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Equivalence between (pre)cat^1-R-algebroids and (pre)crossed modules of R-algebroids

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Abstract

In this study, we introduce (pre)cat^1-R-algebroids, as a generalisation of cat^1-algebras, and prove the equivalence between the categories of (pre)cat^1-R-algebroids and (pre)crossed modules of R-algebroids. Moreover, we look over some immediate consequences of the equivalences and, as an application, for converting precat^1-R-algebroids into cat^1-R-algebroids we develop an equivalent method to the one used for converting precrossed modules into crossed modules of R-algebroids.

Key Words: Cat^1-algebroids, precat^1-algebroids, crossed modules of algebroids, precrossed modules of algebroids, equivalence of categories, Peiffer ideal.
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1 Introduction

Crossed modules, algebraic models of homotopy 2-types, were first introduced by Whitehead in his studies [20, 21] on homotopy groups, and then become a useful tool in both homological and homotopical algebra, with several equivalent descriptions. In this sense, as one of the first published proofs of such equivalences, the equivalence between the categories of crossed modules of groups and $\mathcal{G}$-groupoids was proved by Brown and Spencer in [6], where the equivalence is referred to Duskin's, [9], and Verdier's (1965) unpublished works. Soon after, cat^1-groups, originally named as 1-cat-groups, were introduced and shown to be equivalent to crossed modules of groups, and also to group objects in the category of categories and to simplicial groups whose Moore complex is of lenght 1, by Loday in [13]. An explicit proof of the equivalence between (pre)cat^1-groups and (pre)crossed modules of groups can be found in [5].

On the other hand, cat^1-algebras and cat^n-algebras, in general, were introduced and shown to be equivalent to crossed modules and crossed n-cubes of algebras, respectively, by Ellis in his thesis [10] and in [11]. Shammu gave an explicit proof of the equivalence between cat^1-algebras and crossed modules of algebras in his thesis [19]. Recently, a computer implementation of the equivalence between cat^1-algebras and crossed modules of algebras was made by Arvasi and Odabaş in [1].

As a more general notion, R-algebroids, where R is a commutative ring, were especially studied by Mitchell in [15, 16, 17] and by Amgott in [4]. Mitchell gave a categorical definition of R-algebroids (cf. Definition 1). Later on, as a generalisation of crossed modules of associative R-algebras, Mosa introduced crossed modules of R-algebroids and proved their equivalence to special double R-algebroids with connections in his thesis [18].

In a more recent and similar study [12], using the equivalence between the categories of split epimorphisms and object actions in a semiabelian category C, Janalidze introduced the notion of an internal precrossed module, by describing an internal reflexive graph as an object action equipped with some additional structure, and thus obtained an equivalence between the categories of internal reflexive graphs and internal precrossed modules in C. Likewise, he introduced the notion of an internal crossed module, by describing an internal category as an internal precrossed module satisfying an additional axiom, and thus obtained an upgraded equivalence between the categories of internal categories and internal crossed modules. Using a similar methodology, for a fixed class of spans in a monoidal category, Böhm obtained an equivalent description of a split epimorphism of monoids in terms of a distributive law, and used this equivalence to present equivalent descriptions of some reflexive graphs of monoids in terms of relative precrossed modules of monoids and of some relative categories in the category of monoids in terms of relative crossed modules of monoids, in [7]. Subsequently, Böhm proved in [8] that the last two equivalent categories, the category of relative categories in the category of monoids and the category of relative crossed modules of monoids, are also equivalent to the category of simplicial monoids whose Moore length is 1.

In our study, after giving some basic data on (pre)crossed modules of R-algebroids, in Sect. 2, we introduce *(pre)cat^1-R-algebroids*, as a generalisation of cat^1-algebras, in Sect. 3. Then, in Sect. 4, in order to conclude that the categories $\mathrm{PCat}^1\text{-}\mathrm{Alg}\,(R)$ of precat^1-R-algebroids and $\mathrm{PXAlg}\,(R)$ of precrossed modules of R-algebroids are equivalent, we show that the functor $F:\mathrm{PCat}^1\text{-}\mathrm{Alg}\,(R)\to\mathrm{PXAlg}\,(R)$ defined for each precat^1-R-algebroid $\mathcal{A}=(\mathrm{A},u,v)$ by $F\mathcal{A}=\mathcal{N}_{\mathcal{A}}=(\eta_{\mathcal{A}}:\mathrm{Ker}u\to\mathrm{Im}u)$, where $\eta_{\mathcal{A}}a=va$ on morphisms, and for each morphism $\{f\}:\mathcal{A}\to\mathcal{A}'$ of precat^1-R-algebroids by $F\{f\}=(f_{\mathrm{Ker}u},f_{\mathrm{Im}u})$, where $f_{\mathrm{Ker}u}$ and $f_{\mathrm{Im}u}$ are the restrictions of f on $\mathrm{Ker}u$ and $\mathrm{Im}u$, respectively, is an equivalence of categories.

In Sect. 5, using the restrictions $\tilde{F}$ and $\tilde{G}$ of the functors F and G, we upgrade the equivalence between the categories $\mathrm{PCat}^1\text{-}\mathrm{Alg}\,(R)$ and $\mathrm{PXAlg}\,(R)$ to an equivalence between the categories $\mathrm{Cat}^1\text{-}\mathrm{Alg}\,(R)$ of cat^1-R-algebroids and $\mathrm{XAlg}\,(R)$ of crossed modules of R-algebroids.

Finally, in Sect. 6, we first briefly mention the consequences of the equivalences obtained and simply exemplify some consequences. Then, as an application, for converting a precat^1-R-algebroid into a cat^1-R-algebroid we develop an equivalent method to the one used in [3] for converting a precrossed module into a crossed module of R-algebroids using the Peiffer ideal and we show that the functors $(-)^{ct}$ and $(-)^{cr}$ associating with the two methods, respectively, correspond to each other through the equivalences proved.

It is an undeniable fact that working with a many-object version of a categorical structure is more advantageous than working with its one-object version for several reasons. One reason, for example, is that any results in the many-object case can generally be transferred to the one-object case, while the converse is not always possible. Another reason is that many-object versions generally have more categorically preferred properties. For instance, the category of R-algebroids is monoidally closed, as proved in [18, Proposition 1.1.4], while that of associative R-algebras is not. That is why many categorical structures on groups have been generalised to their many-object versions, namely to groupoids. However, studies on generalisation of associative algebras to algebroids are still insufficient. In particular, many categorical aspects of (pre)crossed modules of R-algebroids have not yet been examined thoroughly. In this respect, the generalisation of cat^1-algebras to (pre)cat^1-R-

algebroids and the equivalences obtained here are of great importance. Because, after now, thanks to these equivalences, we have the opportunity to prefer dealing with (pre)cat^1-R-algebroids, which structurally have just one R-algebroid and thus sometimes may more easily be handled compared to (pre)crossed modules of R-algebroids.

Throughout this paper R will be a fixed commutative ring.

2 Basic data on (pre)crossed modules of R-algebroids

With small differences in essence and in naming, most of the following data come from Mitchell's studies [15, 16] and Mosa's thesis [18] and, despite the differences, for each quoted data below we specified the reference(s) from which the data originated.

Definition 1. *[15, Sect. 11, p. 50][16, Sect. 7, p. 879] A category of which each homset has an R-module structure and of which composition is R-bilinear is called an* R-category*. A small R-category is called an* R-algebroid*. Moreover, if we omit the axiom of the existence of identities from an R-algebroid structure then the remaining structure is called a* pre-R-algebroid.

Note from the definition that every R-algebroid is a pre-R-algebroid.

Remark 1. *Throughout the paper for any pre-R-algebroid* A *we adopt the following notational conventions:*
1. $\mathrm{Ob}\,(\mathrm{A})\,(=\mathrm{A}_0)$ *and* $\mathrm{Mor}\,(\mathrm{A})$ *are the object and morphism sets of* A*, respectively.*
2. $s,t:\mathrm{Mor}\,(\mathrm{A})\to\mathrm{A}_0$ *are the source and target functions. Thus, sa and ta are respectively the source and target of any $a\in\mathrm{Mor}\,(\mathrm{A})$, and a is said to be from sa to ta.*
3. $a\in\mathrm{A}$ *means that $a\in\mathrm{Mor}\,(\mathrm{A})$ and if $a,a'\in\mathrm{A}$ with $ta=sa'$ then their composition is denoted by aa'.*
4. *For each $x,y\in\mathrm{A}_0$, the homset consisting of all morphisms from x to y is denoted by* $\mathrm{A}\,(x,y)$.
5. *The zero morphism of any homset $\mathrm{A}\,(x,y)$ is denoted by $0_{\mathrm{A}(x,y)}$, or only by 0 if there is no ambiguity.*
6. *The identity morphism on any $x\in\mathrm{A}_0$, if exists, is denoted by 1_x, or only by 1 if there is no ambiguity.*

Definition 2. *[15, Sect. 11, p. 51][16, Sect. 7, p. 879] An R-linear functor between two R-categories is called an* R-functor *and an R-functor between two R-algebroids is called an* R-algebroid morphism*. Moreover, an assignment between two pre-R-algebroids satisfying all axioms of an R-functor except for the identity preservation axiom is called a* pre-R-algebroid morphism.

Note from the definition that every R-algebroid morphism is a pre-R-algebroid morphism.

Definition 3. *[18, Chapter I, Definition 1.3.4] Let* A *be a pre-R-algebroid. A pre-R-algebroid* S *is called a* pre-R-subalgebroid *of* A *if* $\mathrm{S}_0\subseteq\mathrm{A}_0$*,* $\mathrm{S}\,(x,y)$ *is an R-submodule of* $\mathrm{A}\,(x,y)$ *for all $x,y\in\mathrm{S}_0$ and the composition of any two composible morphisms of* S *is the same as their composition in* A*. Moreover, if* A *and* S *are both R-algebroids and if the identity morphism on each object of* S *is the same as that of* A *then* S *is said to be an* R-subalgebroid *of* A.

Note in the definition that the source and target functions of S are the restrictions of those of A. Note also that the morphism set of S can be uniquely partitioned into the family $\{\mathrm{S}(x,y) \subseteq \mathrm{A}(x,y) : x,y \in \mathrm{S}_0\}$ of R-submodules. So, the union of a family of R-submodules of a (pre-)R-algebroid A can be a morphism set of at most one (pre-)R-subalgebroid of A and if it is the case then the object set and the source and target functions are uniquely determined by the family. Therefore, if a family $\mathrm{S} = \{\mathrm{S}(x,y) \subseteq \mathrm{A}(x,y) : x,y \in \mathrm{S}_0 \subseteq \mathrm{A}_0\}$ of R-submodules determines a (pre-)R-subalgebroid of A then, by abuse of language, we shall say that S is a (pre-)R-subalgebroid of A.

Definition 4. *[18, Chapter I, Definition 1.3.5] Let* A *be a pre-R-algebroid and* I *be a pre-R-subalgebroid of* A *with* $\mathrm{I}_0 = \mathrm{A}_0$*. If* $ab, ba' \in \mathrm{I}$ *for all* $b \in \mathrm{I}$ *and* $a, a' \in \mathrm{A}$ *with* $ta = sb$, $tb = sa'$ *then* I *is called a* two-sided ideal *of* A*.*

Definition 5. *Given a pre-R-algebroid* A *and two families* $\mathrm{S}_1 = \{\mathrm{S}_1(x,y) \subseteq \mathrm{A}(x,y) : x,y \in \mathrm{A}_0\}$ *and* $\mathrm{S}_2 = \{\mathrm{S}_2(x,y) \subseteq \mathrm{A}(x,y) : x,y \in \mathrm{A}_0\}$ *of subsets, the product* $\mathrm{S}_1\mathrm{S}_2$ *of* S_1 *and* S_2 *is defined as the family* $\{(\mathrm{S}_1\mathrm{S}_2)(x,y) \subseteq \mathrm{A}(x,y) : x,y \in \mathrm{A}_0\}$ *where* $(\mathrm{S}_1\mathrm{S}_2)(x,y) = \{ab : a \in \mathrm{S}_1(x,z), b \in \mathrm{S}_2(z,y), z \in \mathrm{A}_0\}$*.*

Definition 6. *[18, Chapter I, p. 10-11] Let* A *and* N *be two pre-R-algebroids with the same object set* A_0*. A family of maps defined for all* $x,y,z \in \mathrm{A}_0$ *as*

$$\begin{array}{ccc} \mathrm{N}(x,y) \times \mathrm{A}(y,z) & \longrightarrow & \mathrm{N}(x,z) \\ (n,a) & \longmapsto & n^a \end{array}$$

is called a right action *of* A *on* N *if the conditions*

1. $n^{a_1+a_2} = n^{a_1} + n^{a_2}$
2. $(n_1+n_2)^a = n_1^a + n_2^a$
3. $(n^a)^{a'} = n^{aa'}$
4. $(n'n)^a = n'n^a$
5. $r \cdot n^a = (r \cdot n)^a = n^{r \cdot a}$

and the condition $n^{1_{tn}} = n$*, whenever* 1_{tn} *exists, are satisfied for all* $r \in R$, $a, a', a_1, a_2 \in \mathrm{A}$, $n, n', n_1, n_2 \in \mathrm{N}$ *with appropriate sources and targets.*

