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Infinitely many solutions for a class of Dirichlet boundary value problems with impulsive effects

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Abstract

The existence of infinitely many solutions for a class of doubly eigenvalue nonlinear Dirichlet boundary value problems with impulsive effects is established. Our approach is based on variational methods.

Key Words: Infinitely many solutions, Dirichlet boundary value problems with impulsive effects, critical point theory, variational methods.
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1 Introduction

The aim of this paper is to investigate the existence of infinitely many weak solutions for the following doubly eigenvalue nonlinear Dirichlet problem

$$\begin{cases} -u''(x) = \lambda f(x, u(x)) + \mu g(x, u(x)) & a.e.\ x \in [0, T], \\ u(0) = u(T) = 0 \end{cases} \tag{1.1}$$

with the impulsive conditions

$$\Delta u'(x_j) = I_j(u(x_j)), \qquad j = 1, 2, ..., p \tag{1.2}$$

where $T > 0$, $f, g : [0, T] \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, $x_0 = 0 < x_1 < x_2 ... < x_p < x_{p+1} = T, \Delta u'(x_j) = u'(x_j^+) - u'(x_j^-) = lim_{x \to x_j^+} u'(x) - lim_{x \to x_j^-} u'(x), I_j : \mathbb{R} \to \mathbb{R}$ are continuous and λ is a positive parameter and μ is non-negative parameter. We refer to impulsive problem (1.1)-(1.2) as (IP).

Owing to importance of impulsive differential equations arising from the real world describe the dynamics of processes in which sudden, discontinuous jumps occur, many researchers have studied this kind of equations, and here we refer the interested reader to the recent works [13, 18] and the references therein. For a second order differential equation $u'' = f(t, u, u')$ one usually considers impulses in the position u and the velocity u'. However, in the motion of spacecraft one has to consider instantaneous impulses depending on the position that result in jump discontinuities in velocity, but with no change in position [11]. The impulses only on the velocity occurs also in impulsive mechanics [16]. Existence and multiplicity of solutions for impulsive differential equations have been studied by several authors and, for an overview on this subject, we refer the reader to the papers [1, 2, 3, 4, 14, 19, 20, 21, 22]. In the present paper, motivated by [2], employing a smooth version of [7, Theorem 2.1] which is a more precise version of Ricceri's Variational Principle [17, Theorem 2.5] under some hypotheses on the behavior of the nonlinear terms at infinity, under conditions on the potentials of f and g we prove the existence of a definite interval about λ in which the problem (IP) admits a sequence of weak solutions which is unbounded in the Sobolev space $H_0^1(0,T)$(Theorem 3). We also list some consequences of Theorem 3. Replacing the conditions at infinity of the nonlinear terms, by a similar one at zero, the same results hold and, in addition, the sequence of weak solutions uniformly converges to zero; see Remark 4. An example of application is pointed out(see Example 1). To the best of our knowledge, no investigation has been devoted to establishing the existence of infinitely many solutions to such a problem as (IP). We also refer the interested reader to the papers [5, 6, 8, 9, 10, 12] in which Ricceri's Variational Principle [17, Theorem 2.5] and its variants ([7, 15]) have been successfully employed to the existence of infinitely many solutions for boundary value problems.

2 Preliminaries

Our main tool to investigate the existence of infinitely many weak solutions for the problem (IP) is a smooth version of Theorem 2.1 of [7] which is a more precise version of Ricceri's Variational Principle [17, Theorem 2.5] that we now recall here.

Theorem 1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(*a*) *for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(*b*) *If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds:*
either
(b_1) *I_λ possesses a global minimum,*
or

(b_2) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* I_λ *such that*

$$\lim_{n\to+\infty}\Phi(u_n)=+\infty.$$

(c) *If* $\delta<+\infty$ *then, for each* $\lambda\in]0,\frac{1}{\delta}[$, *the following alternative holds: either*

(c_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ,

or

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of* I_λ *which weakly converges to a global minimum of* Φ.

Let $X:=H_0^1(0,T)$. In the Sobolev space X, consider the inner product $\prec u,v\succ=\int_0^T u'(x)v'(x)dx$ and the norm $||u||=\left(\int_0^T |u'(x)|^2dx\right)^{\frac{1}{2}}$. Let $H^2(0,T)=\{u\in C^1[0,T]: u''\in L^2[0,T]\}$.

The following lemma is useful for proving our main result.

