Laguerre isoparametric hypersurfaces in $\mathbb{R}^5$

by

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Abstract

Let $x : M \to \mathbb{R}^n$ be an $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$ and $\mathbf{B}$ be the Laguerre second fundamental form of the immersion x. An eigenvalue of the Laguerre second fundamental form is called *a Laguerre principal curvature* of x. An umbilic free hypersurface $x : M \to \mathbb{R}^n$ with non-zero principal curvatures and vanishing Laguerre form $\mathbf{C} \equiv 0$ is called *a Laguerre isoparametric hypersurface* if the Laguerre principal curvatures of x are constants. The aim of this article is to classify all oriented Laguerre isoparametric hypersurfaces in $\mathbb{R}^5$.

1 Introduction

Li and Wang [7] studied invariants of hypersurfaces in Euclidean space $\mathbb{R}^n$ under the Laguerre transformation group. An oriented hypersurface $x : M \to \mathbb{R}^n$ can be identified as the submanifold $(x, \xi) : M \to U\mathbb{R}^n$, where ξ is the unit normal of x. Two hypersurfaces $x, x^* : M \to \mathbb{R}^n$ are called Laguerre equivalent, if there is a Laguerre transformation $\phi : U\mathbb{R}^n \to U\mathbb{R}^n$ such that $(x^*, \xi^*) = \phi \circ (x, \xi)$ (see [8]). In [7], Li and Wang gave a complete Laguerre invariant system for hypersurfaces in $\mathbb{R}^n$. They proved the following:

Theorem 1. *Two umbilical free oriented hypersurfaces in $\mathbb{R}^n$ with non-zero principal curvatures are Laguerre equivalent if and only if they have the same Laguerre metric g and Laguerre second fundamental form $\mathbf{B}$.*

We notice that the Laguerre geometry of surfaces in $\mathbb{R}^3$ has been studied by Blaschke in [1] and other authors in [2], [3]. From [7], we know that the *Laguerre metric g* of the immersion x can be defined by $g = \langle dY, dY \rangle$. Let $\{E_1, E_2, \ldots, E_{n-1}\}$ be an orthonormal basis for g with dual basis $\{\omega_1, \omega_2, \ldots, \omega_{n-1}\}$. *The Laguerre tensor $\mathbf{L}$, the Laguerre second fundamental form $\mathbf{B}$* and *the Laguerre form $\mathbf{C}$* of the immersion x are defined by

$$\mathbf{L} = \sum_{i,j=1}^{n-1} L_{ij} \omega_i \otimes \omega_j, \quad \mathbf{B} = \sum_{i,j=1}^{n-1} B_{ij} \omega_i \otimes \omega_j, \quad \mathbf{C} = \sum_{i=1}^{n-1} C_i \omega_i, \tag{1.1}$$

respectively, where L_{ij}, B_{ij} and C_i are defined by

$$L_{ij} = \rho^{-2}\big\{\mathrm{Hess}_{ij}(\log\rho) - \tilde{E}_i(\log\rho)\tilde{E}_j(\log\rho) + \frac{1}{2}\left(|\nabla\log\rho|^2 - 1\right)\delta_{ij}\big\}, \qquad (1.2)$$

$$B_{ij} = \rho^{-1}(r_i - r)\delta_{ij}, \qquad (1.3)$$

$$C_i = -\rho^{-2}\big\{\tilde{E}_i(r) - \tilde{E}_i(\log\rho)(r_i - r)\big\}, \qquad (1.4)$$

where $g = \sum_i (r_i - r)^2 III = \rho^2 III$, r_i and r are the curvature radii and mean curvature radius of x respectively, Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to the third fundamental form $III = d\xi \cdot d\xi$ of x (see [7]).

If $\nabla\mathbf{B} = 0$, we call that x is of *parallel Laguerre second fundamental form*, where ∇ is the Levi-Civita connection of the Laguerre metric g. We call an eigenvalue of the Laguerre second fundamental form *a Laguerre principal curvature*, an eigenvalue of the Laguerre tensor *a Laguerre eigenvalue* of x. An umbilic free hypersurface $x : M \to \mathbb{R}^n$ with non-zero principal curvatures and vanishing Laguerre form $\mathbf{C} \equiv 0$ is called *a Laguerre isoparametric hypersurface* if the Laguerre principal curvatures of x are constants. A hypersurface with vanishing Laguerre form is called *a Laguerre isotropic hypersurface*, if the Laguerre eigenvalues of x are equal.