A left action *of* A *on* N *is defined similarly, only with a side difference. Moreover, if* A *has a right and a left action on* N *and if* $({}^a n)^{a'} = {}^a\left(n^{a'}\right)$ *for all* $n \in \mathrm{N}$, $a, a' \in \mathrm{A}$ *with* $ta = sn$, $tn = sa'$ *then* A *is said to have an* associative action *on* N *or to* act on N associatively*.*

Corollary 1. *Given two pre-R-algebroids* A *and* N *with the same object set*
i. *if* A *has a left action on* N *then* ${}^{0_{\mathrm{A}(x,sn)}}n = 0_{\mathrm{A}(x,tn)}$ *and* ${}^{-a}n = {}^a(-n) = -{}^a n$*,*
ii. *if* A *has a right action on* N *then* $n^{0_{\mathrm{A}(tn,y)}} = 0_{\mathrm{A}(sn,y)}$ *and* $n^{-a'} = (-n)^{a'} = -n^{a'}$
for all $n \in \mathrm{N}$, $a, a' \in \mathrm{A}$, $x, y \in \mathrm{A}_0$ *with* $ta = sn, tn = sa'$*.*

Definition 7. *[18, Chapter I, Definition 1.3.2] Let* A *be an R-algebroid and* N *be a pre-R-algebroid with the same object set and let* A *have an associative action on* N*. A pre-R-algebroid morphism* $\eta : \mathrm{N} \to \mathrm{A}$ *is called a* crossed module of R-algebroids *if the conditions*

$$\begin{array}{ll} \text{CM1)} & \eta({}^a n) = a(\eta n) \quad \textit{and} \quad \eta\left(n^{a'}\right) = (\eta n)a' \\ \text{CM2)} & n^{\eta n'} = nn' = {}^{\eta n}n' \end{array}$$

are satisfied for all $a, a' \in \mathrm{A}$, $n, n' \in \mathrm{N}$ *with* $ta = sn$, $tn = sa' = sn'$*.* η *is called a* precrossed module of R-algebroids *if it satisfies CM1.*

Remark 2. *Note from the definition that a crossed module is a precrossed module satisfying CM2. Moreover, if* $\eta : \mathrm{N} \to \mathrm{A}$ *is a (pre)crossed module then each homset of* N *and of* A *is an R-module and so is not empty. Therefore, for each* $x \in \mathrm{A}_0$ *there exist morphisms* $a \in \mathrm{A}$ *and* $n \in \mathrm{N}$ *with* $x = sa$ *and* $ta = sn$. *Hence,* ${}^{a}n \in \mathrm{N}(x, tn)$ *and* $\eta({}^{a}n) \in \mathrm{A}(\eta x, \eta tn)$. *But,* $\eta({}^{a}n) = a(\eta n) \in \mathrm{A}(x, t\eta n)$ *and so* $\eta x = x$, *which means that* η *is the identity on* A_0.

Example 1. *[18, Chapter I, p. 12] If* A *is an R-algebroid and* I *is a two-sided ideal of* A *then the inclusion* $i : \mathrm{I} \to \mathrm{A}$ *is a crossed module, where* A *acts on* I *by composition.*

Proposition 1. *[18, Chapter I, Remark 1.3.6 & Propopsition 1.3.7] Given a pre-R-algebroid morphism* $f : \mathrm{A} \to \mathrm{B}$, *the family* $\mathrm{Ker}f = \{\mathrm{Ker}f(x, y) \subseteq \mathrm{A}(x, y) : x, y \in \mathrm{A}_0\}$, *where* $\mathrm{Ker}f(x, y) = \{a \in \mathrm{A}(x, y) : fa = 0 (= 0_{\mathrm{B}(fx, fy)})\}$, *is a two-sided ideal of* A *and the family* $\mathrm{Im}f = \{f(\mathrm{A}(x, y)) \subseteq \mathrm{B}(fx, fy) : x, y \in \mathrm{A}_0\}$ *is a pre-R-subalgebroid of* B. *If* f *is a morphism of R-algebroids then* $\mathrm{Im}f$ *is an R-subalgebroid and if* f *is a (pre)crossed module then* $\mathrm{Im}f$ *is two-sided ideal of* B.

Definition 8. *[18, Chapter I, Definition 1.3.3] Given two (pre)crossed modules* $\mathcal{N} = (\eta : \mathrm{N} \to \mathrm{A})$ *and* $\mathcal{N}' = (\eta' : \mathrm{N}' \to \mathrm{A}')$ *of R-algebroids, if* $f : \mathrm{N} \to \mathrm{N}'$ *is a pre-R-algebroid morphism,* $g : \mathrm{A} \to \mathrm{A}'$ *is an R-algebroid morphism and the conditions*

$$\begin{array}{ll} \text{CMM1)} & f({}^{a}n) = {}^{ga}(fn) \quad \textit{and} \quad f\left(n^{a'}\right) = (fn)^{ga'} \\ \text{CMM2)} & \eta' f = g\eta \end{array}$$

are satisfied for all $a, a' \in \mathrm{A}$, $n \in \mathrm{N}$ *with* $ta = sn$, $tn = sa'$ *then the pair* (f, g) *is called a* (pre)crossed module morphism, of R-algebroids, *from* $\mathcal{N}$ *to* $\mathcal{N}'$ *and we write* $(f, g) : \mathcal{N} \to \mathcal{N}'$ *for denoting it.*

Remark 3. *Note in the definition above that* $g\eta x = \eta' fx$ *for each* $x \in \mathrm{A}_0$ *by CMM2. But, as explained in Remark 2,* η *and* η' *are equal to the identities on* A_0 *and* A'_0, *respectively. Therefore,* $gx = g\eta x = \eta' fx = fx$ *meaning that if* (f, g) *is a (pre)crossed module morphism then* f *and* g *are equal to each other on the object set.*

Proposition 2. *If* $f : \mathrm{A} \to \mathrm{B}$ *and* $f' : \mathrm{B} \to \mathrm{C}$ *are (pre-)R-algebroid morphisms then the assignment* $f'f : \mathrm{A} \to \mathrm{C}$ *defined by* $(f'f)(x) = f'(fx)$ *on* A_0 *and by* $(f'f)(a) = f'(fa)$ *on* $\mathrm{Mor}(\mathrm{A})$ *is a (pre-)R-algebroid morphism.*

Proposition 3. *All precrossed modules of R-algebroids and their morphisms form a category, denoted by* $\mathrm{PXAlg}(R)$, *in which the composition of any two composible morphisms* (f, g) *and* (f', g') *is defined pointwisely by* $(f', g')(f, g) = (f'f, g'g)$, *where* $f'f$ *and* $g'g$ *are the corresponding composite pre-R-algebroid and R-algebroid morphisms. With the same composition, all crossed modules of R-algebroids and their morphisms form a category, denoted by* $\mathrm{XAlg}(R)$, *which is clearly a full subcategory of* $\mathrm{PXAlg}(R)$.

3 (Pre)cat^1-R-algebroids

Cat^1-algebras and more generally cat^n-algebras were introduced by Ellis in [10, 11]. In this section, as a generalisation of cat^1-algebras, we shall introduce the notion of a *precat*^{1}- and a *cat*^{1}-*R-algebroid*.

Throughout this section and subsequent sections, an *endomorphism* of an R-algebroid A will stand for an *R-algebroid morphism* from A to A:

Definition 9. *Given an R-algebroid* A *and two endomorphisms u and v of* A*, both being the identity on* A_0*, the triple* $\mathcal{A} = (\mathrm{A}, u, v)$ *is called a* cat^1-R-algebroid *if the conditions*

$$\begin{array}{ll} \text{CAT1)} & uv = v \quad \textit{and} \quad vu = u \\ \text{CAT2)} & \mathrm{Ker}u\mathrm{Ker}v = 0_{\mathrm{A}} = \mathrm{Ker}v\mathrm{Ker}u \ , \end{array}$$

where Ker$u$ *and* Ker$v$ *are defined as in Proposition 1 and their product as in Definition 5 and where* $0_{\mathrm{A}} = \left\{0_{\mathrm{A}(x,y)} : x, y \in \mathrm{A}_0\right\}$*, are satisfied.* $\mathcal{A}$ *is called a* precat^1-R-algebroid *if it satisfies CAT1.*

Note from the definition that a cat^1-R-algebroid is a precat^1-R-algebroid satisfying CAT2 and that if $\mathcal{A} = (\mathrm{A}, u, v)$ is a (pre)cat^1-R-algebroid then $(\mathrm{Ker}u)_0 = (\mathrm{Ker}v)_0 = (\mathrm{Im}u)_0 = (\mathrm{Im}v)_0 = \mathrm{A}_0$.

Definition 10. *Given two (pre)cat^1-R-algebroids* $\mathcal{A} = (\mathrm{A}, u, v)$ *and* $\mathcal{A}' = (\mathrm{A}', u', v')$*, a* (pre)cat^1-$R$-algebroid morphism $\{f\} : \mathcal{A} \to \mathcal{A}'$ *consists of an R-algebroid morphism* $f : \mathrm{A} \to \mathrm{A}'$ *satisfying the condition*

$$\text{CATM)} \quad fu = u'f \quad \textit{and} \quad fv = v'f$$

Proposition 4. *All precat^1-R-algebroids and their morphisms form a category, denoted by* PCat^1-Alg(R)*, in which the composition of any two composible morphisms* $\{f\} : \mathcal{A} \to \mathcal{A}'$ *and* $\{f'\} : \mathcal{A}' \to \mathcal{A}''$ *is defined by* $\{f'\}\{f\} = \{f'f\}$*, where $f'f$ is the corresponding composite R-algebroid morphism. Similarly, with the same composition, all cat^1-R-algebroids and their morphisms form a category, denoted by* Cat^1-Alg(R)*. Obviously,* Cat^1-Alg(R) *is a full subcategory of* PCat^1-Alg(R)*.*

Proposition 5. *If* $\mathcal{A} = (\mathrm{A}, u, v)$ *is a precat^1-R-algebroid then for all* $b \in \mathrm{Im}u \cup \mathrm{Im}v$

$$ub = vb = b \tag{3.1}$$

Proof. If $b \in \mathrm{Im}u$ then there exists an $a \in \mathrm{A}$ with $b = ua$, and so $vb = vua$. But, $vua = ua$ by CAT1. Thus, $vb = vua = ua = b$ meaning that the second equality holds. Moreover, $ub = uvb$ by the equality $b = vb$ just verified and $uvb = vb$ by CAT1. So, $ub = uvb = vb$ meaning that the first equality holds as well. A similar argument proves the same result when $b \in \mathrm{Im}v$. □

Corollary 2. *For any precat^1-R-algebroid* $\mathcal{A} = (\mathrm{A}, u, v)$ *the following hold:*

i. $\quad \mathrm{Im}u = \mathrm{Im}v \qquad (3.2)$

ii. $\quad uu = u \quad \textit{and} \quad vv = v \qquad (3.3)$

iii. $\quad a - ua \in \mathrm{Ker}u \quad \textit{and} \quad a - va \in \mathrm{Ker}v \quad \textit{for all } a \in \mathrm{A} \qquad (3.4)$

Proof. ***i.*** $b \in \mathrm{Im}u \Rightarrow b = vb$ by (3.1) $\Rightarrow b \in \mathrm{Im}v \Rightarrow \mathrm{Im}u \subseteq \mathrm{Im}v$. Similarly, $\mathrm{Im}v \subseteq \mathrm{Im}u$ and so $\mathrm{Im}u = \mathrm{Im}v$.
ii. For all $a \in A$, $va \in \mathrm{Im}v$ and so $vva = va$ by (3.1), meaning that $vv = v$. Similarly, $uu = u$.
iii. For all $a \in A$, $uua = ua$ by (3.3). Therefore, $u(a - ua) = ua - uua = ua - ua = 0$, meaning that $a - ua \in \mathrm{Ker}u$. Similarly, $a - va \in \mathrm{Ker}v$. □

4 Equivalence between PCat^1-$\mathrm{Alg}\,(R)$ and $\mathrm{PXAlg}\,(R)$

In this section, we shall first construct two functors, one from PCat^1-$\mathrm{Alg}\,(R)$ to $\mathrm{PXAlg}\,(R)$ and one in the opposite direction, and then prove that each of the functors defined is an equivalence of categories.