Lemma 1. *(see [2]). Let* $u\in X$. *Then*

$$\| u\|_\infty=\max_{t\in[0,T]}|u(t)|\leq\frac{\sqrt{T}}{2}\| u\|. \tag{2.1}$$

By a classical solution of the problem (IP), we mean a function $u\in\{u(x)\in H^1(0,T): u(x)\in H^2(x_j,x_{j+1}),\ j=0,1,...,p\}$ such that u satisfies (1.1)-(1.2). We say that a function $u\in X$ is a weak solution of the problem (IP) if

$$\int_0^T u'(x)v'(x)dx+\sum_{j=1}^p I_j(u(x_j))v(x_j)-\lambda\int_0^T f(x,u(x))v(x)dx$$

$$-\mu\int_0^T g(x,u(x))v(x)dx=0$$

for every $v\in X$. Using standard methods, if f and g are continuous, we see that a weak solution of (1.1) is indeed a classical solution(see [2, Lemma 5]).

In this paper, we assume throughout, and without further mention, that the following condition holds:

(A1) The impulsive functions I_j have sublinear growth, i.e., there exist constants $a_j>0$, $b_j>0$, and $\gamma_j\in[0,1)$, $j=1,2,...,p$ such that

$$|I_j(t)|\leq a_j+b_j|t|^{\gamma_j}\quad\text{for very } t\in\mathbb{R},\ j=1,2,...,p.$$

For the sake of convenience, in the sequel, we define $F(x,t)=\int_0^t f(x,\xi)d\xi$ for all $(x,t)\in[0,T]\times\mathbb{R}$, $C_1=\frac{1}{2}-\sum_{j=1}^p\frac{b_j}{\gamma_j+1}(\frac{\sqrt{T}}{2})^{\gamma_j+1}$, $C_2=\frac{1}{2}+\sum_{j=1}^p\frac{b_j}{\gamma_j+1}(\frac{\sqrt{T}}{2})^{\gamma_j+1}$ and $C_3=\sum_{j=1}^p a_j\frac{\sqrt{T}}{2}+\sum_{j=1}^p\frac{b_j}{\gamma_j+1}(\frac{\sqrt{T}}{2})^{\gamma_j+1}$.

A special case of our main result is the following theorem.

Theorem 2. *Suppose that $C_1 > 0$ and there exist positive constants η and δ with $\eta+\delta < T$. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function and let $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$. Assume that $\liminf_{\xi \longrightarrow +\infty} \frac{F(\xi)}{\frac{4}{T}C_1\xi^2 - \frac{2}{\sqrt{T}}C_3\xi} = 0$ and $\limsup_{\xi \longrightarrow +\infty} \frac{F(\xi)}{\frac{\eta+\delta}{\eta\delta}C_2\xi^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\xi} = +\infty$. Then, for every arbitrary L^1-Carathéodory function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ whose $G(x,t) = \int_0^t g(x,\xi)d\xi$ for every $(x,t) \in [0,T] \times \mathbb{R}$, is a non-negative function satisfying the condition*

$$g_\infty := \lim_{\xi \to +\infty} \frac{\int_0^T \sup_{|t|\le\xi} G(x,t)dx}{\frac{4}{T}C_1\xi^2 - \frac{2}{\sqrt{T}}C_3\xi} < +\infty \tag{2.2}$$

and for every $\mu \in [0, \mu_{g,1}[$ where $\mu_{g,1} := \frac{1}{g_\infty}\left(1 - T\liminf_{\xi\to+\infty} \frac{F(\xi)}{\frac{4}{T}C_1\xi^2 - \frac{2}{\sqrt{T}}C_3\xi}\right)$, the problem

$$\begin{cases} -u''(x) = f(u(x)) + \mu g(x,u(x)) & a.e.\ x \in [0,T], \\ u(0) = u(T) = 0 \end{cases}$$

with the impulsive conditions $\Delta u'(x_j) = I_j(u(x_j)),\ j = 1,2,...,p$, admits a sequence of weak solutions.

3 Main results

We formulate our main result as follows:

Theorem 3. *Assume that*

(A2) $C_1 > 0$ and there exist positive constants η and δ with $\eta + \delta < T$ such that

(A2.1) $F(x,t) \ge 0$ for all $(x,t) \in ([0,\eta] \bigcup [T-\delta, T]) \times \mathbb{R}$;

(A2.2) $\liminf_{\xi \longrightarrow +\infty} \frac{\int_0^T \sup_{|t|\le\xi} F(x,t)dx}{\frac{4}{T}C_1\xi^2 - \frac{2}{\sqrt{T}}C_3\xi} < \limsup_{\xi \longrightarrow +\infty} \frac{\int_\eta^{T-\delta} F(x,\xi)dx}{\frac{\eta+\delta}{\eta\delta}C_2\xi^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\xi}$.

