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Homogenization of a thermoelasticity model for a composite with imperfect interface

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Abstract

The goal of this paper is to obtain a macroscopic model for an ε-periodic thermoelastic composite material formed by two components with imperfect contact at the interface. We consider that on the interface between the two materials the tractions and the temperature fluxes are continuous, but both the temperature and the displacement fields have a jump, proportional to the temperature flux and, respectively, to the normal component of the stress tensor. Under suitable hypotheses on the order of magnitude with respect to ε of the elasticity tensors and of the temperature-displacement tensors in the two components of the medium, we derive, via the periodic unfolding method, the homogenized problem, which contains new coupling terms between the limits of the displacements and, respectively, the temperatures from the two components of the composite material.

Key Words: Homogenization, thermoelasticity, transmission, imperfect interface, periodic unfolding method.
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1 Introduction

This paper deals with the homogenization of a transmission problem in a two-component thermoelastic composite material having an ε-periodic structure. More precisely, we assume that the domain Ω occupied by the thermoelastic composite material is the union of a connected set Ω_1^ε and a disconnected one, Ω_2^ε, consisting of ε-periodic connected sets of size ε. The two components of the composite are separated by a thin layer, modeled here as a surface Γ_ε. We consider that both the displacements and the temperatures have jumps of order ε on the interface Γ_ε. The jump of the displacements is proportional to the normal component of the stress tensor and the jump of the temperature fields is proportional to the temperature flux across Γ_ε. Moreover, we assume that the tractions and the temperature fluxes are continuous across Γ_ε. We consider that the elasticity tensors of the two materials are both of the order of unity, while the temperature-displacement tensor is assumed to be of order one in the connected component of the composite medium and of order ε in the disconnected one.

Under suitable hypotheses on the data, we derive, via the periodic unfolding method, the homogenized problem, which contains new coupling terms between the limits of the displacements and, respectively, of the temperatures from the two components of the medium.

Similar problems to the one we treat here have been addressed, using various methods, in the literature. For the general theory of thermoelasticity, we refer to [12]. For a linear thermoelasticity model, obtained using semigroups theory, we refer to Francfort [11]. In [8], Ene and Paşa study a thermoelasticity model, but without jumps, and they obtain their homogenization results using asymptotic expansions. For other thermoelasticity models, the interested reader is referred to [1]. For transmission problems in composites with imperfect interfaces, see [5], [6], [10], [14] and [15].

As already mentioned, our approach is based on the use of the periodic unfolding method. This method was introduced by Cioranescu, Damlamian and Griso in [2] for the case of fixed domains (see [3] for a general presentation) and, later, it was extended to periodically perforated domains by Cioranescu, Damlamian, Donato, Griso and Zaki in [4]. In [5], Donato *et al.* use the periodic unfolding method for a two-component domain similar to the one considered in this paper and in [7] Donato and Yang introduce a time depending unfolding operator for a wave equation in domains with isolated holes. Also, in [16], Yang defines two time depending unfolding operators for a domain with a similar geometry to the one we consider here and uses them in the homogenization process of a linear hyperbolic problem in a medium with imperfect interfaces.

The structure of the paper is as follows: in Section 2, we formulate the microscopic problem and in Section 3 we prove the existence and uniqueness of a weak solution for this problem. Proper functional spaces are introduced and suitable estimates of the weak solution are obtained. Using the periodic unfolding method, we prove some convergence results in Section 4 and we obtain the homogenized problem.

2 The thermoelasticity problem

Let Ω be an open bounded subset of $\mathbb{R}^N$ ($N \geqslant 2$), with a Lipschitz continuous boundary $\partial\Omega$ and let $Y = (0,1)^N$ be the unit cube in $\mathbb{R}^N$. We suppose that Y_2 is an open connected subset of Y such that $\bar{Y}_2 \subset Y$ and its boundary Γ is Lipschitz continuous. We set $Y_1 = Y \setminus \bar{Y}_2$. One can see that, repeating Y by periodicity, the union of all $\bar{Y}_1$ is a connected set in $\mathbb{R}^N$, which will be denoted by $\mathbb{R}^N_1$. Furthermore, let $\mathbb{R}^N_2 = \mathbb{R}^N \setminus \mathbb{R}^N_1$.

In what follows, the small parameter $\varepsilon \in (0,1)$ represents the characteristic dimension of the periodicity cell and it takes its values in a sequence of real numbers which, in the homogenizing process, will tend to zero. For each $k \in \mathbb{Z}^N$, we define $Y^k = k + Y$ and $Y^k_\alpha = k + Y_\alpha$, where $\alpha \in \{1,2\}$. We also define, for each ε,

$$\mathbb{Z}_\varepsilon = \left\{ k \in \mathbb{Z}^N : \ \ \varepsilon \bar{Y}^k_2 \subset \Omega \right\} \tag{2.1}$$

and we set

$$\Omega^\varepsilon_2 = \bigcup_{k \in \mathbb{Z}_\varepsilon} \left(\varepsilon Y^k_2 \right), \ \ \Omega^\varepsilon_1 = \Omega \setminus \bar{\Omega}^\varepsilon_2. \tag{2.2}$$

The boundary of Ω^ε_2 will be denoted by Γ_ε and n will be the unit normal on Γ_ε, exterior to Ω^ε_1.

We introduce now the fourth order tensors $A^{1\varepsilon}$ and $A^{2\varepsilon}$ which represent the elasticities of the two components Ω_1^ε and, respectively, Ω_2^ε. We consider that for $\alpha \in \{1,2\}$

$$A^{\alpha\varepsilon}(x) = A^\alpha(x/\varepsilon),$$

where A^α are symmetric and positive definite tensors, of components $a_{ijkh}^\alpha \in L^\infty(Y)$. We assume that a_{ijkh}^α are real, smooth and Y-periodic functions.

We also introduce the second order temperature-displacement tensors

$$B^{1\varepsilon}(x) = B^1(x/\varepsilon), \quad B^{2\varepsilon}(x) = \varepsilon B^2(x/\varepsilon)$$

and the thermic-conductivity tensors $K^{\alpha\varepsilon}(x) = K^\alpha(x/\varepsilon)$, where B^α and K^α are symmetric, with the components $b_{ij}^\alpha, k_{ij}^\alpha \in L^\infty(Y)$ being smooth and Y-periodic functions. Moreover, we consider that K^α are positive definite.

Furthermore, T_0 denotes the reference temperature, $\rho^{\alpha\varepsilon}$ are the densities of the two media, defined by $\rho^{\alpha\varepsilon}(x) = \rho^\alpha(x/\varepsilon)$, and $c^{\alpha\varepsilon}(x) = c^\alpha(x/\varepsilon)$ are the specific heats for constant deformation of each of the two media. We also introduce two jump factors $h_\varepsilon^u(x) = h^u(x/\varepsilon)$ and $h_\varepsilon^\theta(x) = h^\theta(x/\varepsilon)$ and we assume that the functions $\rho^\alpha, c^\alpha, h^u, h^\theta \in L^\infty(Y)$ are smooth, Y-periodic and strictly positive.