4.1 Construction of the functors

Given a precat^1-R-algebroid $\mathcal{A} = (\mathrm{A}, u, v)$, we define $\eta_{\mathcal{A}} : \mathrm{Ker}u \to \mathrm{Im}u$ as the identity on A_0 and by

$$\eta_{\mathcal{A}} a = va \tag{4.1}$$

on morphisms.

Proposition 6. *$\mathcal{N}_{\mathcal{A}} = (\eta_{\mathcal{A}} : \mathrm{Ker}u \to \mathrm{Im}u)$ is a precrossed module of R-algebroids.*

Proof. $\mathrm{Ker}u$ is a two-sided ideal, thus a pre-R-subalgebroid, and $\mathrm{Im}u$ is an R-subalgebroid of A by Proposition 1, $\eta_{\mathcal{A}}$ is clearly a pre-R-algebroid morphism and $\mathrm{Im}u$ has an associative action on $\mathrm{Ker}u$ defined as the composition in A. Moreover, for all $a \in \mathrm{Ker}u$ and $b, b' \in \mathrm{Im}u$ with $tb' = sa$ and $ta = sb$

$$\eta_{\mathcal{A}}\left(a^b\right) = v\,(ab) = (va)\,(vb) = (\eta_{\mathcal{A}} a)\,b,$$

where $vb = b$ by (3.1), and similarly $\eta_{\mathcal{A}}({}^{b'}a) = b'\,(\eta_{\mathcal{A}} a)$, meaning that CM1 is satisfied. □

Proposition 7. *Given two precat^1-R-algebroids $\mathcal{A} = (\mathrm{A}, u, v)$ and $\mathcal{A}' = (\mathrm{A}', u', v')$ and a precat^1-R-algebroid morphism $\{f\} : \mathcal{A} \to \mathcal{A}'$, the pair $(f_{\mathrm{Ker}u}, f_{\mathrm{Im}u})$ of f's restrictions $f_{\mathrm{Ker}u} : \mathrm{Ker}u \to \mathrm{Ker}u'$ and $f_{\mathrm{Im}u} : \mathrm{Im}u \to \mathrm{Im}u'$ is a precrossed module morphism from $\mathcal{N}_{\mathcal{A}}$ to $\mathcal{N}_{\mathcal{A}'}$.*

Proof. For all $a \in \mathrm{Ker}u$ and $b \in \mathrm{Im}u$, noting that $fu = u'f$ by CATM for f and that $b = ub$ by (3.1)

$$u' f_{\mathrm{Ker}u} a = u' f a = f u a = f0 = 0 \quad \text{and} \quad f_{\mathrm{Im}u} b = fb = fub = u'fb$$

meaning that $f_{\mathrm{Ker}u} a \in \mathrm{Ker}u'$ and $f_{\mathrm{Im}u} b \in \mathrm{Im}u'$. Hence, $f_{\mathrm{Ker}u}$ and $f_{\mathrm{Im}u}$ are well-defined, since so is f, and the verification that $f_{\mathrm{Ker}u}$ is a pre-R-algebroid morphism and $f_{\mathrm{Im}u}$ is an R-algebroid morphism is straightforward, since all we need are inherited from f. Moreover,

$$f_{\mathrm{Ker}u}\left(a^b\right) = f\,(ab) = fafb = (fa)^{fb} = (f_{\mathrm{Ker}u} a)^{f_{\mathrm{Im}u} b}$$

and similarly $f_{\mathrm{Ker}u}({}^{b'}a) = {}^{f_{\mathrm{Im}u} b'}\,(f_{\mathrm{Ker}u} a)$ for all $a \in \mathrm{Ker}u$ and $b, b' \in \mathrm{Im}u$ with $ta = sb$ and $tb' = sa$, meaning that CMM1 is satisfied. Furthermore, $\eta_{\mathcal{A}'} f_{\mathrm{Ker}u} = f_{\mathrm{Im}u} \eta_{\mathcal{A}}$, since $v'f = fv$ by CATM for f, and thus CMM2 is satisfied. □

Then, a direct calculation proves the following proposition:

Proposition 8. *The assignment $F : \mathrm{PCat}^1\text{-}\mathrm{Alg}\,(R) \to \mathrm{PXAlg}\,(R)$ defined by $F\mathcal{A} = \mathcal{N}_{\mathcal{A}}$ on objects and by $F\{f\} = (f_{\mathrm{Ker}u}, f_{\mathrm{Im}u})$ on morphisms is a functor.*

Now, to develop an inverse functor we define the semidirect product pre-R-algebroid $\mathrm{A} \ltimes \mathrm{N}$ for a precrossed module $\mathcal{N} = (\eta : \mathrm{N} \to \mathrm{A})$ of R-algebroids as in [2], where in brief, we

i. set $\mathrm{A} \ltimes \mathrm{N}$ as the family $\{(\mathrm{A} \ltimes \mathrm{N})(x, y) : x, y \in \mathrm{A}_0\}$ such that $(\mathrm{A} \ltimes \mathrm{N})(x, y) = \{(a, n) : a \in \mathrm{A}(x, y), n \in \mathrm{N}(x, y)\}$,

ii. define on each $(\mathrm{A} \ltimes \mathrm{N})(x, y)$ an addition by $(a, n) + (a_1, n_1) = (a + a_1, n + n_1)$ and an R-action by $r \cdot (a, n) = (r \cdot a, r \cdot n)$,

iii. take $(\mathrm{A} \ltimes \mathrm{N})_0 = \mathrm{A}_0$ and define $s, t : \mathrm{Mor}(\mathrm{A} \ltimes \mathrm{N}) \to (\mathrm{A} \ltimes \mathrm{N})_0$, the source and target functions respectively, by $s(a, n) = sa\,(= sn)$ and $t(a, n) = ta\,(= tn)$,

iv. define a composition on $\mathrm{A} \ltimes \mathrm{N}$ by $(a, n)(a', n') = (aa', n^{a'} + {}^{a}n' + nn')$ for all morphisms (a, n), (a', n') with $t(a, n) = s(a', n')$.

Proposition 9. $\mathrm{A} \ltimes \mathrm{N}$ *is an R-algebroid.*

Proof. As proved in [2] (Sect. 4, Proposition 4), $\mathrm{A} \ltimes \mathrm{N}$ is a pre-R-algebroid. Moreover, a direct calculation shows for each $x \in \mathrm{A}_0$ that the pair $\left(1_x, 0_{\mathrm{N}(x,x)}\right)$ is the identity on x, completing the proof. ∎

Now, in a similar way to that used in [19], we define $u_\eta, v_\eta : \mathrm{A} \ltimes \mathrm{N} \to \mathrm{A} \ltimes \mathrm{N}$ as the identity on objects and by

$$u_\eta(a, n) = (a, 0_{\mathrm{N}(sa,ta)}) \quad \text{and} \quad v_\eta(a, n) = (a + \eta n, 0_{\mathrm{N}(sa,ta)}) \tag{4.2}$$

on morphisms.

Proposition 10. u_η *and* v_η *are endomorphisms of* $\mathrm{A} \ltimes \mathrm{N}$.

Proof. We restrict the proof only to verifying that v_η preserves the composition, since the rest are clear: For all (a, n), $(a', n') \in \mathrm{A} \ltimes \mathrm{N}$ with $t(a, n) = s(a', n')$

$$\begin{aligned} v_\eta((a, n)(a', n')) &= v_\eta(aa', n^{a'} + {}^{a}n' + nn') = (aa' + \eta(n^{a'} + {}^{a}n' + nn'), 0) \\ &= (aa' + (\eta n)a' + a(\eta n') + (\eta n)(\eta n'), 0) \\ &= (a + \eta n, 0)(a' + \eta n', 0) = v_\eta(a, n)\, v_\eta(a', n'), \end{aligned}$$

as required. ∎

Proposition 11. $\mathcal{N}^{\ltimes} = (\mathrm{A} \ltimes \mathrm{N}, u_\eta, v_\eta)$ *is a precat^1-R-algebroid.*

Proof. u_η, v_η are endomorphisms of $\mathrm{A} \ltimes \mathrm{N}$ by Proposition 10. Moreover, for all $(a, n) \in \mathrm{A} \ltimes \mathrm{N}$

$$u_\eta v_\eta(a, n) = u_\eta(a + \eta n, 0) = (a + \eta n, 0) = v_\eta(a, n),$$

by (4.2), meaning that $u_\eta v_\eta = v_\eta$. Similarly, $v_\eta u_\eta = u_\eta$ and thus CAT1 is satisfied. ∎

Proposition 12. *Given two precrossed modules $\mathcal{N}=(\eta:\mathrm{N}\to\mathrm{A})$ and $\mathcal{N}'=(\eta':\mathrm{N}'\to\mathrm{A}')$ of R-algebroids and a precrossed module morphism $(f,g):\mathcal{N}\to\mathcal{N}'$, if $\sigma_g^f:\mathrm{A}\ltimes\mathrm{N}\to\mathrm{A}'\ltimes\mathrm{N}'$ is defined by $\sigma_g^f(x,y)=(gx,fy)$ on objects and by*

$$\sigma_g^f(a,n)=(ga,fn) \tag{4.3}$$

on morphisms then $\{\sigma_g^f\}$ forms a precat^1-R-algebroid morphism from $\mathcal{N}^{\ltimes}$ to $\mathcal{N}'^{\ltimes}$.

Proof. A direct calculation proves that σ_g^f is an R-algebroid morphism. Moreover, for all $(a,n)\in\mathrm{A}\ltimes\mathrm{N}$

$$\begin{aligned}\left(v_{\eta'}\sigma_g^f\right)(a,n)&=v_{\eta'}(ga,fn)=(ga+\eta'fn,0)=(ga+g\eta n,0)\\&=(g(a+\eta n),0)=\sigma_g^f(a+\eta n,0)=\left(\sigma_g^f v_\eta\right)(a,n),\end{aligned}$$

where $\eta'fn=g\eta n$ by CMM2 for (f,g), meaning that $v_{\eta'}\sigma_g^f=\sigma_g^f v_\eta$. Similarly, $u_{\eta'}\sigma_g^f=\sigma_g^f u_\eta$ and thus CATM is satisfied. □

A direct calculation proves the following proposition:

Proposition 13. *For any two precrossed module morphisms $(f,g):\mathcal{N}\to\mathcal{N}'$ and $(f',g'):\mathcal{N}'\to\mathcal{N}''$*

$$\sigma_{g'}^{f'}\sigma_g^f=\sigma_{g'g}^{f'f}.$$

Proposition 14. *The assignment $G:\mathrm{PXAlg}(R)\to\mathrm{PCat}^1\text{-}\mathrm{Alg}(R)$ defined by $G\mathcal{N}=\mathcal{N}^{\ltimes}$ on objects and by $G(f,g)=\{\sigma_g^f\}$ on morphisms is a functor.*

Proof. G is well-defined on both objects and morphisms and it preserves the composition, by Proposition 11, 12 and 13. Moreover, for any precrossed module $\mathcal{N}$ and precrossed module morphism $(f,g):\mathcal{N}\to\mathcal{N}'$ of R-algebroids, obviously $G(f,g)$ is from $G\mathcal{N}$ to $G\mathcal{N}'$ and $G\{id_{\mathcal{N}}\}=id_{G\mathcal{N}}$, as required. □

4.2 Construction of the equivalence

Given a precat^1-R-algebroid $\mathcal{A}=(\mathrm{A},u,v)$, using the two functors F and G, we can get the precat^1-R-algebroid

$$GF\mathcal{A}=G\mathcal{N}_{\mathcal{A}}=(\mathcal{N}_{\mathcal{A}})^{\ltimes}=\left(\mathrm{Im}u\ltimes\mathrm{Ker}u,u_{\eta_{\mathcal{A}}},v_{\eta_{\mathcal{A}}}\right),$$

in which

i. $F\mathcal{A}=\mathcal{N}_{\mathcal{A}}=(\eta_{\mathcal{A}}:\mathrm{Ker}u\to\mathrm{Im}u)$, where $\eta_{\mathcal{A}}a=va$ by (4.1), and $(\mathrm{Im}u\ltimes\mathrm{Ker}u)_0=\mathrm{A}_0$, and

ii. $u_{\eta_{\mathcal{A}}}(b,a)=(b,0)$ and $v_{\eta_{\mathcal{A}}}(b,a)=(b+\eta_{\mathcal{A}}a,0)=(b+va,0)$ for all $(b,a)\in\mathrm{Im}u\ltimes\mathrm{Ker}u$, by (4.2).

