Bull. Math. Soc. Sci. Math. Roumanie
Tome 50(98) No. 2, 2007, 119–130

A note on the generic initial ideal for complete intersections

by
Mircea Cimpoeaş

Abstract

We prove that the d-component of the generic initial ideal, with respect to the reverse lexicographic order, of an ideal generated by a regular sequence of homogeneous polynomials of degree d is revlex in a particular, but important, case. Using this property, we compute the generic initial ideal for several complete intersection with strong Lefschetz property.

Key Words: Complete intersection, generic initial ideal, Lefschetz property.
2000 Mathematics Subject Classification: Primary: 13P10, Secondary 13D40,13C40.

Introduction.

Let K be an algebraically closed field of characteristic zero. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over K. Let $n, d \geq 2$ be two integers. We consider
$I = (f_1, \dots, f_n) \subset S$ an ideal generated by a regular sequence $f_1, \dots, f_n \in S$ of homogeneous polynomials of degree d. We say that $A = S/I$ is a (n, d)-*complete intersection*. Let $J = Gin(I)$ be the generic initial of I, with respect to the reverse lexicographical (revlex) order (see [5, §15.9], for details).

We say that a property (P) holds for a generic sequence of homogeneous polynomials $f_1, f_2, \dots, f_n \in S$ of given degrees $d_1, d_2, \dots, d_n$ if there exists a nonempty open Zariski subset $U \subset S_{d_1} \times S_{d_2} \times \cdots \times S_{d_n}$ such that for every n-tuple $(f_1, f_2, \dots, f_n) \in U$ the property (P) holds. We say that a set of monomials $M \subset S$ is a *revlex set* if, given a monomial $u \in M$, then any other monomial greater than u in revlex order is also in M.

[1]This paper was supported by the CEEX Program of the Romanian Ministry of Education and Research, Contract CEX05-D11-11/2005 and by the Higher Education Commission of Pakistan.

For any nonnegative integer k, we denote by J_k the set of monomials from J of degree k. Conca and Sidman proved that J_d is revlex if $f_1,\dots,f_n$ is a generic regular sequence, (see [4, Theorem 1.2]). In the first part of this paper, we prove that J_d is a revlex set in another case, namely, when $f_i \in k[x_i,\dots,x_n]$. It is likely to be true that J_d is revlex for any (n,d)-complete intersection, but we do not have the means to prove this assertion.

We say that a homogeneous polynomial f of degree s is *semiregular* for S/I if the maps $(S/I)_t \xrightarrow{\cdot f} (S/I)_{t+s}$ are either injective, either surjective for all $t \geq 0$. We say that S/I has the *weak Lefschetz property* (WLP) if there exists a linear form $\ell \in S$, semiregular on S/I. In such case, we say that ℓ is a weak Lefschetz element for S/I. A theorem of Harima-Migliore-Nagel-Watanabe (see [6]) states that S/I has (WLP) in the case $n = 3$. We say that S/I has the *strong Lefschetz property* (SLP) if there exists a linear form $\ell \in S$ such that ℓ^b is semiregular on S/I for all integer $b \geq 1$. In this case, we say that ℓ is a strong Lefschetz element for S/I. Harima-Watanabe [7] and later Herzog-Popescu [8], proved that S/I has (SLP) if $f_i \in k[x_i,\dots,x_n]$, for all $1 \leq i \leq n$.

In the second section of our paper, we compute the generic initial ideal for some particular cases of (n,d)-complete intersections: $(n = 4, d = 2)$, $(n = 5, d = 2)$ and $(n = 4, d = 3)$. In order to do this, we suppose in addition that S/I has (SLP). Note that this property holds for generic complete intersection (see [9]) and also in the case when $f_i \in k[x_i,\dots,x_n]$. It was conjectured that (SLP) holds for any standard complete intersection. A theorem of Wiebe [12] states that S/I has (WLP) (respectively (SLP)) if and only if x_n is a weak (respectively strong) Lefschetz element for S/J, where $J = Gin(I)$. As Example 1.9 show, the hypothesis $char(K) = 0$ and $f_1,\dots,f_n$ is a regular sequence are essentials.