Then, for each $\lambda \in]\lambda_1, \lambda_2[$ where

$$\lambda_1 := \frac{1}{\limsup_{\xi \longrightarrow +\infty} \frac{\int_\eta^{T-\delta} F(x,\xi)dx}{\frac{\eta+\delta}{\eta\delta}C_2\xi^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\xi}}$$

and

$$\lambda_2 := \frac{1}{\liminf_{\xi \longrightarrow +\infty} \frac{\int_0^T \sup_{|t|\le\xi} F(x,t)dx}{\frac{4}{T}C_1\xi^2 - \frac{2}{\sqrt{T}}C_3\xi}},$$

for every arbitrary L^1-Carathéodory function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ whose $G(x,t) = \int_0^t g(x,\xi)d\xi$ for every $(x,t) \in [0,T] \times \mathbb{R}$, is a non-negative function satisfying the condition (2.2) *and for every $\mu \in [0, \mu_{g,\lambda}[$ where*

$$\mu_{g,\lambda} := \frac{1}{g_\infty}\left(1 - \lambda \liminf_{\xi \to +\infty} \frac{\int_0^T \sup_{|t| \le \xi} F(x,t)dx}{\frac{4}{T}C_1\xi^2 - \frac{2}{\sqrt{T}}C_3\xi}\right),$$

the problem (IP) has an unbounded sequence of weak solutions in X.

Proof: Our goal is to apply Theorem 1. Fix $\overline{\lambda} \in]\lambda_1, \lambda_2[$ and let g be a function satisfies the condition (2.2). Since, $\overline{\lambda} < \lambda_2$, one has $\mu_{g,\overline{\lambda}} > 0$. Fix $\overline{\mu} \in [0, \mu_{g,\overline{\lambda}}[$ and put $\nu_1 := \lambda_1$ and $\nu_2 := \frac{\lambda_2}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 g_\infty}$. If $g_\infty = 0$, clearly, $\nu_1 = \lambda_1$, $\nu_2 = \lambda_2$ and $\overline{\lambda} \in]\nu_1, \nu_2[$. If $g_\infty \neq 0$, since $\overline{\mu} < \mu_{g,\overline{\lambda}}$, we obtain $\frac{\overline{\lambda}}{\lambda_2} + \overline{\mu} g_\infty < 1$, and so $\frac{\lambda_2}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2 g_\infty} > \overline{\lambda}$, namely, $\overline{\lambda} < \nu_2$. Hence, bering in mind that $\overline{\lambda} > \lambda_1 = \nu_1$, one has $\overline{\lambda} \in]\nu_1, \nu_2[$. Now, put $Q(x,\xi) = F(x,\xi) + \frac{\overline{\mu}}{\overline{\lambda}} G(x,\xi)$ for all $x \in [0,T]$ and $\xi \in \mathbb{R}$. Consider the functionals Φ, $\Psi : X \to \mathbb{R}$ for each $u \in X$, as follows

$$\Phi(u) = \frac{1}{2}\int_0^T (u'(x))^2 dx + \sum_{j=1}^{p} \int_0^{u(x_j)} I_j(t)dt \quad \text{and} \quad \Psi(u) = \int_0^T Q(x,u(x))dx.$$

It is well known that Ψ is a Gâteaux differentiable functional and sequentially weakly upper semicontinuous whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)v = \int_0^T f(x,u(x))v(x)dx + \frac{\overline{\mu}}{\overline{\lambda}}\int_0^T g(x,u(x))v(x)dx$$

for every $v \in X$, and $\Psi' : X \to X^*$ is a compact operator. Moreover, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)v = \int_0^T u'(x)v'(x)dx + \sum_{j=1}^{p} I_j(u(x_j))v(x_j)$$

for every $v \in X$. Moreover, since I_j for $j = 1, ..., p$ are continuous, we see that $\Phi(u)$ is continuous. Also, as shown in [2], we have $C_1||u||^2 - C_3||u|| \le \Phi(u)$, which follows $\lim_{||u|| \to +\infty} \Phi(u) = +\infty$, namely Φ is coercive. Furthermore, Φ is sequentially weakly lower semicontinuous. Indeed, let $u_n \to u$ be a weakly convergent sequence to u in X. Then, $\liminf_{n \to +\infty} ||u_n|| \ge ||u||$ and $u_n \to u$ uniformly on $[0,T]$. Hence

$$\lim_{n \to +\infty}\Big(\frac{1}{2}\int_0^T (u_n'(x))^2 dx + \sum_{j=1}^{p}\int_0^{u_n(x_j)} I_j(t)dt\Big)$$

$$\ge \frac{1}{2}\int_0^T (u'(x))^2 dx + \sum_{j=1}^{p}\int_0^{u(x_j)} I_j(t)dt,$$