Finally, for $\alpha \in \{1,2\}$ and $u^{\alpha\varepsilon}$ and $\theta^{\alpha\varepsilon}$ defined on $\Omega_\alpha^\varepsilon$, the constitutive laws are given by $\sigma_{ij}^{\alpha\varepsilon} = a_{ijkh}^{\alpha\varepsilon} e_{kh}(u^{\alpha\varepsilon}) - b_{ij}^{\alpha\varepsilon}\theta^{\alpha\varepsilon}$, where $e_{kh}(u^{\alpha\varepsilon}) = \frac{1}{2}\left(\frac{\partial u_k^{\alpha\varepsilon}}{\partial x_h} + \frac{\partial u_h^{\alpha\varepsilon}}{\partial x_k}\right)$ represent the components of the deformation tensor.

Let $T > 0$ be a real number. In what follows, we shall use the notation $\Omega_T = (0,T)\times\Omega$, $\Omega_{T\alpha}^\varepsilon = (0,T)\times\Omega_\alpha^\varepsilon$ and $\Gamma_\varepsilon^T = (0,T)\times\Gamma_\varepsilon$. Our aim is to study the asymptotic behavior of the solution of the problem

$$-\frac{\partial \sigma_{ij}^{\alpha\varepsilon}}{\partial x_j} + \rho^{\alpha\varepsilon}\frac{\partial^2 u_i^{\alpha\varepsilon}}{\partial t^2} = f_i \quad \text{in } \Omega_{T\alpha}^\varepsilon, \tag{2.3}$$

$$-\frac{\partial}{\partial x_i}\left(k_{ij}^{\alpha\varepsilon}\frac{\partial \theta^{\alpha\varepsilon}}{\partial x_j}\right) + T_0 b_{ij}^{\alpha\varepsilon}\frac{\partial e_{ij}(u^{\alpha\varepsilon})}{\partial t} + c^{\alpha\varepsilon}\frac{\partial \theta^{\alpha\varepsilon}}{\partial t} = r \quad \text{in } \Omega_{T\alpha}^\varepsilon, \tag{2.4}$$

$$\sigma_{ij}^{1\varepsilon} n_j = \sigma_{ij}^{2\varepsilon} n_j \quad \text{on } \Gamma_\varepsilon^T, \tag{2.5}$$

$$k_{ij}^{1\varepsilon}\frac{\partial \theta^{1\varepsilon}}{\partial x_j} n_i = k_{ij}^{2\varepsilon}\frac{\partial \theta^{2\varepsilon}}{\partial x_j} n_i \quad \text{on } \Gamma_\varepsilon^T, \tag{2.6}$$

$$\sigma_{ij}^{1\varepsilon} n_j = \varepsilon h_\varepsilon^u (u_i^{2\varepsilon} - u_i^{1\varepsilon}) \quad \text{on } \Gamma_\varepsilon^T, \tag{2.7}$$

$$k_{ij}^{1\varepsilon}\frac{\partial \theta^{1\varepsilon}}{\partial x_j} n_i = \varepsilon h_\varepsilon^\theta (\theta^{2\varepsilon} - \theta^{1\varepsilon}) \quad \text{on } \Gamma_\varepsilon^T, \tag{2.8}$$

$$u^{1\varepsilon} = 0, \quad \theta^{1\varepsilon} = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{2.9}$$

$$u^{\alpha\varepsilon}(0,x) = 0, \quad \dot{u}^{\alpha\varepsilon}(0,x) = 0, \quad \theta^{\alpha\varepsilon}(0,x) = 0, \tag{2.10}$$

where f_i are the components of a vector field $f \in L^2(\Omega)^N$, which represents the forces, and $r \in L^2(\Omega)$ is the exterior energy source.

3 Variational formulation and estimates

We introduce the spaces

$$V_{1\varepsilon} = \left\{ v \in C^{\infty}(0,T;H^1(\Omega_1^{\varepsilon}))),\ v=0 \text{ on } \partial\Omega \text{ and } v=0 \text{ on } \{0\}\times\Omega \right\},$$

$$V_{2\varepsilon} = \left\{ v \in C^{\infty}(0,T;H^1(\Omega_2^{\varepsilon}))),\ v=0 \text{ on } \{0\}\times\Omega \right\}$$

and

$$W_{\varepsilon} = \left(V_{1\varepsilon}^N \times V_{2\varepsilon}^N \right) \times \left(V_{1\varepsilon} \times V_{2\varepsilon} \right). \tag{3.1}$$

An element of W_{ε} will be denoted by $V=(v,w)$, with $v=(v^1,v^2)\in V_{1\varepsilon}^N \times V_{2\varepsilon}^N$ and $w=(w^1,w^2)\in V_{1\varepsilon}\times V_{2\varepsilon}$.

In order to obtain the variational formulation of problem (2.3)-(2.10), we choose $V=(v,w)\in W_{\varepsilon}$ with $v^{\alpha}(0,x)=0$, $\forall x\in\Omega_{\alpha}^{\varepsilon}$ and we multiply the equations (2.3) with $(T-t)\dot{v}_i^{\alpha}$ and the equations (2.4) with $(T-t)\frac{w^{\alpha}}{T_0}$. Adding the obtained relations and integrating by parts, we get the variational formulation of problem (2.3)-(2.10):

Find $U^{\varepsilon}=(u^{\varepsilon},\theta^{\varepsilon})\in W_{\varepsilon}$ such that

$$\mathcal{L}_{\varepsilon}(U^{\varepsilon},V) = \mathcal{D}_{\varepsilon}\left((f,r),V\right),\ \forall V=(v,w)\in W_{\varepsilon}, \tag{3.2}$$

where, for each ε, $\mathcal{L}_{\varepsilon}: W_{\varepsilon}\times W_{\varepsilon}\to\mathbb{R}$ is a bilinear form defined by

$$\begin{aligned}
\mathcal{L}_{\varepsilon}(U,V) = \quad & \sum_{\alpha=1,2}\Bigg[\int_0^T\!\!\int_{\Omega_{\alpha}^{\varepsilon}}(t-T)\Big(\left(-a_{ijkh}^{\alpha\varepsilon}e_{kh}(u^{\alpha})+b_{ij}^{\alpha\varepsilon}\theta^{\alpha}\right)e_{ij}(\dot{v}^{\alpha})+ \\
& +\rho^{\alpha\varepsilon}\dot{u}_i^{\alpha}\ddot{v}_i^{\alpha}++b_{ij}^{\alpha\varepsilon}e_{ij}(u^{\alpha})\dot{w}^{\alpha}+\frac{1}{T_0}c^{\alpha\varepsilon}\theta^{\alpha}\dot{w}^{\alpha}\Big)+\rho^{\alpha\varepsilon}\dot{u}_i^{\alpha}\dot{v}_i^{\alpha}+ \\
& +b_{ij}^{\alpha\varepsilon}e_{ij}(u^{\alpha})w^{\alpha}+\frac{1}{T_0}c^{\alpha\varepsilon}\theta^{\alpha}w^{\alpha}+\frac{1}{T_0}\int_0^t k_{ij}^{\alpha\varepsilon}\frac{\partial\theta^{\alpha}}{\partial x_j}\frac{\partial w^{\alpha}}{\partial x_i}\,\mathrm{d}s\Bigg]- \\
& -\varepsilon\int_0^T\!\!\int_{\Gamma_{\varepsilon}}(t-T)h_{\varepsilon}^{u}(u_i^2-u_i^1)(\dot{v}_i^2-\dot{v}_i^1)- \\
& -\frac{\varepsilon}{T_0}\int_0^T\!\!\int_{\Gamma_{\varepsilon}}(t-T)h_{\varepsilon}^{\theta}(\theta^2-\theta^1)(w^2-w^1),
\end{aligned} \tag{3.3}$$