Now, we define $\alpha_{\mathcal{A}} : \mathrm{A} \to \mathrm{Im}u \ltimes \mathrm{Ker}u$ and $\beta_{\mathcal{A}} : \mathrm{Im}u \ltimes \mathrm{Ker}u \to \mathrm{A}$ as the identity on A_0 and by

$$\alpha_{\mathcal{A}} a = (ua, a - ua) \quad \text{and} \quad \beta_{\mathcal{A}} (b, a) = b + a \tag{4.4}$$

on morphisms.

Lemma 1. *$\{\alpha_{\mathcal{A}}\}$ is a precat^1-R-algebroid morphism from $\mathcal{A}$ to $(\mathcal{N}_{\mathcal{A}})^{\ltimes}$.*

Proof. $a - ua \in \mathrm{Ker}u$ by (3.4) and therefore $\alpha_{\mathcal{A}} a = (ua, a - ua) \in \mathrm{Im}u \ltimes \mathrm{Ker}u$ for all $a \in \mathrm{A}$. In addition, $\alpha_{\mathcal{A}}$ is well-defined, since so is u, and it clearly preserves the addition and R-action. Moreover,

$$\begin{aligned}
\alpha_{\mathcal{A}} (aa') &= (u (aa'), aa' - u (aa')) = ((ua)(ua'), aa' - (ua)(ua')) \\
&= ((ua)(ua'), (a - ua) ua' + ua (a' - ua') + (a - ua)(a' - ua')) \\
&= ((ua)(ua'), (a - ua)^{ua'} + {}^{ua} (a' - ua') + (a - ua)(a' - ua')) \\
&= (ua, a - ua)(ua', a' - ua') = (\alpha_{\mathcal{A}} a)(\alpha_{\mathcal{A}} a')
\end{aligned}$$

and $\alpha_{\mathcal{A}} 1_x = (u1_x, 1_x - u1_x) = (1_x, 1_x - 1_x) = \left(1_x, 0_{\mathrm{N}(x,x)}\right)$ for all $a, a' \in \mathrm{A}$ with $ta = sa'$ and for all $x \in \mathrm{A}_0$, meaning that $\alpha_{\mathcal{A}}$ is an R-algebroid morphism. Furthermore, for all $a \in \mathrm{A}$

$$\begin{aligned}
(u_{\eta_{\mathcal{A}}} \alpha_{\mathcal{A}}) (a) = u_{\eta_{\mathcal{A}}} (ua, a - ua) &= (ua, 0) = (ua, ua - ua) \\
&= (uua, ua - uua) = \alpha_{\mathcal{A}} (ua) = (\alpha_{\mathcal{A}} u)(a),
\end{aligned}$$

where $uua = ua$ by (3.3), and

$$\begin{aligned}
(v_{\eta_{\mathcal{A}}} \alpha_{\mathcal{A}}) (a) = v_{\eta_{\mathcal{A}}} (ua, a - ua) &= (ua + v (a - ua), 0) = (ua + va - vua, va - va) \\
&= (ua + uva - ua, va - uva) = (uva, va - uva) = \alpha_{\mathcal{A}} (va) = (\alpha_{\mathcal{A}} v)(a),
\end{aligned}$$

where $va = uva$ and $vua = ua$ by CAT1 for $\mathcal{A}$. So, we get $u_{\eta_{\mathcal{A}}} \alpha_{\mathcal{A}} = \alpha_{\mathcal{A}} u$ and $v_{\eta_{\mathcal{A}}} \alpha_{\mathcal{A}} = \alpha_{\mathcal{A}} v$, meaning that CATM is satisfied. ∎

Lemma 2. *$\{\beta_{\mathcal{A}}\}$ is a precat^1-R-algebroid morphism from $(\mathcal{N}_{\mathcal{A}})^{\ltimes}$ to $\mathcal{A}$.*

Proof. A direct calculation shows that $\beta_{\mathcal{A}}$ is an R-algebroid morphism from $\mathrm{Im}u \ltimes \mathrm{Ker}u$ to A. Moreover,

$$(v\beta_{\mathcal{A}})(b, a) = v (b + a) = vb + va = b + va = \beta_{\mathcal{A}} (b + va, 0) = (\beta_{\mathcal{A}} v_{\eta_{\mathcal{A}}})(b, a)$$

for all $(b, a) \in \mathrm{Im}u \ltimes \mathrm{Ker}u$, where $vb = b$ by (3.1), meaning that $v\beta_{\mathcal{A}} = \beta_{\mathcal{A}} v_{\eta_{\mathcal{A}}}$. Similarly, $u\beta_{\mathcal{A}} = \beta_{\mathcal{A}} u_{\eta_{\mathcal{A}}}$ and thus CATM is satisfied. ∎

Up to now, given a precat^1-R-algebroid $\mathcal{A} = (\mathrm{A}, u, v)$, we have constructed two precat^1-R-algebroid morphisms, $\{\alpha_{\mathcal{A}}\}$ and $\{\beta_{\mathcal{A}}\}$, the first of which is from $\mathcal{A}$ to $GF\mathcal{A}$ and the latter is in the opposite direction. Below, we shall do the same in the category of precrossed modules of R-algebroids:

Let $\mathcal{N} = (\eta : \mathrm{N} \to \mathrm{A})$ be a precrossed module of R-algebroids. Using the two functors F and G, we can get the precrossed module

$$FG\mathcal{N} = F\mathcal{N}^{\ltimes} = \mathcal{N}_{\mathcal{N}^{\ltimes}} = \left(\eta_{\mathcal{N}^{\ltimes}} : \mathrm{Ker}u_\eta \to \mathrm{Im}u_\eta\right),$$

in which

i. $G\mathcal{N} = \mathcal{N}^{\ltimes} = (\mathrm{A} \ltimes \mathrm{N}, u_\eta, v_\eta)$, where $u_\eta(a,n) = (a,0)$ and $v_\eta(a,n) = (a + \eta n, 0)$ by (4.2), $\mathrm{Ker}u_\eta = \left\{(0,n) = \left(0_{\mathrm{A}(sn,tn)}, n\right) : n \in \mathrm{N}\right\}$, $\mathrm{Im}u_\eta = \left\{(a,0) = \left(a, 0_{\mathrm{N}(sa,ta)}\right) : a \in \mathrm{A}\right\}$ and $(\mathrm{Ker}u_\eta)_0 = (\mathrm{Im}u_\eta)_0 = \mathrm{A}_0$, and

ii. $\eta_{\mathcal{N}^{\ltimes}}(0,n) = v_\eta(0,n) = (0 + \eta n, 0) = (\eta n, 0)$ for all $(0,n) \in \mathrm{Ker}u_\eta$, by (4.1) and (4.2).

Now, we define $\delta^\eta_{\mathrm{N}} : \mathrm{Ker}u_\eta \to \mathrm{N}$, $\delta^\eta_{\mathrm{A}} : \mathrm{Im}u_\eta \to \mathrm{A}$, $\lambda^\eta_{\mathrm{N}} : \mathrm{N} \to \mathrm{Ker}u_\eta$ and $\lambda^\eta_{\mathrm{A}} : \mathrm{A} \to \mathrm{Im}u_\eta$ as the identity on objects and by

$$\delta^\eta_{\mathrm{N}}(0,n) = n, \;\; \delta^\eta_{\mathrm{A}}(a,0) = a, \;\; \lambda^\eta_{\mathrm{N}} n = (0,n), \;\; \lambda^\eta_{\mathrm{A}} a = (a,0) \tag{4.5}$$

on morphisms.
A direct calculation proves the following lemma:

Lemma 3. *The pair $\left(\delta^\eta_{\mathrm{N}}, \delta^\eta_{\mathrm{A}}\right)$ is a precrossed module morphism from $\mathcal{N}_{\mathcal{N}^{\ltimes}}$ to $\mathcal{N}$ and the pair $\left(\lambda^\eta_{\mathrm{N}}, \lambda^\eta_{\mathrm{A}}\right)$ is a precrossed module morphism in the opposite direction.*

Theorem 1. *The functor F (and G) is an equivalence of categories between* $\mathrm{PCat}^1\text{-}\mathrm{Alg}(R)$ *and* $\mathrm{PXAlg}(R)$.

Proof. Given a precat^1-R-algebroid $\mathcal{A} = (\mathrm{A}, u, v)$, $\{\alpha_{\mathcal{A}}\} : \mathcal{A} \to (\mathcal{N}_{\mathcal{A}})^{\ltimes}$ and $\{\beta_{\mathcal{A}}\} : (\mathcal{N}_{\mathcal{A}})^{\ltimes} \to \mathcal{A}$ are precat^1-R-algebroid morphisms by Lemma 1 and 2, respectively. Moreover, by (4.4)

$$\begin{aligned}(\alpha_{\mathcal{A}}\beta_{\mathcal{A}})(b,a) &= \alpha_{\mathcal{A}}(b+a) = (u(b+a), (b+a) - u(b+a)) \\ &= (ub + ua, b + a - ub - ua) = (b,a)\end{aligned}$$

for all $(b,a) \in \mathrm{Im}u \ltimes \mathrm{Ker}u$, where $ub = b$ by (3.1) and $ua = 0$ since $a \in \mathrm{Ker}u$. That is, $\alpha_{\mathcal{A}}\beta_{\mathcal{A}} = id_{\mathrm{Im}u \ltimes \mathrm{Ker}u}$. A similar calculation shows that $\beta_{\mathcal{A}}\alpha_{\mathcal{A}} = id_{\mathrm{A}}$. Therefore, $\alpha_{\mathcal{A}}$ (and $\beta_{\mathcal{A}}$) is an R-algebroid isomorphism and thus $\{\alpha_{\mathcal{A}}\}$ (and $\{\beta_{\mathcal{A}}\}$) is an isomorphism in $\mathrm{PCat}^1\text{-}\mathrm{Alg}(R)$.

Furthermore, for all $\{f\} : \mathcal{A} \to \mathcal{A}'$ in $\mathrm{PCat}^1\text{-}\mathrm{Alg}(R)$ with $\mathcal{A}' = (\mathrm{A}', u', v')$, noting that $GF\{f\} = G(f_{\mathrm{Ker}u}, f_{\mathrm{Im}u}) = \left\{\sigma^{f_{\mathrm{Ker}u}}_{f_{\mathrm{Im}u}}\right\}$ and $fu = u'f$ by CATM for $\{f\}$,

$$\begin{aligned}(\sigma^{f_{\mathrm{Ker}u}}_{f_{\mathrm{Im}u}}\alpha_{\mathcal{A}})(a) &= \sigma^{f_{\mathrm{Ker}u}}_{f_{\mathrm{Im}u}}(ua, a - ua) = (f_{\mathrm{Im}u}ua, f_{\mathrm{Ker}u}(a - ua)) \\ &= (fua, fa - fua) = (u'fa, fa - u'fa) = \left(\alpha_{\mathcal{A}'}f\right)(a)\end{aligned}$$

for all $a \in \mathrm{A}$. That is, $\sigma^{f_{\mathrm{Ker}u}}_{f_{\mathrm{Im}u}}\alpha_{\mathcal{A}} = \alpha_{\mathcal{A}'}f$ and so $(GF\{f\})\{\alpha_{\mathcal{A}}\} = \left\{\alpha_{\mathcal{A}'}\right\}\{f\}$, meaning that the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\{\alpha_{\mathcal{A}}\}"] \arrow[d, "\{f\}"'] & GF\mathcal{A} = (\mathcal{N}_{\mathcal{A}})^{\ltimes} = \left(\mathrm{Im}u \ltimes \mathrm{Ker}u, u_{\eta_{\mathcal{A}}}, v_{\eta_{\mathcal{A}}}\right) \arrow[d, "GF\{f\} = \left\{\sigma^{f_{\mathrm{Ker}u}}_{f_{\mathrm{Im}u}}\right\}"] \\
\mathcal{A}' \arrow[r, "\{\alpha_{\mathcal{A}'}\}"'] & GF\mathcal{A}' = \left(\mathcal{N}_{\mathcal{A}'}\right)^{\ltimes} = \left(\mathrm{Im}u' \ltimes \mathrm{Ker}u', u_{\eta_{\mathcal{A}'}}, v_{\eta_{\mathcal{A}'}}\right)
\end{tikzcd}$$

is commutative. Thus, the family $\boldsymbol{\alpha} = \left\{\{\alpha_{\mathcal{A}}\} : \mathcal{A} \in \mathrm{PCat}^1\text{-}\mathrm{Alg}\,(R)\right\}$ is a natural isomorphism between the identity functor $I_{\mathrm{PCat}^1\text{-}\mathrm{Alg}(R)}$ and the composite functor GF on $\mathrm{PCat}^1\text{-}\mathrm{Alg}\,(R)$.