namely $\liminf_{n\to+\infty}\Phi(u_n)\geq\Phi(u)$ which means Φ is sequentially weakly lower semicontinuous. Put $I_{\overline{\lambda}}:=\Phi-\overline{\lambda}\Psi$. Clearly, the weak solutions of the problem (IP) are exactly the solutions of the equation $I'_{\overline{\lambda}}(u)=0$. Now, we want to show that $\gamma<+\infty$, where γ is defined in Theorem 1. Let $\{\xi_n\}$ be a real sequence such that $\xi_n>\frac{\sqrt{T}C_3}{2C_1}$ for all $n\in\mathbb{N}$ and $\xi_n\to+\infty$ as $n\to\infty$ and

$$\lim_{n\to\infty}\frac{\int_0^T\sup_{|t|\leq\xi_n}Q(x,t)dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n}=\liminf_{\xi\to+\infty}\frac{\int_0^T\sup_{|t|\leq\xi}Q(x,t)dx}{\frac{4}{T}C_1\xi^2-\frac{2}{\sqrt{T}}C_3\xi}.$$

Put $r_n=\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n$ for all $n\in\mathbb{N}$. By the same arguing as given in the proof of [2, Lemma 6], we observe that $r_n>0$ for all $n\in\mathbb{N}$. Taking (2.1) into account that, by the same reasoning as given in [2] we have $\Phi^{-1}(]-\infty,r_n[)\subseteq\{u\in X;||u||_\infty\leq\xi_n\}$. Hence, taking into account that $\Phi(0)=\Psi(0)=0$, for every k large enough, one has

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)}\frac{(\sup_{v\in\Phi^{-1}(]-\infty,r_n[)}\Psi(v))-\Psi(u)}{r_n-\Phi(u)}\\
&\leq \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)}\Psi(v)}{r_n}\\
&\leq \frac{\int_0^T\sup_{|t|\leq\xi_n}Q(x,t)dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n}=\frac{\int_0^T\sup_{|t|\leq\xi_n}\left[F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}G(x,t)\right]dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n}\\
&\leq \frac{\int_0^T\sup_{|t|\leq\xi_n}F(x,t)dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n}+\frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_0^T\sup_{|t|\leq\xi_n}G(x,t)dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n}.
\end{aligned}$$

Moreover, it follows from Assumption (A2) that

$$\liminf_{\xi\longrightarrow+\infty}\frac{\int_0^T\sup_{|t|\leq\xi}F(x,t)dx}{\frac{4}{T}C_1\xi^2-\frac{2}{\sqrt{T}}C_3\xi}<+\infty,$$

so we obtain

$$\lim_{n\to\infty}\frac{\int_0^T\sup_{|t|\leq\xi_n}F(x,t)dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n}<+\infty. \tag{3.1}$$

Then, in view of the condition (2.2) and (3.1), we have

$$\lim_{n\to\infty}\frac{\int_0^T\sup_{|t|\leq\xi_n}F(x,t)dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n}+\lim_{n\to\infty}\frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_0^T\sup_{|t|\leq\xi_n}G(x,t)dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n}<+\infty,$$

which follows

$$\lim_{n\to\infty}\frac{\int_0^T \sup_{|t|\le\xi_n}\left[F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}G(x,t)\right]dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n}<+\infty.$$

Therefore,

$$\gamma\le\liminf_{n\to+\infty}\varphi(r_n)\le\lim_{n\to\infty}\frac{\int_0^T \sup_{|t|\le\xi_n}\left[F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}G(x,t)\right]dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n}<+\infty. \tag{3.2}$$

Since

$$\frac{\int_0^T \sup_{|t|\le\xi_n}Q(x,t)dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n}\le\frac{\int_0^T \sup_{|t|\le\xi_n}F(x,t)dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n}+\frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_0^T \sup_{|t|\le\xi_n}G(x,t)dx}{\frac{4}{T}C_1\xi_n^2-\frac{2}{\sqrt{T}}C_3\xi_n},$$

taking (2.2) into account, one has

$$\liminf_{\xi\to+\infty}\frac{\int_0^T \sup_{|t|\le\xi}Q(x,t)dx}{\frac{4}{T}C_1\xi^2-\frac{2}{\sqrt{T}}C_3\xi}\le\liminf_{\xi\to+\infty}\frac{\int_0^T \sup_{|t|\le\xi}F(x,t)dx}{\frac{4}{T}C_1\xi^2-\frac{2}{\sqrt{T}}C_3\xi}+\frac{\overline{\mu}}{\overline{\lambda}}g_\infty. \tag{3.3}$$