Let σ be the Laguerre embedding $\sigma : U\mathbb{R}_1^n \to U\mathbb{R}^n$ defined by $\sigma(x,\xi) = (x',\xi') \in U\mathbb{R}^n$, $x = (x_0, x_1) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $\xi = (\xi_0, \xi_1) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $x' = \left(-\frac{x_1}{\xi_1}, x_0 - \frac{x_1}{\xi_1}\xi_0\right)$, $\xi' = \left(\frac{1}{\xi_1}, \frac{\xi_0}{\xi_1}\right)$ and let τ be the Laguerre embedding $\tau : U\mathbb{R}_0^n \to U\mathbb{R}^n$ defined by $\tau(x,\xi) = (x',\xi') \in U\mathbb{R}^n$, $x = (x_1, x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$, $\xi = (\xi_1 + 1, \xi_0, \xi_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ and $x' = \left(-\frac{x_1}{\xi_1}, x_0 - \frac{x_1}{\xi_1}\xi_0\right)$, $\xi' = \left(1 + \frac{1}{\xi_1}, \frac{\xi_0}{\xi_1}\right)$ (see [9]). Recently, Li, H. Li and Wang [8] classified the umbilic free hypersurfaces with parallel Laguerre second fundamental form (see Theorem in [8]). Very recently, we notice that Li and Sun [10] proved the following:

Theorem 2. *Let $x : M \to \mathbb{R}^4$ be a Laguerre isoparametric hypersurface in $\mathbb{R}^4$. Then x is Laguerre equivalent to an open part of one of the following hypersurfaces:*
    (1) *the oriented hypersurface $x : S^{k-1} \times H^{4-k} \to \mathbb{R}^4$ given by Example 1, or*
    (2) *the image of τ of the oriented hypersurface $x : \mathbb{R}^3 \to \mathbb{R}_0^4$ given by Example 2.*

The aim of this article is to classify all oriented Laguerre isoparametric hypersurfaces in $\mathbb{R}^5$. We obtain the following:

Theorem 3 (Main Theorem). *Let $x : M \to \mathbb{R}^5$ be a Laguerre isoparametric hypersurface in $\mathbb{R}^5$. Then x is Laguerre equivalent to an open part of one of the following hypersurfaces:*
    (1) *the oriented hypersurface $x : S^{k-1} \times H^{5-k} \to \mathbb{R}^5$ given by Example 1, or*
    (2) *the image of τ of the oriented hypersurface $x : \mathbb{R}^4 \to \mathbb{R}_0^5$ given by Example 2, or*
    (3) *the image of σ of hypersurface $\tilde{x}$ in $\mathbb{R}_1^5$ with mean curvature radius $r = 0$ and $\rho = $ constant, or*
    (4) *the image of τ of hypersurface $\tilde{x}$ in $\mathbb{R}_0^5$ with mean curvature radius $r = 0$ and $\rho = $ constant.*

Remark 1. *From Theorem 1, we see that the examples of case (3) and case (4) in Theorem 3 (Main Theorem) have the same Laguerre second fundamental form $\mathbf{B}$ with the Laguerre isoparametric hypersurfaces $x : M \to \mathbb{R}^5$, thus they are also Laguerre isoparametric hypersurfaces. Li*

and Sun [10] pointed out that, like Laguerre isoparametric surfaces, the Laguerre isoparametric hypersurfaces in $\mathbb{R}^4$ have parallel Laguerre second fundamental form, but for higher dimensional hypersurfaces, the result may not hold. From the proof of our Theorem 3, we see that the Laguerre isoparametric hypersurfaces in $\mathbb{R}^5$ have parallel Laguerre second fundamental form if they are not the Laguerre isotropic hypersurfaces: case (3) and case (4) in Theorem 3. We notice that Hu, H. Li and Wang classified the Möbius isoparametric hypersurfaces in the unit spheres $\mathbb{S}^4$ and $\mathbb{S}^5$ (see [4] and [5]) and the first author and Su [11] classified the conformal isoparametric spacelike hypersurfaces in conformal spaces $\mathbb{Q}_1^4$ and $\mathbb{Q}_1^5$.

