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The weak ideal property in tensor products

by

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Abstract

Let A and B be C*-algebras such that A or B is exact. We describe the largest ideal in $A \otimes B$ which has the weak ideal property. For many C*-algebras A and B as above we characterize when the largest ideal in $A \otimes B$ which has the weak ideal property is the tensor product of the largest ideals in A and B which have the weak ideal property (this is not always true if A or B is exact). Assume that the C*-algebras A and B have the weak ideal property (and one of them is exact). We characterize (in an interesting particular case and also in general) when $A \otimes B$ has the weak ideal property (these two characterizations are totally different in nature).

Key Words: Weak ideal property, tensor product C*-algebra, largest ideal which has the weak ideal property, primitive spectrum, ideal property.
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1 Introduction

In this paper we continue our investigation of the weak ideal property, which was introduced in [8].

Definition 1. *(Definition 8.1 of [8]) Let A be a C*-algebra. We say that A has the* weak ideal property *if whenever $I \subseteq J \subseteq \mathcal{K} \otimes A$ are ideals in $\mathcal{K} \otimes A$ such that $I \neq J$, then J/I contains a non-zero projection.*

The weak ideal property is closely related to two other important properties: the ideal property and the topological dimension zero. A C*-algebra A has the ideal property if any ideal of A is generated, as an ideal, by its projections. A C*-algebra A has topological dimension zero if its primitive spectrum $\mathrm{Prim}(A)$ has a base for its topology consisting of compact open sets (Remark 2.5(vi) of [2]). Note that a separable purely infinite C*-algebra A has real rank zero if and only if A has topological dimension zero and it satisfies a certain K-theoretical condition (see Theorem 4.2 of [12]).

The weak ideal property and the topological dimension zero have good permanence properties (see, e.g., [7], [8], [9], and [2]). It is known that the ideal property $\Rightarrow$ the weak ideal property $\Rightarrow$ the topological dimension zero (the first implication is obvious, the second one is Theorem 2.8 of [9]). These three properties are not identical (see [9]). However, it was shown in [9] that, in many interesting cases, these three concepts coincide. A good understanding of the weak ideal property is important in identifying and studying regularity properties for non-simple C*-algebras, in an attempt to extend Elliott's Classification Program beyond the class of simple C*-algebras.

In this paper, we have been motivated by the following question:

Question 1. *(Question 4.12 of [9]) Let A and B be C*-algebras with A exact. If A and B have the weak ideal property, does $A \otimes B$ have the weak ideal property?*

This question was recently investigated in several papers (see [9], [4], [5], and [6]). This short note could be seen as a natural continuation of these works.

Let A and B be C*-algebras such that A or B is exact. In Proposition 1 we describe the largest ideal in $A \otimes B$ which has the weak ideal property. For many C*-algebras A and B as above we characterize when the largest ideal in $A \otimes B$ which has the weak ideal property is the tensor product of the largest ideals in A and B which have the weak ideal property (this is not always true if A or B is exact, $A \neq 0$, and $B \neq 0$, by Remark 1.9 of [5]) (see Theorem 1). Assume that the C*-algebras A and B have the weak ideal property (and one of them is exact). We prove that $A \otimes B$ has the weak ideal property if and only if $(A/J) \otimes (B/L)$ has the stable quotient property (see Definition 2) for every $J \lhd A$ and every $L \lhd B$ (see Theorem 3). Assume in addition that there is an ideal $I \lhd A$ such that $\mathrm{Prim}(A/I)$ is finite. We prove that in this case $A \otimes B$ has the weak ideal property if and only if $I \otimes B$ has the weak ideal property (see Theorem 2).

Ideals in C*-algebras are assumed to be closed and two sided. If A is a C*-algebra, then $\mathcal{P}(A)$ will denote the set of all projections of A ($\mathcal{P}(A) := \{p \in A\colon p = p^* = p^2\}$), $\mathrm{Prim}(A)$ will denote the primitive spectrum of A, and $I \lhd A$ will denote the fact that I is an ideal of A. If A and B are C*-algebras, then $A \otimes B$ denotes the minimal tensor product of A with B. The C*-algebra of all compact linear bounded operators acting on a separable infinite dimensional Hilbert space is denoted by $\mathcal{K}$.

2 The results

The following lemma and its proof are contained in the proof of Corollary 1.5 of [10]. We include a proof here for the sake of completeness.

Lemma 1. *Let A be a C*-algebra and let $I \lhd A$ and $J \lhd A$. If I and J have the weak ideal property, then $I + J$ has the weak ideal property.*

Proof. Consider the following short exact sequence of C*-algebras:

$$0 \longrightarrow I \longrightarrow I + J \longrightarrow J/(I \cap J) \longrightarrow 0$$

The weak ideal property passes to quotients (Theorem 8.5(5) of [8]), so $J/(I \cap J)$ has the weak ideal property. Extensions of C*-algebras with the weak ideal property have the weak ideal property (Theorem 8.5(5) of [8]), so $I + J$ has the weak ideal property.