Moreover, since G is nonnegative, from Assumption (A1) we obtain

$$\limsup_{\xi\to+\infty}\frac{\int_\eta^{T-\delta}Q(x,\xi)dx}{\frac{\eta+\delta}{\eta\delta}C_2\xi^2+\sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\xi}\ge\limsup_{\xi\to+\infty}\frac{\int_\eta^{T-\delta}F(x,\xi)dx}{\frac{\eta+\delta}{\eta\delta}C_2\xi^2+\sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\xi}. \tag{3.4}$$

Therefore, from (3.3) and (3.4), and from Assumption (A2) and (3.2) we observe

$$\overline{\lambda}\in]\nu_1,\nu_2[\subseteq$$

$$\left]\frac{1}{\limsup_{\xi\longrightarrow+\infty}\frac{\int_\eta^{T-\delta}Q(x,\xi)dx}{\frac{\eta+\delta}{\eta\delta}C_2\xi^2+\sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\xi}},\ \frac{1}{\liminf_{\xi\longrightarrow+\infty}\frac{\int_0^T\sup_{|t|\le\xi}Q(x,t)dx}{\frac{4}{T}C_1\xi^2-\frac{2}{\sqrt{T}}C_3\xi}}\right[\subseteq\left]0,\frac{1}{\gamma}\right[.$$

For the fixed $\overline{\lambda}$, the inequality (3.2) concludes that the condition (b) of Theorem 1 can be applied and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the problem (IP) such that $\lim_{n\to\infty}\|u\|=+\infty$.

The other step is to show that for the fixed $\overline{\lambda}$ the functional $I_{\overline{\lambda}}$ has no global minimum. Let us verify that the functional $I_{\overline{\lambda}}$ is unbounded from below. Assumption (A1) follows that C_2 and C_3 are positive. Since

$$\frac{1}{\overline{\lambda}} < \limsup_{\xi \to +\infty} \frac{\int_{\eta}^{T-\delta} F(x,\xi)dx}{\frac{\eta+\delta}{\eta\delta} C_2 \xi^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}} C_3 \xi},$$

we can consider a real sequence $\{d_n\}$ and a positive constant τ such that $d_n \to +\infty$ as $n \to \infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{\int_{\eta}^{T-\delta} F(x,d_n)dx}{\frac{\eta+\delta}{\eta\delta} C_2 d_n^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}} C_3 d_n} \tag{3.5}$$

for each $n \in \mathbb{N}$ large enough. Let $\{w_n\}$ be a sequence in X defined by

$$w_n(x) = \begin{cases} \frac{d_n}{\eta} x, & x \in [0,\eta], \\ d_n, & x \in [\eta, T-\delta], \\ \frac{d_n}{\delta}(T-x), & x \in [T-\delta, T]. \end{cases} \tag{3.6}$$

For any fixed $n \in \mathbb{N}$, $w_n \in X$ and $||w_n|| = \sqrt{\frac{\eta+\delta}{\eta\delta}} d_n$, and in particular, since $\Phi(u) \leq C_2||u||^2 + C_3||u||$ for all $u \in X$, one has

$$\Phi(w_n) \leq \frac{\eta+\delta}{\eta\delta} C_2 d_n^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}} C_3 d_n. \tag{3.7}$$

On the other hand, bearing Assumption (A1) in mind and since G is nonnegative, from the definition of Ψ, we infer

$$\Psi(w_n) \geq \int_{\eta}^{T-\delta} F(x,d_n)dx. \tag{3.8}$$

So, according to (3.5), (3.7) and (3.8) we obtain

$$I_{\overline{\lambda}}(w_n) \leq \frac{\eta+\delta}{\eta\delta} C_2 d_n^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}} C_3 d_n - \overline{\lambda} \int_{\eta}^{T-\delta} F(x,d_n)dx$$

$$< (1-\overline{\lambda}\tau) \left(\frac{\eta+\delta}{\eta\delta} C_2 d_n^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}} C_3 d_n \right)$$

for every $n \in \mathbb{N}$ large enough. Hence, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, applying Theorem 1 we deduce that there is a sequence $\{u_n\} \subset X$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$, which from the definition of Φ follows that $\lim_{n\to\infty} \|u_n\| = +\infty$. Hence, the conclusion is achieved. $\square$

Remark 1. *Under the conditions* $\liminf_{\xi\longrightarrow+\infty} \frac{\int_0^T \sup_{|t|\le\xi} F(x,t)dx}{\frac{4}{T}C_1\xi^2 - \frac{2}{\sqrt{T}}C_3\xi} = 0$ *and* $\limsup_{\xi\longrightarrow+\infty} \frac{\int_\eta^{T-\delta} F(x,\xi)dx}{\frac{\eta+\delta}{\eta\delta}C_2\xi^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3\xi} = +\infty$, *Theorem 3 concludes that for every* $\lambda > 0$ *and for each* $\mu \in [0, \frac{1}{g_\infty}[$ *the problem (IP) admits infinitely many weak solutions in* X. *Moreover, if* $g_\infty = 0$, *the result holds for every* $\lambda > 0$ *and* $\mu \ge 0$.