2 Laguerre fundamental formulas

We recall the fundamental formulas on Laguerre geometry of hypersurfaces in $\mathbb{R}^n$, for more details, see [7]. Let $x : M \to \mathbb{R}^n$ be an $(n-1)$-dimensional umbilical free hypersurface with vanishing Laguerre form in $\mathbb{R}^n$. Defining the covariant derivative of C_i, L_{ij}, B_{ij} by

$$\sum_j C_{i,j}\omega_j = dC_i + \sum_j C_j\omega_{ji}, \tag{2.1}$$

$$\sum_k L_{ij,k}\omega_k = dL_{ij} + \sum_k L_{ik}\omega_{kj} + \sum_k L_{kj}\omega_{ki}, \tag{2.2}$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}, \tag{2.3}$$

we have from [7] that

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \quad R_{ijkl} = -R_{jikl}, \tag{2.4}$$

$$\sum_i B_{ii} = 0, \quad \sum_{i,j} B_{ij}^2 = 1, \quad \sum_i B_{ij,i} = (n-2)C_j, \quad \text{tr}\mathbf{L} = -\frac{R}{2(n-2)}, \tag{2.5}$$

$$L_{ij,k} = L_{ik,j}, \quad C_{i,j} - C_{j,i} = \sum_k (B_{ik}L_{kj} - B_{jk}L_{ki}), \tag{2.6}$$

$$B_{ij,k} - B_{ik,j} = C_j\delta_{ik} - C_k\delta_{ij}, \tag{2.7}$$

$$R_{ijkl} = L_{jk}\delta_{il} + L_{il}\delta_{jk} - L_{ik}\delta_{jl} - L_{jl}\delta_{ik}. \tag{2.8}$$

Since the Laguerre form $\mathbf{C} \equiv 0$, we have for all indices i, j, k

$$B_{ij,k} = B_{ik,j}, \quad \sum_k B_{ik}L_{kj} = \sum_k B_{kj}L_{ki}. \tag{2.9}$$

Defining the second covariant derivative of B_{ij} by

$$\sum_l B_{ij,kl}\omega_l = dB_{ij,k} + \sum_l B_{lj,k}\omega_{li} + \sum_l B_{il,k}\omega_{lj} + \sum_l B_{ij,l}\omega_{lk}, \tag{2.10}$$

we have the Ricci identity

$$B_{ij,kl} - B_{ij,lk} = \sum_m B_{mj} R_{mikl} + \sum_m B_{im} R_{mjkl}. \qquad (2.11)$$

3   Examples

We review some examples of hypersurfaces in $\mathbb{R}^n$ and calculate their Laguerre invariants, for more details, see [8] and [9] .

Example 1. *Let $x : S^{k-1} \times H^{n-k} \to \mathbb{R}^n$ be an umbilic free hypersurface in $\mathbb{R}^n$ defined by $x(u,v,w) = \left(\frac{u}{w}(1+w), \frac{v}{w} \right)$, where $H^{n-k} = \{ (v,w) \in \mathbb{R}^{n-k+1}_1 | v \cdot v - w^2 = -1, w > 0 \}$ denotes the hyperbolic space embedded in the Minkowski space $\mathbb{R}^{n-k+1}_1$. From [8], we know that x has two distinct Laguerre principal curvatures $B_1 = -\sqrt{\frac{n-k}{(k-1)(n-1)}}$, $B_2 = \sqrt{\frac{k-1}{(n-k)(n-1)}}$, the Laguerre form is vanishing and the Laguerre second fundamental form of x is parallel.*