1 Generic initial ideal for (n,d)-complete intersections.

Let $I = (f_1,\dots,f_n) \subset S = K[x_1,\dots,x_n]$ be an ideal generated by a regular sequence $f_1,\dots,f_n \in S$ of homogeneous polynomials of degree d. Let $J = Gin(I)$ be the generic initial ideal of I, with respect to the revlex order. It is well known that the Hilbert series of S/J is the same as the Hilbert series of S/I and moreover, $H(S/J,t) = H(S/I,t) = (1 + t + \cdots + t^{d-1})^n$. More precisely, we have:

Proposition 1.1. *1.* $H(S/J,k) = \binom{k+n-1}{n-1}$, *for* $0 \leq k \leq d-1$.

2. $H(S/J,k) = \binom{k+n-1}{n-1} - n\binom{j+n-1}{n-1}$, *for* $d \leq k \leq \left\lfloor \frac{n(d-1)}{2} \right\rfloor$ *and* $j = k - d$.

3. $H(S/J,k) = H(S/J, n(d-1) - k)$, *for* $k \geq \left\lceil \frac{n(d-1)}{2} \right\rceil$.

Proof: Use induction on n. Denote $H_n(t) = (1+t+\cdots+t^{d-1})^n$. The case $n = 1$ is trivial. The induction step follows from the equality $H_n(t) = H_{n-1}(t)(1 + t + \cdots + t^{d-1})$. □

Corollary 1.2. *1.* $|J_k| = 0$, *for* $k \leq d-1$.

2. $|J_k| = n\binom{j+n-1}{n-1}$, *for* $d \leq k \leq \left\lfloor \frac{n(d-1)}{2} \right\rfloor$ *and* $j = k-d$.

3. $|J_k| = \binom{\lceil \frac{n(d-1)}{2} \rceil + j + n - 1}{n-1} - \binom{\lfloor \frac{n(d-1)}{2} \rfloor - j + n - 1}{n-1} + n\binom{\lfloor \frac{n(d-1)}{2} \rfloor - d - j - n}{n-1}$, *for* $\left\lceil \frac{n(d-1)}{2} \right\rceil \leq k \leq (n-1)(d-1)-1$, *where* $j = k - \left\lceil \frac{n(d-1)}{2} \right\rceil$

4. $|J_k| = \binom{(n-1)d+j}{n-1} - \binom{n-1+d-1-j}{n-1}$, *for* $(n-1)(d-1) \leq k \leq n(d-1)$, *where* $j = k-(n-1)(d-1)$.

Proof: Using $|J_k| = |S_k| - H(S/J,k)$ the proof follows from 1.1. □

Suppose $f_i = \sum_{k=1}^{N} b_{ik}u_k$ for $1 \leq i \leq n$ where $u_1, u_2, \ldots, u_N \in S$ are all the monomials of degree d decreasing ordered in revlex and $N = \binom{d+n-1}{n-1}$. We denote $u_k = x^{\alpha_k}$. For example, $\alpha_1 = (d, 0, \ldots, 0)$, $\alpha_2 = (d-1, 1, 0, \ldots, 0)$ etc.

We take a generic transformation of coordinates $x_i \mapsto \sum_{j=1}^{n} c_{ij}x_j$ for $i = 1, \ldots, n$. Conca and Sidman proved in [4] that we may assume that c_{ij} are algebraically independents over K. More precisely, if we consider the field extension $K \subset L = K(c_{ij}|i,j = \overline{1,n})$ and if we set

$$F_i = f_i(\sum_{j=1}^{n} c_{1j}x_j, \ldots, \sum_{j=1}^{n} c_{nj}x_j) \in L[x_1, \ldots, x_n],\ i = 1, \ldots, n$$

then $J = Gin(I) = in(F_1, \ldots, F_n) \cap S$.