We now exhibit an example in which the hypotheses of Theorem 3 are satisfied.

Example 1. *Consider the problem*

$$\begin{cases} -u''(x) = \lambda f(x,u(x)) + \mu g(x,u(x)) & a.e.\ x \in [0,4], \\ u(0) = u(4) = 0, \\ \Delta u'(x_1) = \frac{1}{4} + \frac{3}{8}|u(x_1)|^{\frac{1}{2}}, \quad t_1 \in (0,4) \end{cases} \tag{3.9}$$

where

$$f(x,t) = \begin{cases} f^*(x)te^t\Big(2+t+\sin(\ln(|t|)) - (2+t)\cos(\ln(|t|))\Big) \\ \textit{if } (x,t) \in [0,4]\times(\mathbb{R}-\{0\}), \\ 0 \ \textit{if } (x,t) \in [0,4]\times\{0\} \end{cases}$$

where $f^* : [0,4] \to \mathbb{R}$ *is a non-negative continuous function, and* $g(x,t) = e^{x-t^+}(t^+)^{\gamma-1}(\gamma - t^+)$ *where* $t^+ = \max\{t,0\}$ *and* γ *is a positive real number, for all* $x \in [0,4]$ *and* $t \in \mathbb{R}$.
It is obvious that $C_1 = 0.25, C_2 = 0.75$ *and* $C_3 = 0.5$ *Also a direct calculation shows*

$$F(x,t) = \begin{cases} f^*(x)t^2e^t\Big(1-\cos(\ln(|t|))\Big) & \textit{if } (x,t) \in [0,4]\times(\mathbb{R}-\{0\}), \\ 0 & \textit{if } (x,t) \in [0,4]\times\{0\}. \end{cases}$$

So, $\liminf_{\xi\longrightarrow+\infty} \frac{\int_0^4 \sup_{|t|\le\xi} F(x,t)dx}{0.25\xi^2 - 0.5\xi} = 0$ *and* $\limsup_{\xi\longrightarrow+\infty} \frac{\int_1^3 F(x,\xi)dx}{1.5\xi^2 + \frac{\sqrt{2}}{2}\xi} = +\infty$. *Hence, using Theorem 3, since*

$$g_\infty = \lim_{\xi\to+\infty} \frac{\int_0^4 \sup_{|t|\le\xi} e^{x-t^+}(t^+)^\gamma dx}{C_1\xi^2 - C_3\xi} = \lim_{\xi\to+\infty} \frac{(e^4-1)e^{-\xi}\xi^\gamma}{0.25\xi^2 - 0.5\xi} = 0,$$

the problem (3.9) for every $(\lambda,\mu) \in]0,+\infty[\times[0,+\infty[$ *has an unbounded sequence of classical solutions in* $H_0^1(0,4)$.

Remark 2. *Arguing as in [5, Remark 3.3] we explicitly observe that Assumption (A2.2) in Theorem 3 could be replaced by the following more general condition*

(A2.3) there exist two sequence $\{a_n\}$ *and* $\{b_n\}$ *with* $b_n > \frac{\sqrt{T}C_3}{2C_1}$ *for all* $n \in \mathbb{N}$ *and* $\frac{\eta+\delta}{\eta\delta}C_2a_n^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3a_n < \frac{4}{T}C_1b_n^2 - \frac{2}{\sqrt{T}}C_3b_n$ *for every* $n \in \mathbb{N}$ *and* $\lim_{n\to+\infty} b_n = +\infty$ *such that*

$$\lim_{n\to+\infty} \frac{\int_0^T \sup_{|t|\le b_n} F(x,t)dx - \int_\eta^{T-\delta} F(x,a_n)dx}{\frac{4}{T}C_1b_n^2 - \frac{2}{\sqrt{T}}C_3b_n - \frac{\eta+\delta}{\eta\delta}C_2a_n^2 - \sqrt{\frac{\eta+\delta}{\eta\delta}}C_3a_n}$$

$$< \limsup_{\xi \longrightarrow +\infty} \frac{\int_{\eta}^{T-\delta} F(x,\xi)d\xi}{\dfrac{\eta+\delta}{\eta\delta} C_2 \xi^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}} C_3 \xi}.$$