Example 2. *For any positive integers $m_1, \ldots, m_s$ with $m_1 + \cdots + m_s = n - 1$ and any non-zero constants $\lambda_1, \ldots, \lambda_s$, we define $x : \mathbb{R}^{n-1} \to \mathbb{R}^n_0$ to be a spacelike oriented hypersurface in $\mathbb{R}^n_0$ given by*

$$x = \left\{ \frac{\lambda_1 |u_1|^2 + \cdots + \lambda_s |u_s|^2}{2}, u_1, u_2, \ldots, u_s, \frac{\lambda_1 |u_1|^2 + \cdots + \lambda_s |u_s|^2}{2} \right\},$$

where $(u_1, \ldots, u_s) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_s} = \mathbb{R}^{n-1}$ and $|u_i|^2 = u_i \cdot u_i, i = 1, \ldots, s$. Then $\tau \circ (x, \xi) = (x', \xi') : \mathbb{R}^{n-1} \to U\mathbb{R}^n$, and we obtain the hypersurfaces $x' : \mathbb{R}^{n-1} \to \mathbb{R}^n$. From [8], we know that x has $s(s \geq 3)$ distinct Laguerre principal curvatures $B_i = \frac{r_i - r}{\sqrt{\sum_i (r_i - r)^2}}$, $1 \leq i \leq s$, where $r_i = \frac{1}{\lambda_i}$, $r = \frac{\lambda_1 r_1 + \lambda_2 r_2 + \cdots + \lambda_s r_s}{n-1}$ and $\lambda_i \neq 0$ is the constant principal curvature corresponding to e_i. Also from [8], we know that the Laguerre form is vanishing, $L_{ij} = 0$ for $1 \leq i, j \leq n - 1$ and the Laguerre second fundamental form of x is parallel.

Example 3. *Let $y = (v, w) : M^{n-m-1} \to \mathbb{R}^{n-m}_1$ be an umbilic free space-like hypersurface with non-zero principal curvatures in the semi-Euclidean space $\mathbb{R}^{n-m}_1$ and $\xi = (\xi_0, \xi_1) : M^{n-m-1} \to \mathbb{R}^{n-m}_1$ be its unit normal field, where $v, \xi_0 \in \mathbb{R}^{n-m-1}$, $w, \xi_1 \in \mathbb{R}$ and $\xi_0 \cdot \xi_0 - \xi_1^2 = -1$. Let $f : S^m \to \mathbb{R}^{m+1}$ be the canonical embedding. Defining hypersurface $x : S^m \times M^{n-m-1} \to \mathbb{R}^n$ by $x = \left(-\frac{w}{\xi_1} f, v - \frac{w}{\xi_1} \xi_0 \right)$, from [9], by a direct calculation, we know that the Laguerre form is vanishing, x has distinct constant Laguerre principal curvatures*

$$B_i = \frac{H_1}{(n-1)\sqrt{H_2 - \frac{H_1^2}{n-1}}}, 1 \leq i \leq m,$$

$$B_i = \frac{1}{\sqrt{H_2 - \frac{H_1^2}{n-1}}} \left(\frac{H_1}{n-1} - \frac{1}{k_i} \right), \quad m+1 \leq i \leq n-1,$$

$k_i \neq 0$ are the constant principal curvatures of y and the Laguerre second fundamental form of x is not parallel.

4   Proof of Theorem 3 (Main Theorem)

Proof: From (2.5), we know that the number γ of distinct Laguerre principal curvatures can only take the values $\gamma = 2, 3, 4$. From (2.9), we can choose the local orthonormal basis E_1, E_2, E_3, E_4 to diagonalize the matrix (B_{ij}) and (A_{ij}), that is, $B_{ij} = B_i \delta_{ij}$ and $L_{ij} = L_i \delta_{ij}$.