with $U=(u,\theta)$ and $V=(v,w)$, and $\mathcal{D}_{\varepsilon}:\left(L^2(\Omega)^N\times L^2(\Omega)\right)\times W_{\varepsilon}\to\mathbb{R}$ is defined by

$$\mathcal{D}_{\varepsilon}\left((f,r),V\right) = -\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega_{\alpha}^{\varepsilon}}(t-T)\Big(f_i\dot{v}_i^{\alpha}+\frac{1}{T_0}rw^{\alpha}\Big). \tag{3.4}$$

First, we observe that, for any $V = (v, w) \in W_\varepsilon$, we have

$$\mathcal{L}_\varepsilon(V,V) = \frac{1}{2}\sum_{\alpha=1,2}\left[\int_0^T\!\!\int_{\Omega_\alpha^\varepsilon} a_{ijkh}^{\alpha\varepsilon} e_{kh}(v^\alpha)e_{ij}(v^\alpha) + \rho^{\alpha\varepsilon}\dot{v}_i^\alpha\dot{v}_i^\alpha + \right. \\ \left. + \frac{1}{T_0}c^{\alpha\varepsilon}w^\alpha w^\alpha + \frac{2}{T_0}\int_0^t k_{ij}^{\alpha\varepsilon}\frac{\partial\theta^\alpha}{\partial x_j}\frac{\partial w^\alpha}{\partial x_i}\,\mathrm{d}s\right] + \\ + \frac{\varepsilon}{2}\int_0^T\!\!\int_{\Gamma_\varepsilon} h_\varepsilon^u(v_i^2 - v_i^1)^2 + \frac{\varepsilon}{T_0}\int_0^T\!\!\int_{\Gamma_\varepsilon}\int_0^t h_\varepsilon^\theta(w^2 - w^1)^2\,\mathrm{d}s. \tag{3.5}$$

We introduce now the Hilbert space $\mathcal{W}_\varepsilon$, which is the completion of W_ε in the norm $\|\cdot\|$ generated by the scalar product

$$(U,V)_{W_\varepsilon} = \sum_{\alpha=1,2}\left[\int_0^T\!\!\int_{\Omega_\alpha^\varepsilon} \dot{u}_i^\alpha\dot{v}_i^\alpha + e_{ij}(u^\alpha)e_{ij}(v^\alpha) + \theta^\alpha w^\alpha + \right. \\ \left. + \int_0^t \frac{\partial\theta^\alpha}{\partial x_i}\frac{\partial w^\alpha}{\partial x_i}\,\mathrm{d}s\right] + \varepsilon\int_0^T\!\!\int_{\Gamma_\varepsilon}(u_i^2 - u_i^1)(v_i^2 - v_i^1) + \\ + \varepsilon\int_0^T\!\!\int_{\Gamma_\varepsilon}\int_0^t(\theta^2 - \theta^1)(w^2 - w^1)\,\mathrm{d}s. \tag{3.6}$$

Remark 1. *According to Ieşan [12], using Schwarz inequality and Sobolev's embedding theorem [13], one can see that $\mathcal{L}_\varepsilon$ can be extended by continuity to the entire space $\mathcal{W}_\varepsilon \times \mathcal{W}_\varepsilon$ and $\mathcal{D}_\varepsilon$ can also be extended to $\big(L^2(\Omega)^N \times L^2(\Omega)\big) \times \mathcal{W}_\varepsilon$.*

Using now the coercivity of A^α and K^α and the positivity of ρ^α, c^α, h^u, h^θ, it follows immediately that there exists a constant $C > 0$, independent of ε, such that

$$\|V\|^2 \leqslant C\mathcal{L}_\varepsilon(V,V). \tag{3.7}$$

Theorem 1. *The problem* (3.2) *has a unique solution. Moreover, there exists a constant $C > 0$, independent of ε, such that:*

$$\|u_i^{\varepsilon\alpha}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \tag{3.8}$$

$$\|\dot{u}_i^{\varepsilon\alpha}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \quad \|\nabla u_i^{\alpha\varepsilon}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \tag{3.9}$$

$$\|\theta^{\varepsilon\alpha}\|_{L^2(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \quad \left\|\int_0^t (\nabla\theta^{\varepsilon\alpha})^2\right\|_{L^1(\Omega_{T\alpha}^\varepsilon)} \leqslant C, \tag{3.10}$$

$$\|u_i^{2\varepsilon} - u_i^{1\varepsilon}\|_{L^2(\Gamma_\varepsilon^T)} \leqslant C\varepsilon^{-1/2}, \quad \left\|\int_0^t \big(\theta^{2\varepsilon} - \theta^{1\varepsilon}\big)^2\right\|_{L^1(\Gamma_\varepsilon^T)} \leqslant C\varepsilon^{-1/2}. \tag{3.11}$$

Proof: Let $S : \mathcal{W}_\varepsilon \to \mathcal{W}_\varepsilon$ be the correspondence that associates to each $V \in \mathcal{W}_\varepsilon$ the unique $V_0 \in \mathcal{W}_\varepsilon$ such that

$$\mathcal{L}_\varepsilon(U,V) = (U, S(V))_{\mathcal{W}_\varepsilon}, \quad \forall U \in \mathcal{W}_\varepsilon. \tag{3.12}$$

The mapping S is bijective from $\mathcal{W}_\varepsilon$ to its range $R(S)$ and, since $R(S)$ is dense in $\mathcal{W}_\varepsilon$, we can extend S^{-1} to the entire $\mathcal{W}_\varepsilon$. Thus, $S^{-1} : \mathcal{W}_\varepsilon \longrightarrow \mathcal{W}_\varepsilon$ is a bounded operator. One can see that $\mathcal{D}_\varepsilon$ is also bounded in $\mathcal{W}_\varepsilon$. From Riesz-Fréchet theorem, there exists $U^\varepsilon \in \mathcal{W}_\varepsilon$ (which obviously depends on ε, f and r) such that

$$\mathcal{D}_\varepsilon\left((f,r), S^{-1}(\Psi)\right) = (U^\varepsilon, \Psi)_{\mathcal{W}_\varepsilon}, \quad \forall \Psi \in \mathcal{W}_\varepsilon.$$

In particular, for every $V \in \mathcal{W}_\varepsilon$, as $S(V) \in \mathcal{W}_\varepsilon$, we have

$$\mathcal{D}_\varepsilon\left((f,r), V\right) = (U^\varepsilon, S(V))_{\mathcal{W}_\varepsilon},$$

and taking into account (3.12), the existence of a solution is proven. Since $\mathcal{L}_\varepsilon$ is linear, the uniqueness can be easily proven by taking $U^\varepsilon = V = U_1^\varepsilon - U_2^\varepsilon$ in (3.2) and using inequality (3.7).