□

Notation 1. *(see Notation 1.3 of [5]) Proposition 2.1(14) of [10] and Remark 2.2(1) of [10] imply that the weak ideal property admits largest ideals, that is, for every C*-algebra A there is an ideal in A which has the weak ideal property and which contains every ideal in A which has the weak ideal property (see Definition 1.1 of [10]). For a C*-algebra A, we denote by $\mathrm{I}_w(A)$ the largest ideal in A which has the weak ideal property.*

The following result describes the largest ideal in $A \otimes B$ which has the weak ideal property, where A and B are C*-algebras such that A or B is exact.

Proposition 1. *Let A and B be C^*-algebras such that A or B is exact. Then $\mathrm{I}_w(A \otimes B)$ is generated (as an ideal of $A \otimes B$) by the family of rectangular ideals $K \otimes L$ of $A \otimes B$ which have the weak ideal property.*

Proof. Let I be the ideal of $A \otimes B$ generated (as an ideal of $A \otimes B$) by the family of rectangular ideals $K \otimes L$ of $A \otimes B$ which have the weak ideal property. We want to prove that $\mathrm{I}_w(A \otimes B) = I$.

We first prove that if $J \lhd A \otimes B$ and J has the weak ideal property, then $J \subseteq I$. Indeed, a theorem of Kirchberg (see Proposition 2.13 of [3]; see also Theorem 1.3 of [11] and [1]) implies that J is generated (as an ideal of $A \otimes B$) by the family of rectangular ideals $K \otimes L$ contained in J. Since for any such rectangular ideal $K \otimes L$ we have $K \otimes L \subseteq J$, J has the weak ideal property and the weak ideal property passes to ideals (see Theorem 8.5(5) of [8]), it follows that $K \otimes L \lhd A \otimes B$ and $K \otimes L$ has the weak ideal property. Hence $J \subseteq I$.

We now prove that I has the weak ideal property. Let $\mathcal{I}$ be the family of finite sums of rectangular ideals $K \otimes L$ of $A \otimes B$ where any such ideal $K \otimes L$ has the weak ideal property. By Lemma 1 and a standard mathematical induction argument it follows that any element of $\mathcal{I}$ is an ideal of $A \otimes B$ which has the weak ideal property. Then, since $\mathcal{I}$ is directed, we have that $I = \overline{\bigcup_{M \in \mathcal{I}} M} \cong \varinjlim_{M \in \mathcal{I}} M$, and since the weak ideal property is preserved by inductive limits (see Theorem 8.5(4) of [8]), it follows that I has the weak ideal property.

In conclusion, we proved that $\mathrm{I}_w(A \otimes B) = I$.

$\square$

We recall some notations from [5].

Notation 2. *(see Notation 2.2 of [5]) Let A and B be non-zero C^*-algebras and let $I \lhd A \otimes B$. Assume first that $I \neq 0$. Denote by $I(A)$ the ideal of A generated by all $a \in A$ with the property that there is $b \in B$ such that $a \otimes b$ is a non-zero element of I. Similarly, denote by $I(B)$ the ideal of B generated by all $b \in B$ with the property that there is $a \in A$ such that $a \otimes b$ is a non-zero element of I. If $I = 0$ is the zero ideal of $A \otimes B$, denote $I(A) := 0$ and $I(B) := 0$.*

The following result characterizes, in many cases, when $\mathrm{I}_w(A \otimes B) = \mathrm{I}_w(A) \otimes \mathrm{I}_w(B)$.

Theorem 1. *Let A and B be C^*-algebras such that A or B is exact. Let $I := \mathrm{I}_w(A \otimes B)$. Assume that $\mathrm{I}_w(A) \neq 0$ and $\mathrm{I}_w(B) \neq 0$. Assume that any of the following three conditions holds:*

(a) Prim(A) or Prim(B) is finite.

(b) Prim(A) or Prim(B) is Hausdorff.

(c) A and B are separable and A or B has the ideal property.

Then the following are equivalent:

(1) $I = \mathrm{I}_w(A) \otimes \mathrm{I}_w(B)$.

(2) I(A) and I(B) have the weak ideal property.

(3) $\mathrm{I}_w(A) = I(A)$ *and* $\mathrm{I}_w(B) = I(B)$.

Proof. We first prove that (2) $\Rightarrow$ (1).