We write $F_i = \sum_{j=1}^{n} a_{ij}u_j + \cdots$ the monomial decomposition of F_i in

$$L[x_1, \ldots, x_n].$$

With these notations, we have the following elementary lemma:

Lemma 1.3. *J_d is revlex if and only if the following condition is fulfilled:*

$$\Delta = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \neq 0.$$

Proof: Suppose $\Delta \neq 0$. Since $|J_d| = n$, it is enough to show that $u_1, \ldots, u_n \in J$. Let $A = (a_{ij})_{i,j \in \overline{1,n}}$. Since $\Delta = det(A) \neq 0$, A is invertible and we have

$$A^{-1}\begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} = \begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix},$$

where $H_i = u_i +$ small terms in revlex order. Therefore $LM(H_i) = u_i \in J$, for all $1 \leq i \leq n$, where $LM(H_i)$ denotes the leading monomial of H_i in the revlex order.

Conversely, since $u_1, \ldots, u_n \in J_d$, we can find some polynomials

$$H_i \in L[x_1, \dots, x_n],$$

with $LM(H_i) = u_i$, $1 \le i \le n$, as linear combination of F_i's. If we denote $H_i = \sum_{j=1}^N \widetilde{a}_{ij} u_j$ and $\widetilde{A} = (\widetilde{a}_{ij})_{i,j=1,\dots,n}$, it follows that there exists a map $\psi : L^n \to L^n$, given by a matrix $E = (e_{ij})_{i,j=1,\dots,n}$, such that $\widetilde{A} = A \cdot E$. Now, since $det(\widetilde{A}) \neq 0$ it follows that $\Delta = det(A) \neq 0$, as required. □

Remark 1.4. By the changing of variables φ given by $x_i \mapsto \sum_{j=1}^n c_{ij} x_j$, x^{α_k} became

$$m_k := (\sum_{j=1}^n c_{1j} x_j)^{\alpha_{k1}} \cdots (\sum_{j=1}^n c_{nj} x_j)^{\alpha_{kn}} = (\sum_{|t| = \alpha_{k1}} c_1^t x^t) \cdots (\sum_{|t| = \alpha_{kn}} c_n^t x^t),$$

where, for any multiindex $t = (t_1, \dots, t_n)$ we denoted $x^t = x_1^{t_1} \cdots x_n^{t_n}$ and $c_i^t = c_{i1}^{t_1} \cdots c_{in}^{t_n}$. Let g_{kl} be the coefficient in c_{ij}'s of x^{α_l} in the monomial decomposition of m_k. Using the above writing of m_k, we claim that:

$$(1) \quad g_{kl} =$$

$$= \sum_{\substack{|t_1| = \alpha_{k1}, \dots, |t_n| = \alpha_{kn} \\ t_1 + \cdots + t_n = \alpha_l}} \left[\binom{\alpha_{k1}}{t_{11}} \cdots \binom{\alpha_{kn}}{t_{n1}} \right] \left[\binom{\alpha_{k1} - t_{11}}{t_{12}} \cdots \binom{\alpha_{kn} - t_{n1}}{t_{n2}} \right] \cdots$$

$$\left[\binom{\alpha_{k1} - t_{11} - \cdots t_{1n-1}}{t_{1n}} \cdots \binom{\alpha_{kn} - t_{n1} - \cdots - t_{nn-1}}{t_{nn}} \right] \cdot c_1^{t_1} \cdots c_n^{t_n}.$$

Indeed, the monomial $c_1^{t_1} \cdots c_n^{t_n}$ appear in the coefficient of x^{α_l} in the expansion of m_k if and only if $t_1 + \cdots + t_n = \alpha_l$ and $|t_1| = \alpha_{k1}, \dots, |t_n| = \alpha_{kn}$. Moreover, by Newton binomial, the coefficient of $x_1^{t_{i1}} \cdots x_n^{t_{in}}$ in $(\sum_{j=1}^n c_{ij} x_j)^{\alpha_{k1}}$ is $\binom{\alpha_{k1}}{t_{i1}} \binom{\alpha_{k1} - t_{i1}}{t_{i2}} \cdots \binom{\alpha_{k1} - t_{i1} - \cdots t_{i,n-1}}{t_{in}} c_i^{t_i}$ for any $1 \le i \le n$, and thus we proved the claim.