Now, from (3.12) and (3.2),

$$\|U^\varepsilon\|^2 = (U^\varepsilon, U^\varepsilon)_{\mathcal{W}_\varepsilon} = \mathcal{L}_\varepsilon(U^\varepsilon, S^{-1}(U^\varepsilon)) = \mathcal{D}_\varepsilon\left((f,r), S^{-1}(U^\varepsilon)\right) \leqslant$$

$$\leqslant C\|U^\varepsilon\| \cdot \|(f,r)\|_{L^2(\Omega)^N \times L^2(\Omega)},$$

and taking into account the definition of the norm in $\mathcal{W}_\varepsilon$, we can see that the estimates (3.9)-(3.11) hold true.

It remains to prove (3.8). It is known (see, for example, [9]) that there exists a constant $C > 0$, independent of ε, such that for any $v = (v^1, v^2) \in V_{1\varepsilon}^N \times V_{2\varepsilon}^N$, we have

$$\|v_i^2\|_{L^2(\Omega_2^\varepsilon)} \leqslant C\left(\varepsilon\|\nabla v_i^2\|_{L^2(\Omega_2^\varepsilon)} + \varepsilon^{1/2}\|v_i^2\|_{L^2(\Gamma_\varepsilon)}\right), \tag{3.13}$$

$$\varepsilon^{1/2}\|v_i^1\|_{L^2(\Gamma_\varepsilon)} \leqslant C\left(\varepsilon\|\nabla v_i^1\|_{L^2(\Omega_1^\varepsilon)} + \|v_i^1\|_{L^2(\Omega_1^\varepsilon)}\right), \tag{3.14}$$

$$\|v_i^1\|_{L^2(\Omega_1^\varepsilon)} \leqslant C\|\nabla v_i^1\|_{L^2(\Omega_1^\varepsilon)}. \tag{3.15}$$

From (3.15) and (3.9), we conclude that

$$\|u_i^{1\varepsilon}\|_{L^2(\Omega_{T1}^\varepsilon)} \leqslant C\|\nabla u_i^{1\varepsilon}\|_{L^2(\Omega_{T1}^\varepsilon)} \leqslant C. \tag{3.16}$$

In a similar manner, from (3.13) and (3.14), we get

$$\|u_i^{2\varepsilon}\|_{L^2(\Omega_{T2}^\varepsilon)} \leqslant C\left(\varepsilon\|\nabla u_i^{2\varepsilon}\|_{L^2(\Omega_{T2}^\varepsilon)} + \varepsilon^{1/2}\|u_i^{2\varepsilon} - u_i^{1\varepsilon}\|_{L^2(\Gamma_\varepsilon^T)} + \varepsilon^{1/2}\|u_i^{1\varepsilon}\|_{L^2(\Gamma_\varepsilon^T)}\right) \leqslant$$

$$\leqslant C\left(\varepsilon\|\nabla u_i^{2\varepsilon}\|_{L^2(\Omega_{T2}^\varepsilon)} + \varepsilon^{1/2}\|u_i^{2\varepsilon} - u_i^{1\varepsilon}\|_{L^2(\Gamma_\varepsilon^T)} + \varepsilon\|\nabla u_i^{1\varepsilon}\|_{L^2(\Omega_{T1}^\varepsilon)} + \|u_i^{2\varepsilon}\|_{L^2(\Omega_{T2}^\varepsilon)}\right).$$

Taking now into account (3.9), (3.11) and (3.16), one can see that

$$\|u_i^{2\varepsilon}\|_{L^2(\Omega_{T2}^\varepsilon)} \leqslant C.$$

□

4 The homogenization process

In this section, we shall use the notation:

$$W_1 = L^\infty(0,T;H^2(\Omega)\cap H^1_0(\Omega))^N \cap W^{1,\infty}(0,T;H^1(\Omega))^N \cap W^{2,\infty}(0,T;L^2(\Omega))^N,$$

$$W_2 = L^\infty(0,T;H^2(\Omega)\cap H^1_0(\Omega)) \cap W^{1,\infty}(0,T;H^1(\Omega)),$$

$$\begin{aligned} W = \; & W_1 \times L^\infty(0,T;L^2(\Omega;H^1_{per}(Y_1))^N \times W^{2,\infty}(0,T;L^2(\Omega))^N \times W_2 \\ & \times L^\infty(0,T;L^2(\Omega;H^1_{per}(Y_1)) \times W^{1,\infty}(0,T;L^2(\Omega)). \end{aligned}$$

For proving our main convergence result, we shall use the unfolding operators $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$ defined in [16]. These two operators transform functions defined on the oscillating domains $[0,T]\times\Omega_1^\varepsilon$ and $[0,T]\times\Omega_2^\varepsilon$ into functions defined on the fixed domains $[0,T]\times\Omega\times Y_1$ and, respectively, $[0,T]\times\Omega\times Y_2$ and, as a consequence, we can avoid the use of extension operators.

Theorem 2. *If $(u^\varepsilon,\theta^\varepsilon)\in\mathcal{W}_\varepsilon$ is the solution of problem* (2.3)-(2.10)*, with $u^\varepsilon=(u^{1\varepsilon},u^{2\varepsilon})$ and $\theta^\varepsilon=(\theta^{1\varepsilon},\theta^{2\varepsilon})$, then*

$$\begin{aligned}
&\tilde{u}^{\alpha\varepsilon} \overset{*}{\rightharpoonup} |Y_\alpha| u^\alpha \;\; \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega))^N, \\
&\tilde{\theta}^{\alpha\varepsilon} \overset{*}{\rightharpoonup} |Y_\alpha| \theta^\alpha \;\; \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega)), \\
&\mathcal{T}_\alpha^\varepsilon(u^{\alpha\varepsilon}) \overset{*}{\rightharpoonup} u^\alpha \;\; \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega;H^1(Y_\alpha)))^N, \\
&\mathcal{T}_1^\varepsilon(e_{kh}(u^{1\varepsilon})) \overset{*}{\rightharpoonup} e_{kh}(u^1)+e^y_{kh}(\widehat{u}^1) \;\; \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega\times Y_1)), \\
&\mathcal{T}_2^\varepsilon(e_{kh}(u^{2\varepsilon})) \overset{*}{\rightharpoonup} 0 \;\; \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega\times Y_2)), \\
&\mathcal{T}_\alpha^\varepsilon(\theta^{\alpha\varepsilon}) \overset{*}{\rightharpoonup} \theta^\alpha \;\; \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega;H^1(Y_\alpha))), \\
&\mathcal{T}_1^\varepsilon(\nabla\theta^{1\varepsilon}) \overset{*}{\rightharpoonup} \nabla\theta^1+\nabla_y\widehat{\theta}^1 \;\; \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega\times Y_1)), \\
&\mathcal{T}_2^\varepsilon(\nabla\theta^{2\varepsilon}) \overset{*}{\rightharpoonup} 0 \;\; \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega\times Y_2)),
\end{aligned} \tag{4.1}$$