Assume that $I(A)$ and $I(B)$ have the weak ideal property. Using that any of the conditions (a), (b) and (c) holds and results in [9] (see Proposition 4.10 of [9], Proposition 4.11 of [9], and Theorem 4.8 of [9]) we deduce that $\mathrm{I}_w(A) \otimes \mathrm{I}_w(B)$ has the weak ideal property and hence:

$$\mathrm{I}_w(A) \otimes \mathrm{I}_w(B) \subseteq \mathrm{I}_w(A \otimes B) = I. \tag{2.1}$$

By Theorem 2.3(2) of [5] we have:

$$I \subseteq I(A) \otimes I(B) \tag{2.2}$$

Combining (2.1) and (2.2), we get:

$$\mathrm{I}_w(A) \otimes \mathrm{I}_w(B) \subseteq I \subseteq I(A) \otimes I(B) \tag{2.3}$$

Using (2.3), the fact that $\mathrm{I}_w(A) \neq 0$ and $\mathrm{I}_w(B) \neq 0$, and Lemma 1.4 of [5] we deduce that:

$$\mathrm{I}_w(A) \subseteq I(A), \mathrm{I}_w(B) \subseteq I(B) \tag{2.4}$$

Since $I(A)$ and $I(B)$ have the weak ideal property, (2.4) implies that:

$$\mathrm{I}_w(A) = I(A), \mathrm{I}_w(B) = I(B) \tag{2.5}$$

(we used the definitions of $\mathrm{I}_w(A)$ and $\mathrm{I}_w(B)$).

Finally, combining (2.3) and (2.5) we get $I = \mathrm{I}_w(A) \otimes \mathrm{I}_w(B)$, which ends the proof of (2) $\Rightarrow$ (1).

We now prove that (1) $\Rightarrow$ (3).

Assume that $I = \mathrm{I}_w(A) \otimes \mathrm{I}_w(B)$. It was showed in the above proof of (2) $\Rightarrow$ (1) that $\mathrm{I}_w(A) \otimes \mathrm{I}_w(B) \subseteq \mathrm{I}_w(A \otimes B) = I$, which implies that $\mathrm{I}_w(A \otimes B) = I \neq 0$, since $\mathrm{I}_w(A) \neq 0$ and $\mathrm{I}_w(B) \neq 0$. Finally, since $I \neq 0$, Theorem 2.9(2) of [5] implies that $\mathrm{I}_w(A) = I(A)$ and $\mathrm{I}_w(B) = I(B)$.

The proof of (3) $\Rightarrow$ (2) is obvious.

∎

Remark 1. *It is easy to see that the above theorem still holds if in the conditions (a), (b) and (c) we replace* A *by* $\mathrm{I}_w(A)$ *and* B *by* $\mathrm{I}_w(B)$ *(it is well-known that the exactness passes to ideals).*

The following theorem characterizes, in an interesting particular case, when $A \otimes B$ has the weak ideal property, knowing that both factors have the weak ideal property and one of them is exact.

Theorem 2. *Let* A *and* B *be C*-algebras that have the weak ideal property and such that* A *or* B *is exact. Suppose that there exists an ideal* $I \lhd A$ *such that Prim*(A/I) *is finite. Then the following are equivalent:*

(1) $A \otimes B$ *has the weak ideal property.*

(2) $I \otimes B$ has the weak ideal property.

Proof. We first prove that (1) $\Rightarrow$ (2).

Assume that $A \otimes B$ has the weak ideal property. Since $I \otimes B \lhd A \otimes B$ and the weak ideal property passes to ideals (by Theorem 8.5(5) of [8]), it follows that $I \otimes B$ has the weak ideal property.

We now prove that (2) $\Rightarrow$ (1).

Assume that $I \otimes B$ has the weak ideal property. Since A or B is exact, by Proposition 2.17(2) of [1] and Proposition 2.16(iv) of [1] the sequence:

$$0 \longrightarrow I \otimes B \longrightarrow A \otimes B \longrightarrow (A/I) \otimes B \longrightarrow 0$$

is exact. Since Prim(A/I) is finite and since also A/I has the weak ideal property (because A has the weak ideal property and the weak ideal property passes to quotients by Theorem 8.5(5) of [8]), we have that $(A/I) \otimes B$ has the weak ideal property by Proposition 4.10 of [9]. Since the weak ideal property is preserved by extensions (see Theorem 8.5(5) of [8]), $I \otimes B$ has the weak ideal property and $(A/I) \otimes B$ has the weak ideal property, we deduce, using also the above exact sequence of C*-algebras, that $A \otimes B$ has the weak ideal property. □

Corollary 1. *Let A and B be C*-algebras that have the weak ideal property and such that A or B is exact. Suppose that there exist ideals $I \lhd A$ and $J \lhd B$ such that Prim(A/I) and Prim(B/J) are finite. The following are equivalent:*

(1) $A \otimes B$ has the weak ideal property.

(2) $I \otimes J$ has the weak ideal property.