Let B_1, B_2, B_3, B_4 be the constant Laguerre principal curvatures of x. From (2.3), we have

$$\sum_k B_{ij,k}\omega_k = (B_i - B_j)\omega_{ij}. \tag{4.1}$$

We consider three cases:

(1) If $\gamma = 2$, that is, x has two distinct constant Laguerre principal curvatures B_1 and B_2 with multiplicities m_1 and m_2. From (4.1), we have $B_{ij,k} = (B_i - B_j)\Gamma_{ik}^j$, where Γ_{ik}^j is the Levi-Civita connection for the Laguerre metric g given by $\omega_{ij} = \sum_k \Gamma_{ik}^j \omega_k$, $\Gamma_{ik}^j = -\Gamma_{jk}^i$. Thus $B_{ij,k} = 0$, if $1 \leq i, j \leq m_1$ or $m_1 + 1 \leq i, j \leq m_1 + m_2 = 4$. From the symmetry of $B_{ij,k}$, we see that $B_{ij,k} = 0$ for all i, j, k, that is, x has parallel Laguerre second fundamental form. By the Theorem in [8], Example 1 and Example 2, we know that Theorem 3 (Main Theorem) is true.

(2) If $\gamma = 3$, when the Laguerre second fundamental form is parallel, from the Theorem in [8], Example 1 and Example 2, we know that Theorem 3 (Main Theorem) is true.

When the Laguerre second fundamental form is not parallel, we can prove that this case does not occur. In fact, without loss of generality, we may assume that $B_1 \neq B_2 \neq B_3 = B_4$. From (4.1), we have

$$B_{ii,k} = 0, \quad B_{34,k} = 0, \quad \text{for all} \ \ i, k, \tag{4.2}$$

$$\omega_{ij} = \sum_k \frac{B_{ij,k}}{B_i - B_j}\omega_k, \quad \text{for} \ \ B_i \neq B_j. \tag{4.3}$$

From (4.2), (4.3) and (2.10), we have

$$\sum_l B_{13,4l}\omega_l = B_{12,4}\omega_{23} + B_{12,3}\omega_{24} = \frac{2B_{12,3}B_{12,4}}{B_2 - B_3}\omega_1, \tag{4.4}$$

$$\sum_l B_{11,3l}\omega_l = 2B_{12,3}\omega_{21} = \frac{2B_{12,3}^2}{B_2 - B_1}\omega_3 + \frac{2B_{12,3}B_{12,4}}{B_2 - B_1}\omega_4. \tag{4.5}$$

Comparing two sides of (4.4) and (4.5), we have

$$B_{13,41} = \frac{2B_{12,3}B_{12,4}}{B_2 - B_3}, \quad B_{13,42} = B_{13,43} = B_{13,44} = 0, \tag{4.6}$$

$$B_{11,33} = \frac{2B_{12,3}^2}{B_2 - B_1}, \quad B_{11,34} = \frac{2B_{12,3}B_{12,4}}{B_2 - B_1}, \quad B_{11,32} = 0. \tag{4.7}$$

From (2.11), we have $B_{ij,kl} - B_{ij,lk} = (B_i - B_j)R_{ijkl}$. From (2.8), we know that if three of $\{i, j, k, l\}$ are either the same or distinct, then $R_{ijkl} = 0$. Thus, if three of $\{i, j, k, l\}$ are either the same or distinct, then

$$B_{ij,kl} = B_{ij,lk}. \tag{4.8}$$

From (4.8), (2.9), (4.6) and (4.7), we have $B_{12,3}B_{12,4} = 0$. Since the Laguerre second fundamental form is not parallel, without loss of generality, we may assume that $B_{12,3} \neq 0$ and $B_{12,4} = 0$. We may also prove that $B_{12,3}$ is constant. In fact, from (2.10), (4.2) and (4.3), we have

$$\sum_k B_{12,3k}\omega_k = dB_{12,3}, \tag{4.9}$$

$$\sum_k B_{ii,jk}\omega_k = 2\sum_{l \neq i,j} B_{li,j}\omega_{li} = 2\sum_k \sum_{l \neq i,j} \frac{B_{li,j}B_{li,k}}{B_l - B_i}\omega_k, \quad \text{for} \quad B_l \neq B_i. \tag{4.10}$$

Thus

$$B_{ii,jk} = 2\sum_{l \neq i,j} \frac{B_{li,j}B_{li,k}}{B_l - B_i}, \quad \text{for} \quad B_l \neq B_i. \tag{4.11}$$

From (4.2) and (4.11), we know that

$$B_{ii,ji} = B_{ii,jl} = 0, \quad \text{for distinct} \quad i, j, l. \tag{4.12}$$