On the other hand, given a precrossed module $\mathcal{N} = (\eta : \mathrm{N} \to \mathrm{A})$ of R-algebroids, the pair $\left(\delta^\eta_{\mathrm{N}}, \delta^\eta_{\mathrm{A}}\right)$ defined by (4.5) is a precrossed module morphism by Lemma 3. It can also be shown through direct calculations that $\left(\delta^\eta_{\mathrm{N}}, \delta^\eta_{\mathrm{A}}\right)$ is an isomorphism of precrossed modules. Moreover, for all $(f, g) : \mathcal{N} \to \mathcal{N}'$ in $\mathrm{PXAlg}\,(R)$ with $\mathcal{N}' = (\eta' : \mathrm{N}' \to \mathrm{A}')$, noting that $FG\,(f, g) = F\left\{\sigma^f_g\right\} = \left(\left(\sigma^f_g\right)_{\mathrm{Ker}u_\eta}, \left(\sigma^f_g\right)_{\mathrm{Im}u_\eta}\right)$,

$$\left(\delta^{\eta'}_{\mathrm{N}'} \left(\sigma^f_g\right)_{\mathrm{Ker}u_\eta}\right)(0, n) = \delta^{\eta'}_{\mathrm{N}'}\,(0, fn) = fn = \left(f\delta^\eta_{\mathrm{N}}\right)(0, n)$$

for all $(0, n) \in \mathrm{Ker}u_\eta$, meaning that $\delta^{\eta'}_{\mathrm{N}'} \left(\sigma^f_g\right)_{\mathrm{Ker}u_\eta} = f\delta^\eta_{\mathrm{N}}$. A similar calculation shows that $\delta^{\eta'}_{\mathrm{A}'} \left(\sigma^f_g\right)_{\mathrm{Im}u_\eta} = g\delta^\eta_{\mathrm{A}}$. Thus, $\left(\delta^{\eta'}_{\mathrm{N}'}, \delta^{\eta'}_{\mathrm{A}'}\right) FG\,(f, g) = (f, g)\left(\delta^\eta_{\mathrm{N}}, \delta^\eta_{\mathrm{A}}\right)$, i.e., the diagram

$$\begin{tikzcd}
FG\mathcal{N} = \mathcal{N}_{\mathcal{N}^{\ltimes}} = \left(\eta_{\mathcal{N}^{\ltimes}} : \mathrm{Ker}u_\eta \to \mathrm{Im}u_\eta\right) \arrow[r, "\left(\delta^\eta_{\mathrm{N}}, \delta^\eta_{\mathrm{A}}\right)"] \arrow[d, "FG(f,g) = \left(\left(\sigma^f_g\right)_{\mathrm{Ker}u_\eta}, \left(\sigma^f_g\right)_{\mathrm{Im}u_\eta}\right)"'] & \mathcal{N} \arrow[d, "(f,g)"] \\
FG\mathcal{N}' = \mathcal{N}_{\mathcal{N}'^{\ltimes}} = \left(\eta_{\mathcal{N}'^{\ltimes}} : \mathrm{Ker}u_{\eta'} \to \mathrm{Im}u_{\eta'}\right) \arrow[r, "\left(\delta^{\eta'}_{\mathrm{N}'}, \delta^{\eta'}_{\mathrm{A}'}\right)"'] & \mathcal{N}'
\end{tikzcd}$$

is commutative, and so the family $\boldsymbol{\delta} = \left\{\left(\delta^\eta_{\mathrm{N}}, \delta^\eta_{\mathrm{A}}\right) : \mathcal{N} = (\eta : \mathrm{N} \to \mathrm{A}) \in \mathrm{PXAlg}\,(R)\right\}$ is a natural isomorphism between the composite functor FG and the identity functor $I_{\mathrm{PXAlg}(R)}$ on $\mathrm{PXAlg}\,(R)$, as required. □

Conclusion 1. *The categories* $\mathrm{PCat}^1\text{-}\mathrm{Alg}\,(R)$ *and* $\mathrm{PXAlg}\,(R)$ *are equivalent.*

5 Equivalence between $\mathrm{Cat}^1\text{-}\mathrm{Alg}\,(R)$ and $\mathrm{XAlg}\,(R)$

In this section, what we do, in essence, is to upgrade the equivalence between $\mathrm{PCat}^1\text{-}\mathrm{Alg}\,(R)$ and $\mathrm{PXAlg}\,(R)$ obtained in the previous section to an equivalence between $\mathrm{Cat}^1\text{-}\mathrm{Alg}\,(R)$ and $\mathrm{XAlg}\,(R)$.
Assume that we are given a cat^1-R-algebroid $\mathcal{A} = (\mathrm{A}, u, v)$:

Proposition 15. *The following equalities hold for all* $a, a' \in \mathrm{Ker}u$ *with* $sa = ta'$*:*

$$a\,(va') = aa' = (va)\,a' \tag{5.1}$$

Proof. $va' - a' \in \mathrm{Ker}u$ by (3.4) and so $a\,(va') - aa' = a\,(va' - a') = 0$ by CAT2 for $\mathcal{A}$, meaning that $a\,(va') = aa'$. A similar argument proves the right-hand equality $aa' = (va)\,a'$. □

Proposition 16. $\mathcal{N}_{\mathcal{A}} = (\eta_{\mathcal{A}} : \mathrm{Ker}u \to \mathrm{Im}u)$ *is a crossed module.*

Proof. $\mathcal{N}_{\mathcal{A}}$ is a precrossed module by Proposition 6. Moreover, thanks to (5.1),

$$a^{\eta_{\mathcal{A}} a'} = a^{va'} = a\,(va') = aa' = (va)\,a' = {}^{va}a' = {}^{\eta_{\mathcal{A}} a}a'$$

for all $a, a' \in \mathrm{Ker}u$ with $ta = sa'$. So, CM2 is satisfied for $\mathcal{N}_{\mathcal{A}}$, as required. ∎

Now, assume that we are given a crossed module $\mathcal{N} = (\eta : \mathrm{N} \to \mathrm{A})$ of R-algebroids:

Proposition 17. *$\mathcal{N}^{\ltimes} = (\mathrm{A} \ltimes \mathrm{N}, u_\eta, v_\eta)$ is a cat^1-R-algebroid.*

Proof. $\mathcal{N}^{\ltimes}$ is a precat^1-R-algebroid by Proposition 11. Moreover, $\mathrm{Ker}u_\eta = \{(0, n) : n \in \mathrm{N}\}$ and

$$(a, n) \in \mathrm{Ker}v_\eta \Leftrightarrow v_\eta\,(a, n) = (0, 0) \Leftrightarrow (a + \eta n, 0) = (0, 0) \Leftrightarrow a = -\eta n$$

for any $(a, n) \in \mathrm{A} \ltimes \mathrm{N}$, meaning that $\mathrm{Ker}v_\eta = \{(-\eta n, n) : n \in \mathrm{N}\}$. Thus, for all $n, n' \in \mathrm{N}$ with $tn = sn'$

$$(0, n)\,(-\eta n', n') = \left(0\,(-\eta n')\,, n^{-\eta n'} + {}^{0}n' + nn'\right) = (0, -nn' + nn') = (0, 0)\,,$$

where $n^{-\eta n'} = -nn'$ by CM2 for $\mathcal{N}$, meaning that $\mathrm{Ker}u_\eta \mathrm{Ker}v_\eta = 0_{\mathrm{A} \ltimes \mathrm{N}}$. Similarly, $\mathrm{Ker}v_\eta \mathrm{Ker}u_\eta = 0_{\mathrm{A} \ltimes \mathrm{N}}$ and so CAT2 is satisfied for $\mathcal{N}^{\ltimes}$. ∎

Now, we define the functors $\tilde{F} : \mathrm{Cat}^1\text{-}\mathrm{Alg}\,(R) \to \mathrm{XAlg}\,(R)$ and $\tilde{G} : \mathrm{XAlg}\,(R) \to \mathrm{Cat}^1\text{-}\mathrm{Alg}\,(R)$ respectively as the restrictions of the functors F and G defined in Sect. 4.1, i.e., by $\tilde{F}\mathcal{A} = \mathcal{N}_{\mathcal{A}}$ and $\tilde{G}\mathcal{N} = \mathcal{N}^{\ltimes}$ on objects and by $\tilde{F}\,(f) = (f_{\mathrm{Ker}u}, f_{\mathrm{Im}u})$ and $\tilde{G}\,(f, g) = \left\{\sigma_g^f\right\}$ on morphisms.

Theorem 2. *The functor $\tilde{F}$ (and $\tilde{G}$) is an equivalence of categories between* $\mathrm{Cat}^1\text{-}\mathrm{Alg}\,(R)$ *and* $\mathrm{XAlg}\,(R)$.

Proof. Given a precat^1-R-algebroid $\mathcal{A}$ and a precrossed module $\mathcal{N} = (\eta : \mathrm{N} \to \mathrm{A})$ of R-algebroids, we have already constructed in Sect. 4.2 two precat^1-R-algebroid isomorphisms, $\{\alpha_{\mathcal{A}}\}$ from $\mathcal{A}$ to $GF\mathcal{A}$ and $\{\beta_{\mathcal{A}}\}$ in the opposite direction, and two precrossed module isomorphisms, $\left(\delta_{\mathrm{N}}^\eta, \delta_{\mathrm{A}}^\eta\right)$ from $FG\mathcal{N}$ to $\mathcal{N}$ and $\left(\lambda_{\mathrm{N}}^\eta, \lambda_{\mathrm{A}}^\eta\right)$ in the opposite direction. It is clear that $\{\alpha_{\mathcal{A}}\}$ and $\{\beta_{\mathcal{A}}\}$ are also cat^1-R-algebroid isomorphisms from $\mathcal{A}$ to $\tilde{G}\tilde{F}\mathcal{A}$ and from $\tilde{G}\tilde{F}\mathcal{A}$ to $\mathcal{A}$, respectively, and $\left(\delta_{\mathrm{N}}^\eta, \delta_{\mathrm{A}}^\eta\right)$ and $\left(\lambda_{\mathrm{N}}^\eta, \lambda_{\mathrm{A}}^\eta\right)$ are precrossed module isomorphisms from $\tilde{F}\tilde{G}\mathcal{N}$ to $\mathcal{N}$ and from $\mathcal{N}$ to $\tilde{F}\tilde{G}\mathcal{N}$, respectively, in this current case, where $\mathcal{A}$ is a cat^1-R-algebroid and $\mathcal{N}$ is a crossed module of R-algebroids. Therefore, the proof is almost the same as that of Theorem 1. ∎

Conclusion 2. *The categories* $\mathrm{Cat}^1\text{-}\mathrm{Alg}\,(R)$ *and* $\mathrm{XAlg}\,(R)$ *are equivalent.*

6 Consequences and Applications

6.1 Consequences

As stated and proved by Mac Lane in [14] (Sect. 4, Theorem 1) in a more general setting, the equivalence between the categories $\mathrm{Cat}^1\text{-Alg}\,(R)$ and $\mathrm{XAlg}\,(R)$ have the following consequences:

CS1. $\tilde{F}$ is both a left and a right adjoint of the functor $\tilde{G}$.

CS2. Each of the functors $\tilde{F}$ and $\tilde{G}$ is full and faithfull.

CS3. For each $\mathcal{N} \in \mathrm{XAlg}\,(R)$ there exists an $\mathcal{A} \in \mathrm{Cat}^1\text{-Alg}\,(R)$ with $\tilde{F}\mathcal{A} \cong \mathcal{N}$, and such an $\mathcal{A}$ is $\tilde{G}\mathcal{N} = \mathcal{N}^{\ltimes}$.

CS4. For each $\mathcal{A} \in \mathrm{Cat}^1\text{-Alg}\,(R)$ there exists an $\mathcal{N} \in \mathrm{XAlg}\,(R)$ with $\tilde{G}\mathcal{N} \cong \mathcal{A}$, and such an $\mathcal{N}$ is $\tilde{F}\mathcal{A} = \mathcal{N}_{\mathcal{A}}$.

The same consequences apply for the equivalence between the categories $\mathrm{PCat}^1\text{-Alg}$ and $\mathrm{PXAlg}\,(R)$.