Since $a_{il} = \sum_{k=1}^N b_{ik} \cdot g_{kl}$, from the Cauchy-Binet formula we get:

$$\Delta = \sum_{1 \le k_1 < k_2 < \dots < k_n \le N} B_{k_1, k_2, \dots, k_n} G_{k_1, k_2, \dots, k_n}, \; where$$

$$B_{k_1, k_2, \dots, k_n} = \begin{vmatrix} b_{1k_1} & \cdots & b_{1k_n} \\ \vdots & & \vdots \\ b_{nk_1} & \cdots & b_{nk_n} \end{vmatrix} \; and \; G_{k_1, k_2, \dots, k_n} = \begin{vmatrix} g_{k_1 1} & \cdots & g_{k_n 1} \\ \vdots & & \vdots \\ g_{k_1 n} & \cdots & g_{k_n n} \end{vmatrix}.$$

Now, we are able to prove the main result of our paper.

Theorem 1.5. *If $f_i \in K[x_i, \dots, x_n]$ then J_d is revlex. In particular, if S/I is a monomial complete intersection, then J_d is revlex.*

Proof: Let $k_i = \binom{i+d-1}{d}$, for any $i = 1, \ldots, n$. Then $u_{k_i} = x_i^d$. Recall our notation, $u_k = x^{\alpha_k}$. We have $b_{11} \neq 0$, otherwise $I = (f_1, \ldots, f_n) \subset (x_2, \ldots, x_n)$ contradicting the fact that I is an Artinian ideal. Using a similar argument, we get $b_{ik_i} \neq 0$ for all $1 \leq i \leq n$. Thus, multiplying each f_i with $b_{ik_i}^{-1}$, we may assume $b_{ik_i} = 1$ for all $1 \leq i \leq n$. In other words, $f_i = x_i^d + f_i'$, where f_i' contains monomials smaller than x_i^d in the revlex order. Also, since $f_i \in K[x_i, \ldots, x_n]$ we have $b_{i'k_i} = 0$ for any $i' > i$. In particular,$B_{k_1,\ldots,k_n} = 1$.

In the expansion of the determinant $G_{k_1,\ldots,k_n}$, appears the term

$$g_{k_1 1} \cdot g_{k_2 2} \cdots g_{k_n n} = r \cdot (c_{11}^d)(c_{21}^{d-1} c_{22}) \cdots (c_i^{\alpha_i}) \cdots (c_n^{\alpha_n}),$$

where r is a nonzero (positive) integer. Indeed, by (1), we have $g_{11} = c_{11}^d$, $g_{k_2 2} = d c_{21}^{d-1} c_{22}$ and, in general, $g_{k_i i} =$ some binomial coefficient $\cdot c_i^{\alpha_i}$. We claim that $m = (c_{11}^d)(c_{21}^{d-1} c_{22}) \cdots (c_i^{\alpha_i}) \cdots (c_n^{\alpha_n})$ doesn't appear again in the expansion of Δ.