where $(u^1,\widehat{u}^1,u^2,\theta^1,\widehat{\theta}^1,\theta^2)\in W$ is the unique solution of the problem

$$\begin{aligned}
&\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)\left[a^1_{ijkh}\Big(e_{kh}(u^1)+e^y_{kh}(\widehat{u}^1)\Big)-b^1_{ij}\theta^1\right]\Big(\dot{e}_{ij}(\varphi^1)+\dot{e}^y_{ij}(\Phi^1)\Big)+ \\
&\quad+\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)\Big[\rho^\alpha\ddot{u}^\alpha_i\dot{\varphi}^\alpha_i+\frac{1}{T_0}c^\alpha\dot{\theta}^\alpha q^\alpha\Big]+ \\
&\quad+\frac{1}{T_0}\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)k^1_{ij}\Big(\frac{\partial\theta^1}{\partial x_j}+\frac{\partial\widehat{\theta}^1}{\partial y_j}\Big)\Big(\frac{\partial q^1}{\partial x_i}+\frac{\partial Q^1}{\partial y_i}\Big)+ \\
&\quad+\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)b^1_{ij}\Big(\dot{e}_{ij}(u^1)+\dot{e}^y_{ij}(\widehat{u}^1)\Big)q^1+
\end{aligned} \tag{4.2}$$

$$+\int_0^T\int_{\Omega\times\Gamma}(t-T)\Big[h^u(u_i^2-u_i^1)(\dot\varphi_i^2-\dot\varphi_i^1)+\frac{1}{T_0}h^\theta(\theta^2-\theta^1)(q^2-q^1)\Big]=$$

$$=\sum_{\alpha=1,2}\int_0^T\int_{\Omega\times Y_\alpha}(t-T)\Big(f_i\dot\varphi_i^\alpha+\frac{1}{T_0}rq^\alpha\Big),\qquad \forall(\varphi^1,\Phi^1,\varphi^2,q^1,Q^1,q^2)\in W.$$

Moreover, for $\alpha\in\{1,2\}$ and for almost all $x\in\Omega$, we have

$$u^\alpha(0,x)=0,\quad \dot u^\alpha(0,x)=0,\quad \theta^\alpha(0,x)=0. \tag{4.3}$$

Proof: Convergences $(4.1)_{3,4,6,7}$ follow from Yang [16]. It is proven in [7] or [16] that, for $p\in[1,\infty)$ and $q\in(1,\infty]$, if $\varphi_\varepsilon\in L^q(0,T;L^p(\Omega_\alpha^\varepsilon))$ is such that $\|\varphi_\varepsilon\|_{L^q(0,T;L^p(\Omega_\alpha^\varepsilon))}\leqslant C$ and $\mathcal{T}_\alpha^\varepsilon(\varphi_\varepsilon)\rightharpoonup\widehat\varphi$ weakly in $L^q(0,T;L^p(\Omega\times Y_\alpha))$, then

$$\widetilde\varphi_\varepsilon\rightharpoonup|Y_\alpha|\left\langle\widehat\varphi\right\rangle_\alpha\ \text{ weakly in }\ L^q(0,T;L^p(\Omega)).$$

Therefore, from $(4.1)_{3,6}$ it follows that, for $\alpha\in\{1,2\}$,

$$\tilde u^{\alpha\varepsilon}\overset{*}{\rightharpoonup}|Y_\alpha|\left\langle u^\alpha\right\rangle_{Y_\alpha}\ \text{ weakly* in }\ L^\infty(0,T;L^2(\Omega))^N,$$

$$\tilde\theta^{\alpha\varepsilon}\overset{*}{\rightharpoonup}|Y_\alpha|\left\langle\theta^\alpha\right\rangle_{Y_\alpha}\ \text{ weakly* in }\ L^\infty(0,T;L^2(\Omega)).$$

As u^α and θ^α are constant with respect to y, we deduce that convergences $(4.1)_{1,2}$ hold. Also, according to Yang [16], there exist $\widehat u^2\in L^\infty(0,T;L^2(\Omega;H^1(Y_2)))^N$ and $\widehat\theta^2\in L^\infty(0,T;L^2(\Omega;H^1(Y_2)))$ such that

$$\mathcal{T}_2^\varepsilon(e_{kh}(u^{2\varepsilon})\overset{*}{\rightharpoonup}e_{kh}^y\widehat u^2\ \text{ weakly* in }\ L^\infty(0,T;L^2(\Omega\times Y_2)), \tag{4.4}$$

$$\mathcal{T}_2^\varepsilon(\nabla\theta_{2\varepsilon})\overset{*}{\rightharpoonup}\nabla_y\widehat\theta_2\ \text{ weakly* in }\ L^\infty(0,T;L^2(\Omega\times Y_2)). \tag{4.5}$$

Using a suggestive notation, the variational problem (3.2) can be written as

$$\sum_{\alpha=1,2}\Big(I_1^{\alpha\varepsilon}+I_2^{\alpha\varepsilon}+I_3^{\alpha\varepsilon}+I_4^{\alpha\varepsilon}+I_5^{\alpha\varepsilon}+I_6^{\alpha\varepsilon}+I_7^{\alpha\varepsilon}+I_8^{\alpha\varepsilon}\Big)-I_9^\varepsilon-I_{10}^\varepsilon=-\sum_{\alpha=1,2}\Big(I_{11}^{\alpha\varepsilon}+I_{12}^{\alpha\varepsilon}\Big). \tag{4.6}$$

In order to obtain the limit problem, we choose in (4.6) as test functions

$$v_i^\alpha(t,x)=\varphi_i^\alpha(t,x)+\varepsilon\omega_i^\alpha(t,x)\psi_i^{\alpha\varepsilon}(x), \tag{4.7}$$

with no summation of the repeated index, and

$$w^\alpha(t,x)=q^\alpha(t,x)+\varepsilon g^\alpha(t,x)p^{\alpha\varepsilon}(x), \tag{4.8}$$

where $\varphi_i^\alpha,\omega_i^\alpha,q^\alpha,g^\alpha\in\mathcal{D}([0,T]\times\Omega)$ and $\psi_i^\alpha,p^\alpha\in H^1_{per}(Y_\alpha)$ and, obviously, $\psi^{\alpha\varepsilon}(x)=\psi^\alpha(x/\varepsilon)$ and $p^{\alpha\varepsilon}(x)=p^\alpha(x/\varepsilon)$ $(\alpha\in\{1,2\})$.