Proof. Use twice Theorem 2, the fact that the weak ideal property passes to ideals (Theorem 8.5(5) of [8]), and the well-known fact that exactness passes to ideals. □

Remark 2. *Theorem 2 and Corollary 1 hold, in particular, if I and J are maximal ideals of A and B, respectively, since in these cases Prim(A/I) and Prim(B/J) have each only one element.*

Definition 2. *We say that a C*-algebra A has the* stable quotient property *if for every ideal $I \lhd A$ such that $\mathcal{P}(I \otimes \mathcal{K}) \neq \{0\}$, we have that $\mathcal{P}((I/J) \otimes \mathcal{K}) \neq \{0\}$ for every $J \lhd A$, $J \subsetneq I$.*

Remark 3. *(1) Let A be a C*-algebra such that A has the weak ideal property. Then A has the stable quotient property.*

(2) Let A be a non-zero simple C-algebra, such as those classified in [13], for which $\mathcal{P}(A \otimes \mathcal{K}) = \{0\}$. Then A has the stable quotient property.*

Notation 3. *(see Notation 2.1 of [5]; see also Lemma 2.13(i) of [1]) Let A and B be C*-algebras and let $I \lhd A \otimes B$. Denote $I_A := \{a \in A : a \otimes B \subseteq I\}$ and $I_B := \{b \in B : A \otimes b \subseteq I\}$. When I is a prime ideal of $A \otimes B$, this notation was introduced in Lemma 2.13(i) of [1].*

The following theorem characterizes when a tensor product of C*-algebras has the weak ideal property, knowing that both factors have the weak ideal property and one of the factors is exact.

Theorem 3. *Let A and B be two C*-algebras that have the weak ideal property. Assume that A or B is exact. Then the following are equivalent:*

(1) $A \otimes B$ has the weak ideal property.

(2) $(A/J) \otimes (B/L)$ has the stable quotient property for every $J \lhd A$ and every $L \lhd B$.

Proof. We begin with an observation. Let $J \lhd A$ and $L \lhd B$ be arbitrary ideals. Since A or B is exact, Proposition 2.17 of [1], Proposition 2.16(ii) of [1] and Lemma 2.12(iii) of [1] imply that there is a *-isomorphism:

$$(A \otimes B)/(J \otimes B + A \otimes L) \cong (A/J) \otimes (B/L) \tag{2.6}$$

We first prove that (1) $\Rightarrow$ (2).

Assume that $A \otimes B$ has the weak ideal property. Since the weak ideal property passes to quotients (see Theorem 8.5(5) of [8]), using also the *-isomorphism from (2.6) we deduce that $(A/J) \otimes (B/L)$ has the weak ideal property, and hence $(A/J) \otimes (B/L)$ has the stable quotient property.

We now prove that (2) $\Rightarrow$ (1).

Assume that $(A/J) \otimes (B/L)$ has the stable quotient property for every $J \lhd A$ and every $L \lhd B$. Assume also that $0 \neq S \lhd (A \otimes B)/I$, where $I \lhd A \otimes B$. The obvious inclusion of ideals of $A \otimes B$

$$I_A \otimes B + A \otimes I_B \subseteq I \tag{2.7}$$

(see Theorem 2.3(1) of [5] and also [1]) canonicaliy induces a surjective *-homomorphism Φ defined on $(A \otimes B)/(I_A \otimes B + A \otimes I_B)$ and with values in $(A \otimes B)/I$. Note that in order to prove that $A \otimes B$ has the weak ideal property it is enough to show that $\mathcal{P}(S \otimes \mathcal{K}) \neq \{0\}$. Now let $T := \Phi^{-1}(S)$. Then $T \lhd (A \otimes B)/(I_A \otimes B + A \otimes I_B)$, $\Phi(T) = S$ and hence $T \neq 0$ (since Φ is linear and $S \neq 0$). By Corollary 4 of [4] (and Definition 9 of [4]) it follows that:

$$\mathcal{P}(T \otimes \mathcal{K}) \neq \{0\} \tag{2.8}$$

On the other hand $S \cong T/\ker(\Phi|T)$ and hence:

$$S \otimes \mathcal{K} \cong (T/\ker(\Phi|T)) \otimes \mathcal{K} \tag{2.9}$$

Finally, since $(A \otimes B)/(I_A \otimes B + A \otimes I_B)$ has the stable quotient property (use (2.6) and the fact that $(A/I_A) \otimes (B/I_B)$ has the stable quotient property by our hypothesis), using also (2.8) and the fact that $T/\ker(\Phi|T) \neq 0$ (since $T/\ker(\Phi|T) \cong S \neq 0$), we deduce that $\mathcal{P}((T/\ker(\Phi|T)) \otimes \mathcal{K}) \neq 0$, which implies that $\mathcal{P}(S \otimes \mathcal{K}) \neq \{0\}$ (use (2.9)). This ends the proof of (2) $\Rightarrow$ (1).

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