Obviously, from (A2.3) we obtain (A2.2), by choosing $a_k = 0$ for all $n \in \mathbb{N}$. Moreover, if we assume (A2.3) instead of (A2.2) and set $r_n = \dfrac{4}{T} C_1 b_n^2 - \dfrac{2}{\sqrt{T}} C_3 b_n$ for all $n \in \mathbb{N}$, by the same arguing as inside in Theorem 3, we obtain

$$\begin{aligned} \varphi(r_n) &\leq \frac{\sup_{v \in \Phi^{-1}(]-\infty, r_n])} \Psi(v) - \int_0^T F(x, w_n(x))dx}{r_n - \frac{1}{2}\int_0^T (w_n'(x))^2 dx + \sum_{j=1}^p \int_0^{w_n(x_j)} I_j(t)dt} \\ &\leq \frac{\int_0^T \sup_{|t| \leq b_n} F(x,t)dx - \int_{\eta}^{T-\delta} F(x, a_n)dx}{\dfrac{4}{T} C_1 b_n^2 - \dfrac{2}{\sqrt{T}} C_3 b_n - \dfrac{\eta+\delta}{\eta\delta} C_2 a_n^2 - \sqrt{\frac{\eta+\delta}{\eta\delta}} C_3 a_n} \end{aligned}$$

where w_n as given in (3.6) with a_n instead of d_n. We have the same conclusion as in Theorem 3 with Λ replaced by

$$\Lambda' := \left] \frac{1}{\limsup_{\xi \longrightarrow +\infty} \frac{\int_{\eta}^{T-\delta} F(x,\xi)d\xi}{\frac{\eta+\delta}{\eta\delta} C_2 \xi^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}} C_3 \xi}}, \frac{1}{\lim_{n \to +\infty} \frac{\int_0^T \sup_{|t| \leq b_n} F(x,t)dx - \int_{\eta}^{T-\delta} F(x,a_n)dx}{\frac{4}{T} C_1 b_n^2 - \frac{2}{\sqrt{T}} C_3 b_n - \frac{\eta+\delta}{\eta\delta} C_2 a_n^2 - \sqrt{\frac{\eta+\delta}{\eta\delta}} C_3 a_n}} \right[$$

Here, we point out a simple consequence of Theorem 3.

Corollary 1. *Assume that all assumptions in Theorem 3 except Assumption (A2.2) hold and*

(B1) $\liminf_{\xi \longrightarrow +\infty} \frac{\int_0^T \sup_{|t| \leq \xi} F(x,t)dx}{\frac{4}{T} C_1 \xi^2 - \frac{2}{\sqrt{T}} C_3 \xi} < 1;$

(B2) $\limsup_{\xi \longrightarrow +\infty} \frac{\int_{\eta}^{T-\delta} F(x,\xi)dx}{\frac{\eta+\delta}{\eta\delta} C_2 \xi^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}} C_3 \xi} > 1.$

Then, for every arbitrary L^1-Carathéodory function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ whose $G(x,t) = \int_0^t g(x,\xi)d\xi$ for every $(x,t) \in [0,T] \times \mathbb{R}$, is a non-negative function satisfying the condition (2.2) *and for every $\mu \in [0, \mu_{g,1}[$ where*

$$\mu_{g,1} := \frac{1}{g_\infty} \left(1 - \liminf_{\xi \to +\infty} \frac{\int_0^T \sup_{|t| \leq \xi} F(x,t)dx}{\dfrac{4}{T} C_1 \xi^2 - \dfrac{2}{\sqrt{T}} C_3 \xi} \right),$$

the problem

$$\begin{cases} -u''(x) = f(x, u(x)) + \mu g(x, u(x)) & a.e. \ x \in [0,T], \\ u(0) = u(T) = 0 \end{cases}$$

with the impulsive conditions $\Delta u'(x_j) = I_j(u(x_j))$, $j = 1, 2, ..., p$, has an unbounded sequence of weak solutions in X.

Remark 3. *Theorem 2 is an immediately consequence of Corollary 1.*

The following result is special case of Theorem 3 with $\mu = 0$.

Theorem 4. *Assume that the assumptions in Theorem 3 hold. Then, for each $\lambda \in]\lambda_1, \lambda_2[$ where*

$$\lambda_1 := \frac{1}{\limsup_{\xi \longrightarrow +\infty} \frac{\int_\eta^{T-\delta} F(x,\xi)dx}{\frac{\eta+\delta}{\eta\delta} C_2 \xi^2 + \sqrt{\frac{\eta+\delta}{\eta\delta}} C_3 \xi}}$$

and

$$\lambda_2 := \frac{1}{\liminf_{\xi \longrightarrow +\infty} \frac{\int_0^T \sup_{|t| \le \xi} F(x,t)dx}{\frac{4}{T} C_1 \xi^2 - \frac{2}{\sqrt{T}} C_3 \xi}},$$

the problem

$$\begin{cases} -u''(x) = \lambda f(x, u(x)) & a.e.\ x \in [0, T], \\ u(0) = u(T) = 0 \end{cases}$$

with the impulsive conditions $\Delta u'(x_j) = I_j(u(x_j))$, $j = 1, 2, ..., p$, has an unbounded sequence of weak solutions in X.

