mathbb{Z}_4$ as $\mathbb{Z}_{24}$-module and $T = Spec(M)$. The split graph $G(\tau_T) = P_4$ and $\gamma_t(P_4) = \gamma(P_4) = 2$.

Theorem 7. *Let $\bar{M}$ be a faithful module and $|Min(R)| < \infty$. If $\gamma(G(\tau_{Spec(M)})) > 1$, then $\gamma_t(G(\tau_{Spec(M)})) = \gamma(G(\tau_{Spec(M)})) = |Min(R)|$.*

Proof. Since $\bar{M}$ is a faithful module and $\gamma(G(\tau_{Spec(M)})) > 1$, then R is a reduced ring and $|Min(R)| > 1$. Suppose that $Min(R) = \{p_1, \ldots, p_n\}$. If $n = 2$, the result follows from Theorem 1. Therefore, suppose that $n \geq 3$. We define $\widehat{p_i\bar{M}} = p_1 \ldots p_{i-1}p_{i+1} \ldots p_n\bar{M}$, for every $i = 1, \ldots, n$. Clearly, $\widehat{p_i\bar{M}} \neq \bar{0}$, for every $i = 1, \ldots, n$. Since R is reduced, we deduce that $\widehat{p_i\bar{M}}p_i\bar{M} = \bar{0}$. Therefore, every $p_i\bar{M}$ is a vertex of $G(\tau_{Spec(M)})$. If K is a vertex of $G(\tau_{Spec(M)})$, then by [8, Corollary 3.5], $(K : M) \subseteq Z(R) = \cup_{i=1}^{n} p_i$. It follows from the Prime Avoidance Theorem that $(K : M) \subseteq p_i$, for some i, $1 \leq i \leq n$. Thus p_iM is a maximal element of $V(G(\tau_{Spec(M)}))$, for every $i = 1, \ldots, n$. From Theorem 4, $\gamma_t(G(\tau_{Spec(M)})) = |Min(R)|$. Now, we show that $\gamma(G(\tau_{Spec(M)})) = n$. Assume to the contrary, that $B = \{J_1, \ldots, J_{n-1}\}$ is a dominating set for $G(\tau_{Spec(M)})$. Since $n \geq 3$, the submodules $p_i\bar{M}$ and $p_j\bar{M}$, for $i \neq j$ are not adjacent (from $p_ip_j = 0 \subseteq p_k$ it would follow that $p_i \subseteq p_k$ or $p_j \subseteq p_k$ which is not true). Because of that, we may assume that for some $k < n-1$, $J_i = p_iM$ for $i = 1, \ldots, k$, but none of the other of submodules from B are equal to some p_sM (if $B = \{p_1M, \ldots, p_{n-1}M\}$, then p_nM would be adjacent to some p_iM, for $i \neq n$). So every submodule in $\{p_{k+1}M, ..., p_nM\}$ is adjacent to a submodule in $\{J_{k+1}, ..., J_{n-1}\}$. It follows that for some $s \neq t$, there is an l such that $(p_sM)J_l = 0 = (p_tM)J_l$. Since $p_s \nsubseteq p_t$, it follows that $J_l \subseteq p_tM$, so $J_l^2 = 0$, which is impossible, since the ring R is reduced. So $\gamma_t(G(\tau_{Spec(M)})) = \gamma(G(\tau_{Spec(M)})) = |Min(R)|$. □

By Theorem 7, we have the following corollary.

Corollary 5. *Let $\bar{M}$ is a faithful module and $|Min(R)| < \infty$. If $\gamma(G(\tau_{Spec(M)})) > 1$, then the following are equivalent.*

(a) $\gamma(G(\tau_{Spec(M)})) = 2$.

(b) $G(\tau_{Spec(M)}) = B_{n,m}$ *such that* $n, m \geq 2$.

(c) $G(\tau_{Spec(M)}) = K_{n,m}$ *such that* $n, m \geq 2$.

(d) R *has exactly two minimal primes.*

Proof. Follows from Theorem 1 and Theorem 7. □

In the following theorem the domination number of bipartite Zariski topology-graphs is given.

Theorem 8. *Let* $\bar{M}$ *be a faithful module. If* $G(\tau_T)$ *is a bipartite graph, then* $\gamma(G(\tau_T)) \leq 2$.

Proof. Assume that $\bar{M}$ is a faithful module. If $G(\tau_T)$ is a bipartite graph, then from Theorem 1, either R is a reduced ring with exactly two minimal prime ideals, or $G(\tau_T)$ is a star graph with more than one vertex. If R is a reduced ring with exactly two minimal prime ideals and $\gamma(G(\tau_T)) = 1$, then we are done. If R is a reduced ring with exactly two minimal prime ideals and $\gamma(G(\tau_T)) > 1$, then the result follows by Corollary 5. If $G(\tau_T)$ is a star graph with more than one vertex, then we are done. □

Theorem 9. *If* R *is a Notherian ring and* $\bar{M}$ *a finitely generated faithful module, then* $\gamma(G(\tau_{Spec(M)})) \leq |Ass(\bar{M})| < \infty$.