Remark 4. *Since $\tilde{F}$ is full, by* CS2, *for any morphism $(f_1, f_2) : \tilde{F}\mathcal{A} \to \tilde{F}\mathcal{A}'$ there exists a morphism $\{f\} : \mathcal{A} \to \mathcal{A}'$ with $\tilde{F}\{f\} = (f_1, f_2)$, and from the commutative diagram below, such a morphism $\{f\}$ must satisfy the equalities $\{f\} = \{\beta_{\mathcal{A}'}\}\,(\tilde{G}\tilde{F}\{f\})\,\{\alpha_{\mathcal{A}}\} = \{\beta_{\mathcal{A}'}\sigma_{f_1}^{f_2}\alpha_{\mathcal{A}}\}$. So, the map $f : \mathrm{A} \to \mathrm{A}'$ should be defined by $fa = \beta_{\mathcal{A}'}\sigma_{f_2}^{f_1}\alpha_{\mathcal{A}}a = \beta_{\mathcal{A}'}\sigma_{f_2}^{f_1}(ua, a - ua) = \beta_{\mathcal{A}'}(f_2ua, f_1(a - ua)) = f_2ua + f_1(a - ua)$ on morphisms:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\{\alpha_{\mathcal{A}}\}"] \arrow[d, "\{f\}"'] & \tilde{G}\tilde{F}\mathcal{A} = (\mathcal{N}_{\mathcal{A}})^{\ltimes} = (\mathrm{Im}u \ltimes \mathrm{Ker}u, u_{\eta_{\mathcal{A}}}, v_{\eta_{\mathcal{A}}}) \arrow[d, "\tilde{G}\tilde{F}\{f\} = \left\{\sigma_{f_2}^{f_1}\right\}"] \\
\mathcal{A}' & \tilde{G}\tilde{F}\mathcal{A}' = \left(\mathcal{N}_{\mathcal{A}'}\right)^{\ltimes} = (\mathrm{Im}u' \ltimes \mathrm{Ker}u', u_{\eta_{\mathcal{A}'}}, v_{\eta_{\mathcal{A}'}}) \arrow[l, "\{\beta_{\mathcal{A}'}\}"]
\end{tikzcd}$$

Example 2. *Each R-algebroid* A *determines a cat^1-R-algebroid $\mathcal{A}_{cat} = (\mathrm{A}, id_{\mathrm{A}}, id_{\mathrm{A}})$ and thus the crossed module $\tilde{F}\mathcal{A}_{cat} = \mathcal{N}_{\mathcal{A}_{cat}} = \left(\eta_{\mathcal{A}_{cat}} : \mathrm{Ker}id_{\mathrm{A}} \to \mathrm{Im}id_{\mathrm{A}}\right) = \left(\eta_{\mathcal{A}_{cat}} : 0_{\mathrm{A}} \to \mathrm{A}\right)$. Therefore, given any morphism $(f_1, f_2) : \tilde{F}\mathcal{A}_{cat} \to \tilde{F}\mathcal{A}'_{cat}$ of such crossed modules, the map $f : \mathrm{A} \to \mathrm{A}'$ specified in Remark 4 is to be defined by $fa = f_2a$ for each $a \in \mathrm{A}$, since $u = id_{\mathrm{A}}$.*

Example 3. *Given a two-sided ideal* I *of an R-algebroid* A, *$\mathcal{I} = (i : \mathrm{I} \to \mathrm{A})$ is a crossed module, where i is the inclusion, as stated in Example 1. Thus, $\tilde{G}\mathcal{I} = \mathcal{I}^{\ltimes} = (\mathrm{A} \ltimes \mathrm{I}, u_i, v_i)$ and $\tilde{F}\mathcal{I}^{\ltimes} = \left(\eta_{\mathcal{I}^{\ltimes}} : \mathrm{Ker}u_i \to \mathrm{Im}u_i\right)$, where $u_i(a, b) = (a, 0)$ and $v_i(a, b) = (a + b, 0)$ by (4.2),* $\mathrm{Ker}u_i = \{(0, b) \in \mathrm{A} \ltimes \mathrm{I}\}$, $\mathrm{Im}u_i = \{(a, 0) \in \mathrm{A} \ltimes \mathrm{I}\}$ *and $\eta_{\mathcal{I}^{\ltimes}}(0, b) = v_i(0, b) = (b, 0)$ by (4.1). Consequently, we observe that the pair (ρ_1, ρ_2) of maps $\rho_1 : \mathrm{Ker}u_i \to \mathrm{I}$ and $\rho_2 : \mathrm{Im}u_i \to \mathrm{I}$, which are respectively defined by $\rho_1(0, b) = b$ and $\rho_2(a, 0) = a$, is an isomorphism in* $\mathrm{XAlg}\,(R)$ *from $\tilde{F}\mathcal{I}^{\ltimes}$ to $\mathcal{I}$, as required by* CS3.

Example 4. *The functor $\tilde{G}$ maps the crossed module $\mathcal{N}_{\mathcal{A}_{cat}} = \left(\eta_{\mathcal{A}_{cat}} : 0_{\mathrm{A}} \to \mathrm{A}\right)$ of Example 2 to the cat^1-R-algebroid $\tilde{G}\mathcal{N}_{\mathcal{A}_{cat}} = (\mathrm{A} \ltimes 0_{\mathrm{A}}, u_{\eta_{\mathcal{A}_{cat}}}, v_{\eta_{\mathcal{A}_{cat}}})$, where $u_{\eta_{\mathcal{A}_{cat}}}(a, 0) = (a, 0)$ and $v_{\eta_{\mathcal{A}_{cat}}}(a, 0) = \left(a + \eta_{\mathcal{A}_{cat}}0, 0\right) = (a, 0)$. Then, we see that $\mathrm{A} \ltimes 0_{\mathrm{A}} \cong \mathrm{A}$, $u_{\eta_{\mathcal{A}_{cat}}} = v_{\eta_{\mathcal{A}_{cat}}} = id_{\mathrm{A} \ltimes 0_{\mathrm{A}}}$ and so $\tilde{G}\mathcal{N}_{\mathcal{A}_{cat}} \cong \mathcal{A}_{cat}$, as required by* CS4.

6.2 An application: The correspondence of the functors $(-)^{cr}$ and $(-)^{ct}$

It is sometimes required to get a crossed module from a precrossed module and mostly we do this by means of Peiffer subgroups or Peiffer ideals. In our study [3], we introduced Peiffer ideal for a precrossed module of R-algebroids and used it to get a crossed module, and the procedure gave us the functor $(-)^{cr}$ as sketched out in Sect. 6.2.1 below. In this respect, the question is that "how can we equivalently convert a precat^1-R-algebroid into a cat^1-R-algebroid?". As detailed below, we can do this by using the functors F, $(-)^{cr}$ and $\tilde{G}$ successively. But, our ultimate aim is to develop a shortcut functor, which is naturally isomorphic to $\tilde{G}\,(-)^{cr}\,F$.

6.2.1 Constructing a cat^1-R-algebroid via the functor $(-)^{cr}$

We recall from Sect. 5 of [3] that the functor $(-)^{cr} : \mathrm{PXAlg}\,(R) \to \mathrm{XAlg}\,(R)$ assigns to each precrossed module $\mathcal{N} = (\eta : \mathrm{N} \to \mathrm{A})$ the crossed module $\mathcal{N}^{cr} = (\eta^{cr} : \mathrm{N}^{cr} \to \mathrm{A})$ and to each precrossed module morphism $(f,g) : \mathcal{N} \to \mathcal{N}'$ the crossed module morphism $(f,g)^{cr} : \mathcal{N}^{cr} \to \mathcal{N}'^{cr}$ such that

i. $\mathrm{N}^{cr} = \frac{\mathrm{N}}{[\![\mathrm{N},\mathrm{N}]\!]} = \left\{ \frac{\mathrm{N}}{[\![\mathrm{N},\mathrm{N}]\!]}\,(x,y) = \frac{\mathrm{N}(x,y)}{[\![\mathrm{N},\mathrm{N}]\!](x,y)} : x, y \in \mathrm{A}_0 \right\}$, where

- ***i₁.*** the family $[\![\mathrm{N},\mathrm{N}]\!] = \{[\![\mathrm{N},\mathrm{N}]\!]\,(x,y) : x,y \in \mathrm{A}_0\}$ is the Peiffer ideal of N,
- ***i₂.*** $[\![\mathrm{N},\mathrm{N}]\!]\,(x,y)$ is the subgroup, and an R-submodule, of $\mathrm{N}\,(x,y)$ generated by the set of Peiffer commutators $[\![\mathrm{N},\mathrm{N}]\!]_{\mathrm{g}}\,(x,y) = \{[\![n,n']\!]_1, [\![n,n']\!]_2 : n,n' \in \mathrm{N}, sn = x, tn = sn', tn' = y\}$,
- ***i₃.*** the Peiffer commutators of $n, n' \in \mathrm{N}$ with $tn = sn'$ are defined by $[\![n,n']\!]_1 = n^{\eta n'} - nn'$ and $[\![n,n']\!]_2 = {}^{\eta n}n' - nn'$,

ii. η^{cr} is defined as the identity on $\mathrm{A}_0 (= \mathrm{N}_0)$ and by $\eta^{cr}\overline{n} = \eta n$ on morphisms, where $\overline{n} = n + [\![\mathrm{N},\mathrm{N}]\!]\,(sn,tn)$, and

iii. $(f,g)^{cr} = (f^{cr}, g)$, where f^{cr} is defined by $f^{cr}\overline{n} = \overline{fn}$ on morphisms.

Now, assume that we are given a precat^1-R-algebroid $\mathcal{A} = (\mathrm{A}, u, v)$. Then, the functor F from PCat^1-Alg$\,(R)$ to PXAlg$\,(R)$ gives us the precrossed module $F\mathcal{A} = \mathcal{N}_{\mathcal{A}} = (\eta_{\mathcal{A}} : \mathrm{Ker}u \to \mathrm{Im}u)$, where $\eta_{\mathcal{A}}a = va$ on morphisms, and the functor $(-)^{cr}$ gives the crossed module

$$(\mathcal{N}_{\mathcal{A}})^{\mathrm{cr}} = (\eta^{\mathrm{cr}}_{\mathcal{A}} : (\mathrm{Ker}u)^{\mathrm{cr}} \to \mathrm{Im}u)\,,$$

where $(\mathrm{Ker}u)^{\mathrm{cr}} = \frac{\mathrm{Ker}u}{[\![\mathrm{Ker}u,\mathrm{Ker}u]\!]}$, in which $[\![\mathrm{Ker}u,\mathrm{Ker}u]\!] = \{[\![\mathrm{Ker}u,\mathrm{Ker}u]\!]\,(x,y) : x,y \in \mathrm{A}_0\}$ where each homset $[\![\mathrm{Ker}u,\mathrm{Ker}u]\!]\,(x,y)$ is the R-submodule of $\mathrm{Ker}u\,(x,y)$ generated by the set $[\![\mathrm{Ker}u,\mathrm{Ker}u]\!]_{\mathrm{g}}\,(x,y) = \{[\![a,a']\!]_1, [\![a,a']\!]_2 : a,a' \in \mathrm{Ker}u, sa = x, ta = sa', ta' = y\}$ of Peiffer commutators $[\![a,a']\!]_1 = a^{\eta_{\mathcal{A}}a'} - aa' = a^{va'} - aa'$ and $[\![a,a']\!]_2 = {}^{va}a' - aa'$, and where $\eta^{\mathrm{cr}}_{\mathcal{A}}\overline{a} = \eta_{\mathcal{A}}a = va$ on morphisms.

Then, the functor $\tilde{G}$ from XAlg$\,(R)$ to Cat^1-Alg$\,(R)$ gives us the cat^1-R-algebroid

$$\tilde{G}\,(\mathcal{N}_{\mathcal{A}})^{\mathrm{cr}} = (\mathcal{N}_{\mathcal{A}})^{\mathrm{cr}\text{-}\ltimes} = (\mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}}\,, u_{\eta^{\mathrm{cr}}_{\mathcal{A}}}, v_{\eta^{\mathrm{cr}}_{\mathcal{A}}}),$$

where

i. $u_{\eta^{\mathrm{cr}}_{\mathcal{A}}}(b,\overline{a}) = (b,\overline{0})$ and $v_{\eta^{\mathrm{cr}}_{\mathcal{A}}}(b,\overline{a}) = (b+\eta^{\mathrm{cr}}_{\mathcal{A}}\overline{a},\overline{0}) = (b+va,\overline{0})$ for all $(b,\overline{a}) \in \mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}}$,

ii. $\mathrm{Ker}u_{\eta^{\mathrm{cr}}_{\mathcal{A}}} = \left\{(b,\overline{a}) \in \mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}} : u_{\eta^{\mathrm{cr}}_{\mathcal{A}}}(b,\overline{a}) = (0,\overline{0})\right\} = \{(0,\overline{a}) : a \in \mathrm{Ker}u, 0 \in \mathrm{Im}u\,(sa,ta)\}$, and

iii. $\mathrm{Ker}v_{\eta^{\mathrm{cr}}_{\mathcal{A}}} = \left\{(b,\overline{a}) \in \mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}} : v_{\eta^{\mathrm{cr}}_{\mathcal{A}}}(b,\overline{a}) = (0,\overline{0})\right\} = \{(-va,\overline{a}) : a \in \mathrm{Ker}u\}$.