From (4.12), (4.8) and (2.9), we have

$$B_{12,31} = B_{11,23} = 0, \quad B_{12,32} = B_{22,13} = 0, \quad B_{12,33} = B_{33,12} = 0. \tag{4.13}$$

On the other hand, from (4.2), (4.3) and $B_{12,4} = 0$, we have $\sum_k B_{34,1k}\omega_k = B_{12,3}\omega_{24} = \sum_k \frac{B_{12,3}B_{24,k}}{B_2 - B_4}\omega_k$. Thus $B_{34,1k} = \frac{B_{12,3}B_{24,k}}{B_2 - B_4}$ and we have $B_{34,12} = 0$. From (4.8) and (2.9), we have

$$B_{12,34} = B_{34,12} = 0. \tag{4.14}$$

From (4.9), (4.13) and (4.14), we see that $dB_{12,3} = 0$. Therefore, we know that $B_{12,3}$ is constant. From (4.2) and (4.3), we have

$$\omega_{12} = \frac{B_{12,3}}{B_1 - B_2}\omega_3, \quad \omega_{13} = \frac{B_{12,3}}{B_1 - B_3}\omega_2, \quad \omega_{23} = \frac{B_{12,3}}{B_2 - B_3}\omega_1, \quad \omega_{14} = \omega_{24} = 0. \tag{4.15}$$

From (4.15), (2.4), and by a simple calculation, we have

$$\frac{1}{2}\sum_{k,l} R_{12kl}\omega_k \wedge \omega_l = \frac{2B_{12,3}^2}{(B_1 - B_3)(B_2 - B_3)}\omega_1 \wedge \omega_2 + \frac{B_{12,3}}{B_1 - B_2}\omega_4 \wedge \omega_{34}, \tag{4.16}$$

$$\frac{1}{2}\sum_{k,l} R_{13kl}\omega_k \wedge \omega_l = \omega_{12} \wedge \omega_{23} - d\omega_{13} = \frac{2B_{12,3}^2}{(B_1 - B_2)(B_3 - B_2)}\omega_1 \wedge \omega_3, \tag{4.17}$$

$$\frac{1}{2}\sum_{k,l} R_{14kl}\omega_k \wedge \omega_l = \omega_{13} \wedge \omega_{34} = \frac{B_{12,3}}{B_1 - B_3}\omega_2 \wedge \omega_{34}, \tag{4.18}$$

$$\frac{1}{2}\sum_{k,l} R_{23kl}\omega_k \wedge \omega_l = \omega_{21} \wedge \omega_{13} - d\omega_{23} = \frac{2B_{12,3}^2}{(B_2 - B_1)(B_3 - B_1)}\omega_2 \wedge \omega_3, \tag{4.19}$$

$$\frac{1}{2}\sum_{k,l} R_{24kl}\omega_k \wedge \omega_l = \omega_{23} \wedge \omega_{34} - d\omega_{24} = \frac{B_{12,3}}{B_2 - B_3}\omega_1 \wedge \omega_{34}. \tag{4.20}$$

Let $\omega_{34} = \sum_k \Gamma_{k4}^3 \omega_k$, $\Gamma_{k4}^3 = -\Gamma_{k3}^4$. Comparing two sides of (4.16)– (4.20), we have

$$R_{1212} = \frac{2B_{12,3}^2}{(B_1 - B_3)(B_2 - B_3)}, \quad R_{1313} = \frac{2B_{12,3}^2}{(B_1 - B_2)(B_3 - B_2)}, \tag{4.21}$$

$$R_{2323} = \frac{2B_{12,3}^2}{(B_2 - B_1)(B_3 - B_1)}, \quad R_{1414} = R_{2424} = 0. \tag{4.22}$$

From (4.16), (4.18) and (4.20), we know that

$$\frac{1}{2}R_{12k4} = \frac{B_{12,3}}{B_2 - B_1}\Gamma_{k4}^3, \quad \frac{1}{2}R_{142k} = \frac{B_{12,3}}{B_1 - B_3}\Gamma_{k4}^3, \quad \frac{1}{2}R_{24k1} = \frac{B_{12,3}}{B_3 - B_2}\Gamma_{k4}^3. \tag{4.23}$$