Since $f_i \in k[x_i, \ldots, x_n]$, in the monomials in (c_{tl}) of a_{ij} there are no c_{tl}'s with $t < i$. Also, all the monomials of f_i' contain variables x_t with $t > i$. Corresponding to them, in a_{ij}'s there are c_{tj}'s with $t > i$. Thus in a_{il} the only monomials in $c_{i1}, \ldots, c_{in}$ of degree d comes from $\varphi(x_i^d) = (\sum_{j=1}^n c_{ij} x_j)^d$, the other monomials being multiples of some c_{tl} with $t > i$. Consequently, in the expansion of Δ, the monomials of the type $c_1^{\beta_1} \cdots c_n^{\beta_n}$, where $\beta_1, \ldots, \beta_n$ are multiindices with $|\beta_1| = \cdots = |\beta_n| = d$ comes only from $\varphi(x_1^d), \ldots, \varphi(x_n^d)$.

On the other hand, for any $1 \leq i \leq n$, $c_i^{\alpha_i}$ is unique between the monomials in c_{tl}'s from $\varphi(x_n^d)$, because they are of the type c_i^γ, where γ is a multiindex with $|\gamma| = d$. From these facts, it follows that the monomial m is unique in the monomial expansion of Δ and occurs there with a nonzero coefficient. Thus $\Delta \neq 0$ and by applying Lemma 1.3 we complete the proof of the theorem. □

Remark 1.6. In the case $n = 2$ and $n = 3$, J_d is revlex for any (n, d)-complete intersection. Indeed, in the case $n = 2$, J itself is revlex since it is strongly stable. In the case $n = 3$, since $|J_d| = 3$ and J is strongly stable, it follows that either (a) $J_d = (x_1^d, x_1^{d-1} x_2, x_1^{d-2} x_2^2)$, either (b) $J_d = (x_1^d, x_1^{d-1} x_2, x_1^{d-1} x_3)$. But in the case (b), the map $(S/J)_{d-1} \xrightarrow{\cdot x_3} (S/J)_d$ is not injective, because $x_1^{d-1} \neq 0$ in $(S/J)_{d-1}$ and $x_1^{d-1} x_3 = 0$ in $(S/J)_d$. This is a contradiction with the fact that x_3 is a weak Lefschetz element on S/J and therefore, J_d is revlex.

Lemma 1.7. *(a)* $a_{i1} = f_i(c_{11}, \ldots, c_{n1})$ *for all* $1 \leq i \leq n$.

(b) If $1 \leq l \leq n$ *is an integer then the sequence* $a_{1l}, a_{2l}, \ldots, a_{nl}$ *is regular as a sequence of polynomials in* $K[c_{ij} | \ 1 \leq i, j \leq n]$.

Proof: Substituting $x_j = 0$ for $j \neq 1$ in F_i we get (a). In order to prove (b), firstly notice that $a_{11}, a_{21}, \ldots, a_{n1}$ is a regular sequence on $K[c_{11}, c_{21}, \ldots, c_{n1}]$, since $f_1, \ldots, f_n$ is a regular sequence on $K[x_1, \ldots, x_n]$ and $c_{11}, c_{21}, \ldots, c_{n1}$ are algebraically independent over K.

Let $1 \leq l \leq n$ be an integer. We claim that

$$(*)\ \frac{K[c_{ij}|\ 1 \leq i,j \leq n]}{(a_{1l},\ldots,a_{nl},c_{i1}-c_{ij}\ ,\ 1 \leq i \leq n,\ 2 \leq j \leq n)} \cong \frac{K[c_{11},c_{21},\ldots,c_{n1}]}{(a_{11},a_{21},\ldots,a_{n1})}.$$

Indeed, by (1), if we put $c_{ij} = c_{i1}$ for all $1 \leq i \leq n$ and $2 \leq j \leq n$ in the expansion of g_{kl} we obtain $r_l \cdot g_{k1}$, where r_l is a strictly positive integer, which depends only on l, and therefore, a_{il} became $r_l \cdot a_{i1}$. From (*) it follows that $a_{1l},\ldots,a_{nl},c_{i1}-c_{ij}$ *for* $1 \leq i \leq n,\ 2 \leq j \leq n$ is a system of parameters for $K[c_{ij}|\ 1 \leq i,j \leq n]$ and thus $a_{1l},\ldots,a_{nl}$ is a regular sequence on $K[c_{ij}|\ 1 \leq i,j \leq n]$, so we proved (b). □