First, one can see that $\varepsilon\omega^\alpha\psi^{\alpha\varepsilon}\longrightarrow0$ strongly in $L^\infty(0,T;L^2(\Omega))^N$ and $\varepsilon g^\alpha p^{\alpha\varepsilon}\longrightarrow0$ strongly in $L^\infty(0,T;L^2(\Omega))$. Therefore, according to Yang [16] or Donato and Yang [7],

$$\mathcal{T}_\alpha^\varepsilon(\varepsilon\omega^\alpha\psi^{\alpha\varepsilon})\longrightarrow0\ \text{ strongly in }\ L^\infty(0,T;L^2(\Omega\times Y_\alpha))^N, \tag{4.9}$$

$$\mathcal{T}^\varepsilon_\alpha(\varepsilon g^\alpha p^{\alpha\varepsilon}) \longrightarrow 0 \;\text{ strongly in }\; L^\infty(0,T;L^2(\Omega\times Y_\alpha)). \tag{4.10}$$

Moreover, as $e_{ij}(\varepsilon\omega^\alpha\psi^{\alpha\varepsilon})(x) = \varepsilon\psi^\alpha_i(x/\varepsilon)e_{ij}(\omega^\alpha)(x) + \omega^\alpha_i(x)e^y_{ij}(\psi^\alpha)(x/\varepsilon)$, for $\alpha \in \{1,2\}$, one can easily see that

$$\mathcal{T}^\varepsilon_\alpha(e_{ij}(\varepsilon\omega^\alpha\psi^{\alpha\varepsilon})) = \varepsilon\psi^\alpha_i\mathcal{T}^\varepsilon_\alpha(e_{ij}(\omega^\alpha)) + e^y_{ij}(\psi^\alpha)\mathcal{T}^\varepsilon_\alpha(\omega^\alpha_i) \longrightarrow e^y_{ij}(\Phi^\alpha) \tag{4.11}$$

strongly in $L^\infty(0,T;L^2(\Omega\times Y_\alpha))$, where $\Phi^\alpha_i(t,x,y) = \omega^\alpha_i(t,x)\psi^\alpha_i(y)$ (no summation). With a similar justification,

$$\mathcal{T}^\varepsilon_\alpha(\nabla(\varepsilon g^\alpha p^{\alpha\varepsilon})) = \varepsilon p^\alpha\mathcal{T}^\varepsilon_\alpha(\nabla g^\alpha) + \nabla_y p^\alpha\mathcal{T}^\varepsilon_\alpha(g^\alpha) \longrightarrow \nabla_y(Q^\alpha) \tag{4.12}$$

strongly in $L^\infty(0,T;L^2(\Omega\times Y_\alpha))$ where $Q^\alpha(t,x,y) = g^\alpha(t,x)p^\alpha(y)$.

The limit problem is obtained by applying the the corresponding unfolding operator to each term of (4.6) and passing to limit with $\varepsilon\to 0$. For integrals over Γ_ε we will use a lemma similar to Lemma 2.16 in [5]. Thus, we get:

$$\begin{gathered}
\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)\Big[-a^1_{ijkh}\Big(e_{kh}(u^1)+e^y_{kh}(\widehat{u}^1)\Big)+b^1_{ij}\theta^1\Big]\Big(e_{ij}(\dot{\varphi}^1)+e^y_{ij}(\dot{\Phi}^1)\Big)+\\
-\int_0^T\!\!\int_{\Omega\times Y_2}(t-T)a^2_{ijkh}e^y_{kh}(\widehat{u}^2)\Big(e_{ij}(\dot{\varphi}^2)+e^y_{ij}(\dot{\Phi}^2)\Big)+\\
+\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)\rho^\alpha\dot{u}^\alpha_i\ddot{\varphi}^\alpha_i+\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)b^1_{ij}\Big(e_{ij}(u^1)+e^y_{ij}(\widehat{u}^1)\Big)\dot{q}^1+\\
+\frac{1}{T_0}\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)c^\alpha\theta^\alpha\dot{q}^\alpha+\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}\rho^\alpha\dot{u}^\alpha_i\dot{\varphi}^\alpha_i+\\
+\int_0^T\!\!\int_{\Omega\times Y_1}b^1_{ij}\Big(e_{ij}(u^1)+e^y_{ij}(\widehat{u}^1)\Big)q^1+\frac{1}{T_0}\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}c^\alpha\theta^\alpha q^\alpha+ \qquad (4.13)\\
+\frac{1}{T_0}\int_0^T\!\!\int_{\Omega\times Y_1}\int_0^t k^1_{ij}\Big(\frac{\partial\theta^1}{\partial x_j}+\frac{\partial\widehat{\theta}^1}{\partial y_j}\Big)\Big(\frac{\partial q^1}{\partial x_i}+\frac{\partial Q^1}{\partial y_i}\Big)\,\mathrm{d}s+\\
+\frac{1}{T_0}\int_0^T\!\!\int_{\Omega\times Y_2}\int_0^t k^2_{ij}\frac{\partial\widehat{\theta}^2}{\partial y_j}\Big(\frac{\partial q^2}{\partial x_i}+\frac{\partial Q^2}{\partial y_i}\Big)\,\mathrm{d}s-\\
-\int_0^T\!\!\int_{\Omega\times\Gamma}(t-T)h^u(u^2_i-u^1_i)(\dot{\varphi}^2_i-\dot{\varphi}^1_i)-\frac{1}{T_0}\int_0^T\!\!\int_{\Omega\times\Gamma}(t-T)h^\theta(\theta^2-\theta^1)(q^2-q^1)=\\
=-\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)\Big(f_i\dot{\varphi}^\alpha_i+\frac{1}{T_0}rq^\alpha\Big).
\end{gathered}$$

In order to bring equation (4.13) to the form (4.2), we integrate by parts with respect to t some terms of (4.13) and after a multiplication by -1, we get:

$$\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)\Big[a^1_{ijkh}\Big(e_{kh}(u^1)+e^y_{kh}(\widehat{u}^1)\Big)-b^1_{ij}\theta^1\Big]\Big(e_{ij}(\dot{\varphi}^1)+e^y_{ij}(\dot{\Phi}^1)\Big)+$$

$$+\int_0^T\!\!\int_{\Omega\times Y_2}(t-T)a^2_{ijkh}e^y_{kh}(\widehat{u}^2)\Big(e_{ij}(\dot{\varphi}^2)+e^y_{ij}(\dot{\Phi}^2)\Big)+\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)\rho^\alpha\ddot{u}^\alpha_i\dot{\varphi}^\alpha_i+$$

$$+\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)b^1_{ij}\Big(\dot{e}_{ij}(u^1)+\dot{e}^y_{ij}(\widehat{u}^1)\Big)q^1+\frac{1}{T_0}\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)c^\alpha\dot{\theta}^\alpha q^\alpha+$$

$$+\frac{1}{T_0}\int_0^T\!\!\int_{\Omega\times Y_1}(t-T)k^1_{ij}\Big(\frac{\partial\theta^1}{\partial x_j}+\frac{\partial\widehat{\theta}^1}{\partial y_j}\Big)\Big(\frac{\partial q^1}{\partial x_i}+\frac{\partial Q^1}{\partial y_i}\Big)+ \tag{4.14}$$

$$+\frac{1}{T_0}\int_0^T\!\!\int_{\Omega\times Y_2}(t-T)k^2_{ij}\frac{\partial\widehat{\theta}^2}{\partial y_j}\Big(\frac{\partial q^2}{\partial x_i}+\frac{\partial Q^2}{\partial y_i}\Big)$$

$$+\int_0^T\!\!\int_{\Omega\times\Gamma}(t-T)h^u(u^2_i-u^1_i)(\dot{\varphi}^2_i-\dot{\varphi}^1_i)-\frac{1}{T_0}\int_0^T\!\!\int_{\Omega\times\Gamma}(t-T)h^\theta(\theta^2-\theta^1)(q^2-q^1)=$$

$$=\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)\Big(f_i\dot{\varphi}^\alpha_i+\frac{1}{T_0}rq^\alpha\Big).$$