Since we know that the Bianchi identities of curvature tensors R_{ijkl} are $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ and $R_{ijkl} = R_{klij}$, $R_{ijlk} = R_{jikl}$, we have $R_{142k} + R_{12k4} + R_{24k1} = 0$. Thus, from (4.23), we have $\Gamma_{k4}^3 = 0$ for all k. Thus $\omega_{34} = 0$. From (4.15) and (2.4), we have $-\frac{1}{2}\sum_{k,l} R_{34kl}\omega_k \wedge \omega_l = d\omega_{34} - \sum_k \omega_{3k} \wedge \omega_{k4} = 0$, this implies that

$$R_{3434} = 0. \tag{4.24}$$

From (2.8), (4.21), (4.22) and (4.24), we have

$$-L_1 - L_2 = \frac{2B_{12,3}^2}{(B_1 - B_3)(B_2 - B_3)}, \quad -L_1 - L_3 = \frac{2B_{12,3}^2}{(B_1 - B_2)(B_3 - B_2)}, \tag{4.25}$$

$$-L_2 - L_3 = \frac{2B_{12,3}^2}{(B_2 - B_1)(B_3 - B_1)}, \quad L_1 + L_4 = L_2 + L_4 = L_3 + L_4 = 0, \tag{4.26}$$

this implies that $L_1 = L_2 = L_3$ and $L_1 = \frac{2B_{12,3}^2}{(B_2-B_1)(B_3-B_1)}$, $L_2 = \frac{2B_{12,3}^2}{(B_1-B_2)(B_3-B_2)}$, $L_3 = \frac{2B_{12,3}^2}{(B_1-B_3)(B_2-B_3)}$. By a simple calculation, we must have $B_1 = B_2 = B_3$, a contradiction. Thus, the case that the Laguerre second fundamental form is not parallel does not occur.

(3) If $\gamma = 4$, when the Laguerre second fundamental form is parallel, from the Theorem in [8], Example 1 and Example 2, we know that Theorem 3 (Main Theorem) is true.

When the Laguerre second fundamental form is not parallel, since $B_1 \neq B_2 \neq B_3 \neq B_4$, from (4.1), we have $B_{ii,k} = 0$, for all i, k. We denote by i, j, k, l the four distinct elements of $\{1, 2, 3, 4\}$ with order arbitrarily given, from (2.3), we have

$$\omega_{ij} = \frac{B_{ij,k}\omega_k + B_{ij,l}\omega_l}{B_i - B_j}, \quad \text{for } i \neq j. \tag{4.27}$$

From (4.27) and (2.4), by a simple and similar calculation (see [5]), we have

$$-\frac{1}{2}\sum_{s,t} R_{ijst}\omega_s \wedge \omega_t = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} - \omega_{il} \wedge \omega_{lj}$$

$$\equiv -\left(\frac{2B_{ij,k}^2}{(B_i - B_k)(B_j - B_k)} + \frac{2B_{ij,l}^2}{(B_i - B_l)(B_j - B_l)}\right)\omega_i \wedge \omega_j$$

$$\mod (\omega_s \wedge \omega_t, \quad (s,t) \neq (i,j), (j,i)).$$

Comparing two sides of the above equation, we have

$$R_{ijij} = \frac{2B_{ij,k}^2}{(B_i - B_k)(B_j - B_k)} + \frac{2B_{ij,l}^2}{(B_i - B_l)(B_j - B_l)}. \tag{4.28}$$

Since the Laguerre second fundamental form is not parallel, by the symmetry of $B_{ij,k}$, we may consider two cases:

$Case(i)$ If at least two of $\{B_{12,3}, B_{12,4}, B_{13,4}, B_{23,4}\}$ are nonzero, without loss of generality, we may assume that $B_{12,3} \neq 0$ and $B_{12,4} \neq 0$. From (2.2) and (2.3), we have

$$L_{ij,k} = E_k(L_i)\delta_{ij} + \Gamma_{ik}^j(L_i - L_j),$$

$$B_{ij,k} = E_k(B_i)\delta_{ij} + \Gamma_{ik}^j(B_i - B_j).$$

Thus, we see that

$$L_{12,3} = \Gamma_{32}^1(L_1 - L_2) = \Gamma_{23}^1(L_1 - L_3) = \Gamma_{13}^2(L_2 - L_3), \tag{4.29}$$

$$B_{12,3} = \Gamma_{32}^1(B_1 - B_2) = \Gamma_{23}^1(B_1 - B_3) = \Gamma_{13}^2(B_2 - B_3) \neq 0, \tag{4.30}$$

$$L_{12,4} = \Gamma_{42}^1(L_1 - L_2) = \Gamma_{24}^1(L_1 - L_4) = \Gamma_{14}^2(L_2 - L_4), \tag{4.31}$$

$$B_{12,4} = \Gamma_{42}^1(B_1 - B_2) = \Gamma_{24}^1(B_1 - B_4) = \Gamma_{14}^2(B_2 - B_4) \neq 0, \tag{4.32}$$

and therefore

$$\frac{L_{12,3}}{B_{12,3}} = \frac{L_1 - L_2}{B_1 - B_2} = \frac{L_1 - L_3}{B_1 - B_3} = \frac{L_2 - L_3}{B_2 - B_3},$$

$$\frac{L_{12,4}}{B_{12,4}} = \frac{L_1 - L_2}{B_1 - B_2} = \frac{L_1 - L_4}{B_1 - B_4} = \frac{L_2 - L_4}{B_2 - B_4}.$$

Thus, there is a function λ such that

$$\frac{L_1 - L_2}{B_1 - B_2} = \frac{L_1 - L_3}{B_1 - B_3} = \frac{L_2 - L_3}{B_2 - B_3} = \frac{L_1 - L_4}{B_1 - B_4} = \frac{L_2 - L_4}{B_2 - B_4} = -\lambda, \tag{4.33}$$

and there is another function μ such that

$$L_1 + \lambda B_1 = L_2 + \lambda B_2 = L_3 + \lambda B_3 = L_4 + \lambda B_4 = \mu, \tag{4.34}$$

that is, we have

$$L_{ij} + \lambda B_{ij} = \mu\delta_{ij}, \quad \text{for any } 1 \leq i, j \leq 4. \tag{4.35}$$

By the similar method in [6] (see page 8 of [6]), we can prove that λ and μ are constants. From (4.34), we see that L_1, L_2, L_3, L_4 are constants. If $L_1 = L_2 = L_3 = L_4$, then x is a Laguerre isotropic hypersurface in $\mathbb{R}^5$. From the Theorem 1.1 in [9], we see that Theorem 3 (Main Theorem) is true. From Remark 1, we know that these two isotropic hypersurfaces are also Laguerre isoparametric.

If at least two of $\{L_1, L_2, L_3, L_4\}$ are not equal, that is, x is non-Laguerre isotropic hypersurface, by the Proposition 6.1 in [9], we know that $\nabla \mathbf{L} = 0$, that is, the Laguerre tensor of x is parallel. From (4.29)–(4.32), we see that $L_1 = L_2 = L_3 = L_4$, a contradiction.

$Case(ii)$ If exactly one of $\{B_{12,3}, B_{12,4}, B_{13,4}, B_{23,4}\}$ is nonzero, without loss of generality, we may assume that $B_{12,3} \neq 0$. From (2.8) and (4.28), we see that

$$-L_1 - L_2 = \frac{2B_{12,3}^2}{(B_1 - B_3)(B_2 - B_3)}, \quad -L_1 - L_3 = \frac{2B_{12,3}^2}{(B_1 - B_2)(B_3 - B_2)},$$

$$-L_2 - L_3 = \frac{2B_{12,3}^2}{(B_2 - B_1)(B_3 - B_1)}, \quad -L_1 - L_4 = -L_2 - L_4 = -L_3 - L_4 = 0,$$

this implies that $L_1 = L_2 = L_3$. By a simple calculation, we must have $B_1 = B_2 = B_3$, a contradiction. Thus, $Case(ii)$ does not occur. This completes the proof of Theorem 3 (Main Theorem). □

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