A direct calculation shows in $(\mathcal{N}_{\mathcal{A}})^{\mathrm{cr}\text{-}\ltimes}$ that the equalities $u_{\eta^{\mathrm{cr}}_{\mathcal{A}}} v_{\eta^{\mathrm{cr}}_{\mathcal{A}}} = v_{\eta^{\mathrm{cr}}_{\mathcal{A}}}$ and $v_{\eta^{\mathrm{cr}}_{\mathcal{A}}} u_{\eta^{\mathrm{cr}}_{\mathcal{A}}} = u_{\eta^{\mathrm{cr}}_{\mathcal{A}}}$ hold and thus CAT1 is satisfied. Moreover, for all $a, a' \in \mathrm{Ker}u$ with $ta = sa'$

$$(0,\overline{a})(-va',\overline{a'}) = \left(0(-va), \overline{a}^{-va'} + {}^{0}(-va') + \overline{a}\overline{a'}\right) = \left(0, \overline{-a^{va'} + aa'}\right) = (0,\overline{0}),$$

where $\overline{-a^{va'} + aa'} = -a^{va'} + aa' + [\![\mathrm{Ker}u, \mathrm{Ker}u]\!](sa,ta') = [\![\mathrm{Ker}u, \mathrm{Ker}u]\!](sa,ta') = \overline{0}(= 0_{(\mathrm{Ker}u)^{\mathrm{cr}}(sa,ta')})$ since $-a^{va'} + aa' = -[\![a,a']\!]_1$ and $-[\![a,a']\!]_1 \in [\![\mathrm{Ker}u, \mathrm{Ker}u]\!](sa,ta')$, meaning that $\mathrm{Ker}u_{\eta^{\mathrm{cr}}_{\mathcal{A}}} \mathrm{Ker}v_{\eta^{\mathrm{cr}}_{\mathcal{A}}} = 0_{\mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}}}$. It can similarly be shown that $\mathrm{Ker}v_{\eta^{\mathrm{cr}}_{\mathcal{A}}} \mathrm{Ker}u_{\eta^{\mathrm{cr}}_{\mathcal{A}}} = 0_{\mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}}}$ and thus CAT2 is satisfied, as required.

6.2.2 The functor $(-)^{ct}$

In this part, for any precat^1-R-algebroid $\mathcal{A} = (\mathrm{A}, u, v)$ in order to directly obtain a cat^1-R-algebroid, which is isomorphic to $\tilde{G}(-)^{cr} F\mathcal{A} = (\mathcal{N}_{\mathcal{A}})^{\mathrm{cr}\text{-}\ltimes}$, we shall develop a shortcut functor from $\mathrm{PCat}^1\text{-}\mathrm{Alg}(R)$ to $\mathrm{Cat}^1\text{-}\mathrm{Alg}(R)$ and to this end we shall use the ideal $(\!|\mathrm{A},\mathrm{A}|\!)$ of A, which is the ideal corresponding to the Peiffer ideal $[\![\mathrm{Ker}u, \mathrm{Ker}u]\!]$ of $F\mathcal{A}$ and is obtained by redescribing the generators of $[\![\mathrm{Ker}u, \mathrm{Ker}u]\!]$ for precat^1-R-algebroids. Now, assume that we are given a precat^1-R-algebroid $\mathcal{A} = (\mathrm{A}, u, v)$:

Lemma 4. *$[\![\mathrm{Ker}u, \mathrm{Ker}u]\!]_{\mathrm{g}}(x,y) = (\mathrm{Ker}u\mathrm{Ker}v)(x,y) \cup (\mathrm{Ker}v\mathrm{Ker}u)(x,y)$ for all $x, y \in \mathrm{A}_0$.*

Proof. For any $[\![a,a']\!]_1, [\![a,a']\!]_2 \in [\![\mathrm{Ker}u, \mathrm{Ker}u]\!]_{\mathrm{g}}(x,y)$

$$[\![a,a']\!]_1 = a^{va'} - aa' = a(va') - aa' = a(va' - a') \in (\mathrm{Ker}u\mathrm{Ker}v)(x,y),$$

since $va' - a' \in \mathrm{Ker}v$ by (3.4). Similarly, $[\![a,a']\!]_2 \in (\mathrm{Ker}v\mathrm{Ker}u)(x,y)$ and thus

$$[\![\mathrm{Ker}u, \mathrm{Ker}u]\!]_{\mathrm{g}}(x,y) \subseteq (\mathrm{Ker}u\mathrm{Ker}v)(x,y) \cup (\mathrm{Ker}v\mathrm{Ker}u)(x,y) \tag{6.1}$$

Note on the other hand that any $a \in (\mathrm{Ker}u\mathrm{Ker}v)(x,y)$ is of the form $a_u a_v$ for some $a_u \in \mathrm{Ker}u$ and $a_v \in \mathrm{Ker}v$ with $sa_u = x, ta_u = sa_v, ta_v = y$ and so

$$a = a_u a_v = a_u(a_v - va_v - ua_v + vua_v) = a_u(v(ua_v - a_v) - (ua_v - a_v)) = [\![a_u, ua_v - a_v]\!]_1,$$

where $va_v = 0$ since $a_v \in \mathrm{Ker}v$, $vua_v = ua_v$ by CAT1 and $ua_v - a_v \in \mathrm{Ker}u$ by (3.4). Hence, $a \in [\![\mathrm{Ker}u, \mathrm{Ker}u]\!]_{\mathrm{g}}(x,y)$ and thus $(\mathrm{Ker}u\mathrm{Ker}v)(x,y) \subseteq [\![\mathrm{Ker}u, \mathrm{Ker}u]\!]_{\mathrm{g}}(x,y)$. A similar calculation shows that $(\mathrm{Ker}v\mathrm{Ker}u)(x,y) \subseteq [\![\mathrm{Ker}u, \mathrm{Ker}u]\!]_{\mathrm{g}}(x,y)$ and therefore

$$(\mathrm{Ker}u\mathrm{Ker}v)(x,y) \cup (\mathrm{Ker}v\mathrm{Ker}u)(x,y) \subseteq [\![\mathrm{Ker}u, \mathrm{Ker}u]\!]_{\mathrm{g}}(x,y) \tag{6.2}$$

Consequently, we get from (6.1) and (6.2) that

$$[\![\mathrm{Ker}u, \mathrm{Ker}u]\!]_{\mathrm{g}}(x,y) = (\mathrm{Ker}u\mathrm{Ker}v)(x,y) \cup (\mathrm{Ker}v\mathrm{Ker}u)(x,y)$$

□

Now, let us denote the subset $(\mathrm{Ker}u\mathrm{Ker}v)(x,y) \cup (\mathrm{Ker}v\mathrm{Ker}u)(x,y)$ of $\mathrm{A}(x,y)$ by $(\!|\mathrm{A},\mathrm{A}|\!)_{\mathrm{g}}(x,y)$ and let $(\!|\mathrm{A},\mathrm{A}|\!)(x,y)$ be the subgroup of $\mathrm{A}(x,y)$ generated by $(\!|\mathrm{A},\mathrm{A}|\!)_{\mathrm{g}}(x,y)$. Since each of the sets $(\mathrm{Ker}u\mathrm{Ker}v)(x,y)$ and $(\mathrm{Ker}v\mathrm{Ker}u)(x,y)$ is closed under the R-action, so is $(\!|\mathrm{A},\mathrm{A}|\!)_{\mathrm{g}}(x,y)$ and thus $(\!|\mathrm{A},\mathrm{A}|\!)(x,y)$ is an R-submodule of $\mathrm{A}(x,y)$. Then, it can be verified through direct calculations that the family

$$(\!|\mathrm{A},\mathrm{A}|\!) = \{(\!|\mathrm{A},\mathrm{A}|\!)(x,y) : x,y \in \mathrm{A}_0\}$$

is a two-sided ideal of A and the family

$$\mathrm{A}^{ct} = \frac{\mathrm{A}}{(\!|\mathrm{A},\mathrm{A}|\!)} = \left\{ \frac{\mathrm{A}}{(\!|\mathrm{A},\mathrm{A}|\!)}(x,y) = \frac{\mathrm{A}(x,y)}{(\!|\mathrm{A},\mathrm{A}|\!)(x,y)} : x,y \in \mathrm{A}_0 \right\}$$

form an R-algebroid, of which

i. the object set is A_0,

ii. the source and target functions, the addition and the composition are all induced by those of A,

and on which

iii. the R-action is induced by that defined on A.

Then, denoting each morphism $a + (\!|\mathrm{A},\mathrm{A}|\!)(sa,ta)$ of A^{ct} by $[a]$, and noting that $s[a] = sa$ and $t[a] = ta$, we define the maps $u^{ct}, v^{ct} : \mathrm{A}^{ct} \to \mathrm{A}^{ct}$ as the identity on A_0 and by

$$u^{ct}[a] = [ua] \quad \text{and} \quad v^{ct}[a] = [va] \tag{6.3}$$

on morphisms. Note that $u(\!|\mathrm{A},\mathrm{A}|\!) = v(\!|\mathrm{A},\mathrm{A}|\!) = 0_{\mathrm{A}}$ and thus u^{ct} and v^{ct} are well-defined, since so are u and v. Moreover, a direct calculation shows that they are both R-algebroid morphisms and $\mathcal{A}^{ct} = (\mathrm{A}^{ct}, u^{ct}, v^{ct})$ is a precat^1-R-algebroid.

Remark 5. *Since* $\mathrm{Ker}u(x,y)$ *is an R-submodule of* $\mathrm{A}(x,y)$*, the equality* $[\![\mathrm{Ker}u,\mathrm{Ker}u]\!]_{\mathrm{g}}(x,y) = (\!|\mathrm{A},\mathrm{A}|\!)_{\mathrm{g}}(x,y)$ *of generating sets (from Lemma 4) gives the equality*

$$[\![\mathrm{Ker}u,\mathrm{Ker}u]\!](x,y) = (\!|\mathrm{A},\mathrm{A}|\!)(x,y) \tag{6.4}$$

of R-submodules for all $x,y \in \mathrm{A}_0$ *and thus the equality*

$$[\![\mathrm{Ker}u,\mathrm{Ker}u]\!] = (\!|\mathrm{A},\mathrm{A}|\!) \tag{6.5}$$

of ideals generated.

Proposition 18. $[a] \in \mathrm{Ker}u^{ct} \Leftrightarrow a \in \mathrm{Ker}u$ *and* $[a] \in \mathrm{Ker}v^{ct} \Leftrightarrow a \in \mathrm{Ker}v$ *for all* $a \in \mathrm{A}$.

Proof. That "$a \in \mathrm{Ker}u \Rightarrow [a] \in \mathrm{Ker}u^{ct}$" is clear. In the opposite direction

$$[a] \in \mathrm{Ker}u^{ct} \Rightarrow u^{ct}[a] = [0] \Rightarrow [ua] = [0] \Rightarrow ua \in (\!|\mathrm{A},\mathrm{A}|\!) \Rightarrow vua = 0 \Rightarrow ua = 0 \Rightarrow a \in \mathrm{Ker}u,$$

where $vua = 0$ since $v(\!|\mathrm{A},\mathrm{A}|\!) = 0_{\mathrm{A}}$ and where $vua = ua$ by CAT1, and this completes the proof of the first biconditional statement. The proof of the second biconditional statement is the same. □

Proposition 19. *$\mathcal{A}^{ct} = (\mathrm{A}^{ct}, u^{ct}, v^{ct})$ is a cat^1-R-algebroid.*

Proof. $\mathrm{Ker}u^{ct} = \{[a] : a \in \mathrm{Ker}u\}$ and $\mathrm{Ker}v^{ct} = \{[a] : a \in \mathrm{Ker}v\}$ by Proposition 18. Then, for any $a \in \mathrm{Ker}u$ and $a' \in \mathrm{Ker}v$ with $ta = sa'$, noting that $aa' \in (\!|\mathrm{A}, \mathrm{A}|\!)\,(sa, ta')$

$$[a]\,[a'] = [aa'] = aa' + (\!|\mathrm{A}, \mathrm{A}|\!)\,(sa, ta') = (\!|\mathrm{A}, \mathrm{A}|\!)\,(sa, ta') = 0_{\mathrm{A}^{ct}(sa,ta')},$$

meaning that $\mathrm{Ker}u^{ct}\mathrm{Ker}v^{ct} = 0_{\mathrm{A}^{ct}}$. It can similarly be shown that $\mathrm{Ker}v^{ct}\mathrm{Ker}u^{ct} = 0_{\mathrm{A}^{ct}}$, as required. ∎

The construction above gives us a functor $(-)^{ct} : \mathrm{PCat}^1\text{-Alg}\,(R) \to \mathrm{Cat}^1\text{-Alg}\,(R)$, which assigns to each precat^1-R-algebroid $\mathcal{A}$ the cat^1-R-algebroid $\mathcal{A}^{ct}$ and to each precat^1-R-algebroid morphism $\{f\}$ the cat^1-R-algebroid morphism $\{f\}^{ct} = \{f^{ct}\}$ such that $f^{ct}\,[a] = [fa]$ for all $a \in \mathrm{A}$.