Choosing in (4.14) $\varphi^1_i=\varphi^2_i=\Phi^1_i=\Phi^2_i=q^1=q^2=Q^1=0$ and taking into account that $Q^2(t,x,y)=g^2(t,x)p^2(y)$, with $g^2\in\mathcal{D}((0,T)\times\Omega)$ and $p^2\in H^1_{per}(Y_2)$, we obtain

$$\int_{Y_2}k^2_{ij}\frac{\partial\widehat{\theta}^2}{\partial y_j}\frac{\partial p^2}{\partial y_i}=0. \tag{4.15}$$

Since K^2 is coercive, if we take $p^2=\widehat{\theta}^2$ in (4.15), we get

$$0=\int_{Y_2}K^2\nabla_y\widehat{\theta}^2\nabla_y\widehat{\theta}^2\geqslant\lambda\int_{Y_2}\nabla_y\widehat{\theta}^2,$$

which means that $\nabla_y\widehat{\theta}^2=0$. In a similar manner, one can prove that $e^y_{kh}(\widehat{u}^2)=0$. Thus, using some density arguments, convergences $(4.1)_{5,10}$ and problem (4.2) are finally proven.

In order to prove that the limits u^α, $\dot{u}^\alpha$ and θ^α ($\alpha\in\{1,2\}$) are also zero at $t=0$ one can choose test functions of the form

$$v^\alpha_i(t,x)=\varphi^\alpha(t)\eta^\alpha_i(x)+\varepsilon\omega^\alpha_i(t,x)\psi^{\alpha\varepsilon}_i(x), \tag{4.16}$$

$$w^\alpha(t,x)=q^\alpha(t)\zeta^\alpha(x)+\varepsilon g^\alpha(t,x)p^{\alpha\varepsilon}(x), \tag{4.17}$$

with $\varphi^\alpha,q^\alpha\in\mathcal{C}^\infty([0,T])$ such that $\varphi^\alpha(0)=q^\alpha(0)=1$ and $\varphi^\alpha(T)=0$, $\eta^\alpha_i,\zeta^\alpha\in\mathcal{D}(\Omega)$, $\omega^\alpha_i,g^\alpha\in\mathcal{D}([0,T]\times\Omega)$ and $\psi^\alpha_i,p^\alpha\in H^1_{per}(Y_\alpha)$ and integrate by parts with respect to t, before passing to the limit, in terms $I^{\alpha\varepsilon}_5$ and $I^{\alpha\varepsilon}_4+I^{\alpha\varepsilon}_7$, respectively. For an example of applying such a method, we refer the reader to [7].

If we assume that problem (4.2) possesses two distinct solutions, it is not difficult to see that, by linearity, their difference verifies the same equation, but with the right-hand side zero. Thus, the uniqueness of the solution of the limit problem can be easily proven since A^1,K^1 are coercive, $\rho^\alpha,c^\alpha,h^u,h^\theta$ are strictly positive and $u^1(0,x)=u^2(0,x)=0$. □

We are going now to decouple the limit problem (4.2). We choose as test functions $\varphi_i^1 = \varphi_i^2 = q^1 = q^2 = Q^1 = 0$ and, keeping in mind that $\Phi_i^1(t,x,y) = \omega_i^1(t,x)\psi_i^1(y)$ (no summation) with $\omega_i^1 \in \mathcal{D}([0,T] \times \Omega)$ and $\psi_i^1 \in H^1_{per}(Y_1)$, we shall get a problem with the unknown $\widehat{u}^1$:

$$\int_{Y_1} a^1_{ijkh} \frac{\partial \widehat{u}^1_k}{\partial y_h} \frac{\partial \psi^1_i}{\partial y_j} = -\frac{\partial u^1_k}{\partial x_h} \int_{Y_1} a^1_{ijkh} \frac{\partial \psi^1_i}{\partial y_j} + \theta^1 \int_{Y_1} b^1_{ij} \frac{\partial \psi^1_i}{\partial y_j}. \tag{4.18}$$

We introduce now the unique solution $z^1 \in \tilde{H}^1_{per}(Y_1)^N$, of problem

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\Big(a^1_{ijkh}\dfrac{\partial z^1_k}{\partial y_h} - b^1_{ij}\Big) = 0 \quad \text{in } Y_1 \\ \Big(a^1_{ijkh}\dfrac{\partial z^1_k}{\partial y_h} - b^1_{ij}\Big)n_j = 0 \quad \text{on } \Gamma. \end{cases} \tag{4.19}$$

and also, for $l, m = 1, \ldots, N$, we introduce the unique solutions $w_1^{lm} \in \tilde{H}^1_{per}(Y_1)^N$ of problems

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\Big(a^1_{ijlm} + a^1_{ijkh}\dfrac{\partial w^{lm}_{1k}}{\partial y_h}\Big) = 0 \quad \text{in } Y_1 \\ \Big(a^1_{ijlm} + a^1_{ijkh}\dfrac{\partial w^{lm}_{1k}}{\partial y_h}\Big)n_j = 0 \quad \text{on } \Gamma. \end{cases} \tag{4.20}$$

Therefore, from (4.18), one can see that

$$\widehat{u}^1_k(t,x,y) = \frac{\partial u^1_l}{\partial x_m}(t,x) w^{lm}_{1k}(y) + \theta^1(t,x) z^1_k(y). \tag{4.21}$$

Choosing proper test functions, we can find a similar problem for $\widehat{\theta}^1$:

$$\int_{Y_1} k^1_{ij} \frac{\partial \widehat{\theta}^1}{\partial y_j} \frac{\partial p^1}{\partial y_i} = -\frac{\partial \theta^1}{\partial x_j} \int_{Y_1} k^1_{ij} \frac{\partial p^1}{\partial y_i}. \tag{4.22}$$

Now, for $k = 1, \ldots, N$, let $\chi^\alpha \in \tilde{H}^1_{per}(Y_\alpha)^N$ be the solutions of the problems

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\Big(k^1_{ik} + k^1_{ij}\dfrac{\partial \chi^1_k}{\partial y_j}\Big) = 0 \quad \text{in } Y_1 \\ \Big(k^1_{ik} + k^1_{ij}\dfrac{\partial \chi^1_k}{\partial y_j}\Big)n_i = 0 \quad \text{on } \Gamma. \end{cases} \tag{4.23}$$