Remark 4. *We observe in Theorem 3 we can replace $\xi \to +\infty$ with $\xi \to 0^+$, that by the same arguing as in the proof of Theorem 3 but using conclusion (c) of Theorem 1 instead of (b), the problem (IP) has a sequence of pairwise distinct weak solutions, which strongly converges to 0 in X.*

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References

[1] R.P. Agarwal, D. O'Regan, Multiple nonnegative solutions for second order impulsive differential equations, *Appl. Math. Comput.* **114** (2000), 51–59.

[2] L. Bai, B. Dai, An application of variational method to a class of Dirichlet boundary value problems with impulsive effects, *J. Franklin Inst.* **348** (2011), 2607–2624.

[3] L. Bai, B. Dai, Existence and multiplicity of solutions for impulsive boundary value problem with a parameter via critical point theory, *Math. Comput. Modelling* **53** (2011), 1844–1855.

[4] L. Bai, B. Dai, Three solutions for a p-Laplacian boundary value problem with impulsive effects, *Appl. Math. Comput.* **217** (2011), 9895–9904.

[5] G. Bonanno, B. Di Bella, Infinitely many solutions for a fourth-order elastic beam equation, *Nonl. Diff. Eq. Appl. NoDEA* **18** (2011), 357–368.

[6] G. Bonanno, G. Molica Bisci, A remark on perturbed elliptic Neumann problems, *Studia Univ. "Babeş-Bolyai", Mathematica*, Volume **LV**, Number 4, December 2010.

[7] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.* **2009**, (2009) 1–20.

[8] G. Bonanno, G. Molica Bisci, V. Rădulescu, Arbitrarily small weak solutions for a nonlinear eigenvalue problem in Orlicz-Sobolev spaces, *Monatsh. Math.* **165** (3-4), (2012), 305–318.

[9] G. Bonanno, G. Molica Bisci, V. Rădulescu, Infinitely many solutions for a class of nonlinear eigenvalue problems in Orlicz-Sobolev spaces, *C.R. Acad. Sci. Paris*, Ser. I **349** (2011), 263–268.

[10] G. Bonanno, G. Molica Bisci, V. Rădulescu, Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz-Sobolev spaces, *Nonlinear Anal. TMA* **75** (2012), 4441–4456.

[11] T.E. Carter, Necessary and sufficient conditions for optimal impulsive rendezvous with linear equations of motion, *Dyn. Control* **10** (2000), 219–227.

[12] S. Heidarkhani, Infinitely many solutions for systems of n two-point boundary value Kirchhoff-type problems, *Annales Polonici Mathematici* **107.2** (2013), 133–152.

[13] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of impulsive differential equations, Series in Modern Applied Mathematics,* vol.6, World Scientific, Teaneck, NJ, 1989.

[14] X.N. Lin, D.Q. Jiang, Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equations, *J. Math. Anal. Appl.* **321** (2006), 501–514.

[15] S. A. Marano, D. Motreanu, Infinitely many critical points of non-differentiable functions and applications to a Neumann type problem involving the p-Laplacian, *J. Diff. Eq.* **182** (2002), 108–120.

[16] S. Pasquero, On the simultaneous presence of unilateral and kinetic constraints in time-dependent impulsive mechanics, *J. Math. Phys.* **47** (2006), 082903, 19 pp.

[17] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.* **113** (2000), 401–410.

[18] A.M. Samoilenko, N.A. Perestyuk, *Impulsive differential equations*, World Scientific, Singapore, 1995.

[19] Y. Tian, W. Ge, Variational methods to Sturm-Liouville boundary value problem for impulsive differential equations, *Nonlinear Anal. TMA* **72** (2010), 277–287.

[20] J. Xiao, J.J. Nieto, Variational approach to some damped Dirichlet nonlinear impulsive differential equations, *J. Franklin Inst.* **348** (2011), 369–377.

[21] D. ZHANG, B. DAI, Existence of solutions for nonlinear impulsive differential equations with Dirichlet boundary conditions, *Math. Comput. Modelling* **53** (2011), 1154–1161.

[22] Z. ZHANG, R. YUAN, An application of variational methods to Dirichlet boundary value problem with impulses, *Nonlinear Anal. RWA* **11** (2010), 155–162.

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