Proof. from [19], since R is a Notherian ring and $\bar{M}$ a finitely generated module, $|Ass(\bar{M})| < \infty$. Let $Ass(\bar{M}) = \{p_1, ..., p_n\}$, where $p_i = (\bar{0} : R\bar{m}_i)$ for some $\bar{m}_i \in \bar{M}$ for every $i = 1, \ldots, n$. Set $A = \{Rm_i | 1 \leq i \leq n\}$. We show that A is a dominating set of $G(\tau_{Spec(M)})$. Clearly, every Rm_i is a vertex of $G(\tau_{Spec(M)})$, for $i = 1, \ldots, n$ $((p_i\bar{M})(\bar{m}_iR) = \bar{0})$. If K is a vertex of $G(\tau_{Spec(M)})$, then [19, Corollary 9.36] implies that $(\bar{K} : \bar{M}) \subseteq Z(\bar{M}) = \cup_{i=1}^{n} p_i$. It follows from the Prime Avoidance Theorem that $(\bar{K} : \bar{M}) \subseteq p_i$, for some i, $1 \leq i \leq n$. Thus $\bar{K}(R\bar{m}_i) = (\bar{0})$, as desired. □

Acknowledgement. The authors would like to thank the referee for his/her very helpful comments and valuable suggestions.

References

[1] D. F. Anderson, P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra*, **217**, 434–447 (1999).

[2] W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, New York-Heidelberg-Berlin, Springer-Verlag (1974).

[3] H. Ansari-Toroghy, F. Farshadifar, Product and dual product of submodules, *Far East J. Math. Sci*, **25 (3)**, 447–455 (2007).

[4] H. Ansari-Toroghy, S. Habibi, The Zariski topology-graph of modules over commutative rings, *Comm. Algebra*, **42**, 3283–3296 (2014).

[5] H. Ansari-Toroghy, S. Habibi, The Zariski topology-graph of modules over commutative rings II, *Arab. J. Math. https://doi.org/10.1007/s40065-020-00301-1* (2020).

[6] H. Ansari-Toroghy, S. Habibi, The annihilating-submodule graph of modules over commutative rings, *Math. Reports*, **20 (70)**, 245–262 (2018).

[7] H. Ansari-Toroghy, S. Habibi, The annihilating-submodule graph of modules over commutative rings II, *Arab. J. Math. DOI 10.1007/s40065-016-0154-0* (2016).

[8] M. F. Atiyah, I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley (1969).

[9] R. Balakrishnan, K. Renganathan, *A Textbook of Graph Theory*, Universitext, Springer (2000).

[10] I. Beck, Coloring of commutative rings, *J. Algebra*, **116**, 208–226 (1988).

[11] M. Behboodi, Z. Rakeei, The annihilating-ideal graph of commutative rings I, *J. Algebra Appl*, **10 (4)**, 727–739 (2011).

[12] R. Diestel, *Graph Theory*, Grad. Texts in Math., Springer, NJ (2005).

[13] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Domination in Graphs Advanced Topics*, Marcel Dekker Inc., New York (1998).

[14] T. Y. Lam, *A First Course in Non-Commutative Rings*, Springer - Verlag, New York (1991).

[15] C.-P. Lu, Prime submodules of modules, *Comment. Math. Univ. St. Paul*, **33 (1)**, 61–69 (1984).

[16] C.-P. Lu, The Zariski topology on the prime spectrum of a module, *Houston J. Math*, **25 (3)**, 417–432 (1999).

[17] D. A. Mojdeh, A. M. Rahimi, Dominating sets of some graphs associated to commutative rings, *Comm. Algebra*, **40**, 3389–3396 (2012).

[18] R. Nikandish, H. R. Maimani, S. Kiani, Domination number in the annihilating-ideal graphs of commutative rings, *Publ. Inst. Math. N. S.*, **97 (111)**, 225–231 (2015).

[19] R. Y. Sharp, *Steps in Commutative Algebra*, Cambridge University Press (1991).

[20] T. Tamizh Chelvam, K. Selvakumar, Central sets in the annihilating-ideal graph of commutative rings, *J. Combin. Math. Combin. Comput*, **88**, 277-288 (2014).

Received: 06.04.2021
Accepted: 09.05.2021

(1) Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Guilan,
P. O. Box 41335-19141 Rasht, Iran
E-mail: habibishk@gmail.com

(2) Department of Mathematics,
Payame Noor University, Tehran, Iran
E-mail: Z_habibi@pnu.ac.ir

(3) Department of Mathematics,
Payame Noor University, Tehran, Iran
E-mail: masoomeh.hezarjaribi@pnu.ac.ir