In a similar manner, the linearity of (4.22) implies that

$$\widehat{\theta}^1_k(t,x,y) = \frac{\partial \theta^1}{\partial x_k}(t,x)\chi^1_k(y). \tag{4.24}$$

We define now the homogenized coefficients

$$\begin{aligned} a^{1*}_{ijlm} &= \int_{Y_1}\Big(a^1_{ijlm} + a^1_{ijkh}\frac{\partial w^{lm}_{1k}}{\partial y_h}\Big), \qquad & b^{1*}_{lm} &= \int_{Y_1}\Big(b^1_{lm} + b^1_{ij}\frac{\partial w^{lm}_{1i}}{\partial y_j}\Big), \\ k^{1*}_{ik} &= \int_{Y_1}\Big(k^1_{ik} + k^1_{ij}\frac{\partial \chi^1_k}{\partial y_j}\Big), & \beta^{1*}_{ij} &= \int_{Y_1}\Big(a^1_{ijkh}\frac{\partial z^1_k}{\partial y_h} - b^1_{ij}\Big), \end{aligned} \tag{4.25}$$

$$\gamma^{1*} = \int_{Y_1} b^1_{ij}\frac{\partial z^1_i}{\partial y_j}. \tag{4.26}$$

Let us notice that for any $l, m = 1, \dots, N$, we have $\beta^{1*}_{lm} = -b^{1*}_{lm}$.

Theorem 3. *If $(u^\varepsilon, \theta^\varepsilon) \in \mathcal{W}_\varepsilon$ is the solution of (2.3)-(2.10), where $u^\varepsilon = (u^{1\varepsilon}, u^{2\varepsilon})$ and $\theta^\varepsilon = (\theta^{1\varepsilon}, \theta^{2\varepsilon})$, then, for $\alpha \in \{1,2\}$, we have*

$$\tilde{u}^{\alpha\varepsilon} \overset{*}{\rightharpoonup} |Y_\alpha| u^\alpha \quad \textit{weakly* in} \ \ L^\infty(0,T;L^2(\Omega))^N, \tag{4.27}$$

$$\tilde{\theta}^{\alpha\varepsilon} \overset{*}{\rightharpoonup} |Y_\alpha| \theta^\alpha \quad \textit{weakly* in} \ \ L^\infty(0,T;L^2(\Omega)), \tag{4.28}$$

where (u,θ), with $u = (u^1, u^2)$ and $\theta = (\theta^1, \theta^2)$, is the unique solution of the problem

$$-\frac{\partial}{\partial x_j}\Big(a^{1*}_{ijkh}\frac{\partial u^1_k}{\partial x_h} - b^{1*}_{ij}\theta^1\Big) + \langle \rho^1\rangle_{Y_1}\frac{\partial^2 u^1_i}{\partial t^2} - H^u(u^2_i - u^1_i) = |Y_1|\, f_i \quad \textit{in } \Omega_T, \tag{4.29}$$

$$\langle \rho^2\rangle_{Y_2}\frac{\partial^2 u^2_i}{\partial t^2} + H^u(u^2_i - u^1_i) = |Y_2|\, f_i \quad \textit{in } \Omega_T, \tag{4.30}$$

$$\begin{aligned} -\frac{\partial}{\partial x_i}\Big(k^{1*}_{ij}\frac{\partial \theta^1}{\partial x_j}\Big) + T_0 b^{1*}_{ij}\frac{\partial e_{ij}(u^1)}{\partial t} + \Big(T_0\gamma^{1*} + \langle c^1\rangle_{Y_1}\Big)\frac{\partial \theta^1}{\partial t} - \\ -H^\theta(\theta^2 - \theta^1) = |Y_1|\, r \ \textit{ in } \Omega_T, \end{aligned} \tag{4.31}$$

$$\langle c^2\rangle_{Y_2}\frac{\partial \theta^2}{\partial t} + H^\theta(\theta^2 - \theta^1) = |Y_2|\, r \quad \textit{in } \Omega_T, \tag{4.32}$$

$$u^1 = 0, \quad \theta^1 = 0 \qquad \textit{on } (0,T)\times\partial\Omega, \tag{4.33}$$

$$u^\alpha(0,x) = 0, \quad \dot{u}^\alpha(0,x) = 0, \quad \theta^\alpha(0,x) = 0, \tag{4.34}$$

with $H^u = \int_\Gamma h^u$ and $H^\theta = \int_\Gamma h^\theta$.

Proof: Since the convergences $(4.1)_{1,2}$ hold, in order to prove (4.27) and (4.28), we only need to show that the limits u^α and θ^α from Theorem 2 verify the problem (4.29)-(4.32). This can be easily proven by inserting the expressions of $\widehat{u}^\alpha$ and $\widehat{\theta}^\alpha$, (4.21) and (4.24), in (4.2) and using the cell problems (4.23)-(4.19) and the homogenized coefficients formulas (4.25)-(4.26). Thus, (4.2) becomes

$$\int_0^T\!\!\int_\Omega (t-T)\Big(a^{1*}_{ijkh}\frac{\partial u^1_k}{\partial x_k} - b^{1*}_{ij}\theta^1\Big)\frac{\partial \dot{\varphi}^1_i}{\partial x_j} +$$

$$+\sum_{\alpha=1,2}\int_0^T\!\!\int_\Omega (t-T)\left[\langle\rho^\alpha\rangle_{Y_\alpha}\ddot{u}_i^\alpha\dot{\varphi}_i^\alpha+\frac{1}{T_0}\langle c^\alpha\rangle_{Y_\alpha}\dot{\theta}^\alpha q^\alpha\right]+$$
$$+\int_0^T\!\!\int_\Omega (t-T)\left[\frac{1}{T_0}k_{ij}^{1*}\frac{\partial\theta^1}{\partial x_j}\frac{\partial q^1}{\partial x_i}+\left(b_{ij}^{1*}\dot{e}_{ij}(u^1)+\gamma^{1*}\dot{\theta}^1\right)q^1\right]+ \tag{4.35}$$
$$+\int_0^T\!\!\int_\Omega (t-T)\left[H^u(u_i^2-u_i^1)(\dot{\varphi}_i^2-\dot{\varphi}_i^1)+\frac{1}{T_0}H^\theta(\theta^2-\theta^1)(q^2-q^1)\right]=$$
$$=\sum_{\alpha=1,2}\int_0^T\!\!\int_\Omega (t-T)\,|Y_\alpha|\left(f_i\dot{\varphi}_i^\alpha+\frac{1}{T_0}rq^\alpha\right),$$

which holds for any $\varphi_i^1, q^1 \in \mathcal{D}(0,T;H_0^1(\Omega))$ and $\varphi_i^2, q^2 \in \mathcal{D}(0,T;L^2(\Omega))$. It is not difficult to see that (4.35) is the variational formulation of (4.29)-(4.32). ∎

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