6.2.3 The correspondence of the functors $(-)^{cr}$ and $(-)^{ct}$

Up to now, given a precat^1-R-algebroid $\mathcal{A} = (\mathrm{A}, u, v)$, we have first constructed the cat^1-R-algebroid $(\mathcal{N}_{\mathcal{A}})^{\mathrm{cr}\text{-}\ltimes} = (\mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}}, u_{\eta^{\mathrm{cr}}_{\mathcal{A}}}, v_{\eta^{\mathrm{cr}}_{\mathcal{A}}})$, using the composite functor $\tilde{G}\,(-)^{cr}\,F$, and then the cat^1-R-algebroid $\mathcal{A}^{ct} = (\mathrm{A}^{ct}, u^{ct}, v^{ct})$, using the functor $(-)^{ct}$. In this final stage, we shall prove that $(\mathcal{N}_{\mathcal{A}})^{\mathrm{cr}\text{-}\ltimes}$ and $\mathcal{A}^{ct}$ are isomorphic to each other and the composite functor $\tilde{G}\,(-)^{cr}\,F$ is naturally isomorphic to the functor $(-)^{ct}$, and we shall say, by abuse of language, that the functor $(-)^{ct}$ corresponds to the functor $(-)^{cr}$ and vice versa. Now, assume that we are given a precat^1-R-algebroid $\mathcal{A} = (\mathrm{A}, u, v)$:

Lemma 5. *For any $b, b' \in \mathrm{Im}u$ and $a, a' \in \mathrm{Ker}u$, all with common sources and targets*

$$b = b' \;\; \text{and} \;\; \overline{a} = \overline{a'} \quad \Leftrightarrow \quad \overline{b + a} = \overline{b' + a'}\;,$$

where $\overline{a}, \overline{a'}, \overline{b + a}, \overline{b' + a'} \in (\mathrm{Ker}u)^{\mathrm{cr}}$.

Proof. The implication "$b = b'$ and $\overline{a} = \overline{a'} \;\Rightarrow\; \overline{b + a} = \overline{b' + a'}$" is clear. Conversely, noting that $u[\![\mathrm{Ker}u, \mathrm{Ker}u]\!]\,(x, y) = \{0_{\mathrm{A}(x,y)}\}$ for all $x, y \in \mathrm{A}_0$,

$$\begin{aligned}
\overline{b + a} = \overline{b' + a'} &\Rightarrow b + a + [\![\mathrm{Ker}u, \mathrm{Ker}u]\!]\,(sa, ta) = b' + a' + [\![\mathrm{Ker}u, \mathrm{Ker}u]\!]\,(sa, ta) \\
&\Rightarrow b - b' + a - a' \in [\![\mathrm{Ker}u, \mathrm{Ker}u]\!]\,(sa, ta) \\
&\Rightarrow u\,(b - b' + a - a') = 0 \\
&\Rightarrow ub - ub' + 0 - 0 = 0 \\
&\Rightarrow ub = ub' \Rightarrow b = b',
\end{aligned}$$

where the last implication holds by (3.1). Thus, the backward implication of the biconditional statement is partially satisfied. On the other hand, the equality $\overline{b} = \overline{b'}$, which is due to $b = b'$, and the equality $\overline{b + a} = \overline{b' + a'}$ together gives the needed second equality $\overline{a} = \overline{a'}$, proving that the backward implication is completely satisfied. ∎

Theorem 3. *The cat^1-R-algebroids $(\mathcal{N}_{\mathcal{A}})^{\mathrm{cr}\text{-}\ltimes}$ and $\mathcal{A}^{ct}$ are isomorphic to each other.*

Proof. Define the map $\gamma_{\mathcal{A}} : \mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}} \to \mathrm{A}^{ct}$ by $\gamma_{\mathcal{A}}(b, \overline{a}) = [b + a]$. Clearly, $\gamma_{\mathcal{A}}(b, \overline{a}) \in \mathrm{A}^{ct}$ for all $(b, \overline{a}) \in \mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}}$. Besides, for all $(b, \overline{a}), \left(b', \overline{a'}\right) \in \mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}}$

$$\begin{aligned}
(b, \overline{a}) = \left(b', \overline{a'}\right) &\Leftrightarrow \overline{b + a} = \overline{b + a'} && \text{by Lemma 5} \\
&\Leftrightarrow b + a + [\![\mathrm{Ker}u, \mathrm{Ker}u]\!]\,(sa, ta) = b' + a' + [\![\mathrm{Ker}u, \mathrm{Ker}u]\!]\,(sa, ta) \\
&\Leftrightarrow b + a + (\!|\mathrm{A}, \mathrm{A}|\!)\,(sa, ta) = b' + a' + (\!|\mathrm{A}, \mathrm{A}|\!)\,(sa, ta) && \text{by (6.4)} \\
&\Leftrightarrow [b + a] = [b' + a'] \\
&\Leftrightarrow \gamma_{\mathcal{A}}(b, \overline{a}) = \gamma_{\mathcal{A}}\left(b', \overline{a'}\right),
\end{aligned}$$

meaning that $\gamma_{\mathcal{A}}$ is well-defined and 1-1. In addition, for any $[a] \in \mathrm{A}^{ct}$, $(ua, \overline{a - ua}) \in \mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}}$ and $\gamma_{\mathcal{A}}(ua, \overline{a - ua}) = [a]$, meaning that $\gamma_{\mathcal{A}}$ is onto and so is a bijection. Moreover, direct calculations show that $\gamma_{\mathcal{A}}$ is an R-algebroid morphism and thus is an R-algebroid isomorphism. Furthermore,

$$\gamma_{\mathcal{A}} v_{\eta^{\mathrm{cr}}_{\mathcal{A}}}(b, \overline{a}) = \gamma_{\mathcal{A}}\left(b + va, \overline{0}\right) = [b + va + 0] = [vb + va] = v^{ct}[b + a] = v^{ct}\gamma_{\mathcal{A}}(b, \overline{a})$$

for all $(b, \overline{a}) \in \mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}}$, where $b = vb$ by (3.1), meaning that $\gamma_{\mathcal{A}} u_{\eta^{\mathrm{cr}}_{\mathcal{A}}} = u^{ct}\gamma_{\mathcal{A}}$. Similarly, $\gamma_{\mathcal{A}} v_{\eta^{\mathrm{cr}}_{\mathcal{A}}} = v^{ct}\gamma_{\mathcal{A}}$ and thus $\{\gamma_{\mathcal{A}}\}$ is a cat^1-R-algebroid isomorphism from $(\mathcal{N}_{\mathcal{A}})^{\mathrm{cr}\text{-}\ltimes}$ to $\mathcal{A}^{ct}$, as required. ∎

Theorem 4. *The functor $\tilde{G}(-)^{cr} F$ is naturally isomorphic to the functor $(-)^{ct}$.*

Proof. Given a precat^1-R-algebroid $\mathcal{A} = (\mathrm{A}, u, v)$, $\{\gamma_{\mathcal{A}}\}$ is a cat^1-R-algebroid isomorphism from $(\mathcal{N}_{\mathcal{A}})^{\mathrm{cr}\text{-}\ltimes}$ to $\mathcal{A}^{ct}$ by Theorem 3. In addition, given a precat^1-R-algebroid morphism $\{f\} : \mathcal{A} \to \mathcal{A}'$, noting that $\tilde{G}(-)^{cr} F\{f\} = \tilde{G}(f_{\mathrm{Ker}u}, f_{\mathrm{Im}u})^{cr} = \tilde{G}(f^{cr}_{\mathrm{Ker}u}, f_{\mathrm{Im}u}) = \left\{\sigma^{f^{cr}_{\mathrm{Ker}u}}_{f_{\mathrm{Im}u}}\right\}$, for all $(b, \overline{a}) \in \mathrm{Im}u \ltimes (\mathrm{Ker}u)^{\mathrm{cr}}$

$$\gamma_{\mathcal{A}'}\sigma^{f^{cr}_{\mathrm{Ker}u}}_{f_{\mathrm{Im}u}}(b, \overline{a}) = \gamma_{\mathcal{A}'}\left(f_{\mathrm{Im}u}b, f^{cr}_{\mathrm{Ker}u}\overline{a}\right) = \gamma_{\mathcal{A}'}\left(fb, \overline{fa}\right) = [fb + fa] = f^{ct}[b + a] = f^{ct}\gamma_{\mathcal{A}}(b, \overline{a}).$$

So, $\gamma_{\mathcal{A}'}\sigma^{f^{cr}_{\mathrm{Ker}u}}_{f_{\mathrm{Im}u}} = f^{ct}\gamma_{\mathcal{A}}$ and thus $\{\gamma_{\mathcal{A}'}\}\left(\tilde{G}(-)^{cr} F\{f\}\right) = \left((-)^{ct}\{f\}\right)\{\gamma_{\mathcal{A}}\}$, i.e., the diagram

$$\begin{tikzcd}
\tilde{G}(-)^{cr} F\mathcal{A} = (\mathcal{N}_{\mathcal{A}})^{\mathrm{cr}\text{-}\ltimes} \arrow[r, "\tilde{G}(-)^{cr}F\{f\}=\left\{\sigma^{f^{cr}_{\mathrm{Ker}u}}_{f_{\mathrm{Im}u}}\right\}"] \arrow[d, "\{\gamma_{\mathcal{A}}\}"'] & \tilde{G}(-)^{cr} F\mathcal{A}' = \left(\mathcal{N}_{\mathcal{A}'}\right)^{\mathrm{cr}\text{-}\ltimes} \arrow[d, "\{\gamma_{\mathcal{A}'}\}"] \\
(-)^{ct}\mathcal{A} = \mathcal{A}^{ct} \arrow[r, "(-)^{ct}\{f\}=\{f\}^{ct}=\left\{f^{ct}\right\}"'] & (-)^{ct}\mathcal{A}' = \mathcal{A}'^{ct}
\end{tikzcd}$$

is commutative, and as a consequence the family $\boldsymbol{\gamma} = \left\{\{\gamma_{\mathcal{A}}\} : \mathcal{A} \in \mathrm{PCat}^1\text{-}\mathrm{Alg}(R)\right\}$ is a natural isomorphism between the functors $\tilde{G}(-)^{cr} F$ and $(-)^{ct}$, as required. ∎

As a final remark, we recall from [3, Proposition 16] that the functor $(-)^{cr}$ is the left adjoint of the natural functor in the opposite direction. Therefore, by the natural isomorphism just proved we have the following result:

Corollary 3. *The functor* $(-)^{ct}$ *is the left adjoint of the inclusion functor* $\mathrm{Cat}^1\text{-}\mathrm{Alg}\,(R) \to \mathrm{PCat}^1\text{-}\mathrm{Alg}\,(R)$.

Proof. For any precat^1-R-algebroid $\mathcal{A}$ and cat^1-R-algebroid $\mathcal{B}$, we have the consecutive natural bijections

$$\begin{aligned}
\mathrm{Cat}^1\text{-}\mathrm{Alg}\,(R)\left(\mathcal{A}^{ct},\mathcal{B}\right) &\cong \mathrm{Cat}^1\text{-}\mathrm{Alg}\,(R)\left(\tilde{G}\,(F\mathcal{A})^{cr},\mathcal{B}\right)\\
&\cong \mathrm{XAlg}\,(R)\left((F\mathcal{A})^{cr},\tilde{F}\mathcal{B}\right)\\
&\cong \mathrm{PXAlg}\,(R)\left(F\mathcal{A},\tilde{F}\mathcal{B}\right)\\
&\cong \mathrm{PCat}^1\text{-}\mathrm{Alg}\,(R)\left(\mathcal{A},G\tilde{F}\mathcal{B}\right)\\
&\cong \mathrm{PCat}^1\text{-}\mathrm{Alg}\,(R)\left(\mathcal{A},\mathcal{B}\right),
\end{aligned}$$

the first of which holds because $(-)^{ct}$ is naturally isomorphic to $\tilde{G}\,(-)^{cr}\,F$ by Theorem 4, the second because $\tilde{F}$ and $\tilde{G}$ are left adjoints of each other as pointed out in Sect. 6.1, the third because $(-)^{cr}$ is the left adjoint of the natural functor $\mathrm{XAlg}\,(R) \to \mathrm{PXAlg}\,(R)$ as proved in [3, Proposition 16], the fourth because F and G are left adjoints of each other by Sect. 6.1 and the fifth because $\tilde{F}\mathcal{B} = F\mathcal{B}$ and because GF is naturally isomorphic to $id_{\mathrm{PCat}^1\text{-}\mathrm{Alg}(R)}$ by Theorem 1. □

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