As we noticed in Remark 1.6, for $n = 3$, the conclusion of Theorem 1.5 holds for any regular sequence f_1, f_2, f_3 of homogeneous polynomials of degree d. In the following, we give another proof of this, without using the fact that $S/(f_1, f_2, f_3)$ has the (WLP), i.e. x_3 is a weak Lefschetz element for S/J. Also, we get the same conclusion for the case $n = 4$ and $d = 2$. However, this approach do not works in the general case.

Proposition 1.8. *(a) If $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ is a regular sequence of homogeneous polynomials of degree $d \geq 2$, $I = (f_1, f_2, f_3)$ and $J = Gin(I)$, the generic initial ideal of I, with respect to the reverse lexicographical order, then J_d is a revlex set.*

(b) If $f_1, f_2, f_3, f_4 \in K[x_1, x_2, x_3, x_4]$ is a regular sequence of homogeneous polynomials of degree 2, $I = (f_1, f_2, f_3, f_4)$ and $J = Gin(I)$, the generic initial ideal of I, with respect to the reverse lexicographical order, then J_2 is a revlex set.

Proof: (a) Let $A = (a_{ij})_{i,j=\overline{1,3}}$. Since $Gin(f_1, f_2)$ is strongly stable, it follows by Lemma 1.3 that $\Delta_3 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$. Analogously, $\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \neq 0$ and $\Delta_1 = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \neq 0$. We have $\Delta = a_{13}\Delta_1 - a_{23}\Delta_2 + a_{33}\Delta_3$. Suppose $\Delta = 0$. It follows $a_{13}\Delta_1 = a_{23}\Delta_2 - a_{33}\Delta_3$ and therefore, since a_{13}, a_{23}, a_{33} is a regular sequence in $K[c_{ij}|i,j = \overline{1,3}]$, we get $\Delta_1 \in (a_{23}, a_{33})$. The first three monomials of degree d in revlex order are x_1^d, $x_1^{d-1}x_2$ and $x_1^{d-2}x_2^2$. It follows that the degree of a_{i1}, a_{i2} and a_{i3} in c_{21}, c_{22}, c_{23} is 0, 1, respectively 2, for any $1 \leq i \leq 3$. Therefore, the degree of Δ_1 in the variables c_{21}, c_{22}, c_{23} is 1, but the degree of a_{23} and a_{33} in c_{21}, c_{22}, c_{23} is 2, which is impossible, since $\Delta_1 \in (a_{23}, a_{33})$.

(b) Let $A = (a_{ij})_{i,j=\overline{1,4}}$. Since any three polynomials from f_1, f_2, f_3, f_4 form a regular sequence, it follows from (a) that any 3×3 minor of the matrix $\widetilde{A} = (a_{ij})_{\substack{i=\overline{1,4}\\ j=\overline{1,3}}}$ is nonzero. Let Δ_i be the minor obtained from $\widetilde{A}$ by erasing the i-row. Suppose $\Delta = 0$. It follows that $a_{14}\Delta_1 = a_{24}\Delta_2 - a_{34}\Delta_3 + a_{44}\Delta_4$ and therefore, since $a_{14}, a_{24}, a_{34}, a_{44}$ is a regular sequence in $K[c_{ij}|i,j = \overline{1,4}]$, we get $\Delta_1 \in (a_{24}, a_{34}, a_{44})$. Since the first 4 monomials in revlex are $x_1^2, x_1x_2, x_2^2, x_1x_3$, we get a contradiction from the fact that the degree of Δ_1 in the variables $c_{31}, c_{32}, c_{33}, c_{34}$ is zero, but the degree of a_{24}, a_{34}, a_{44} in $c_{31}, c_{32}, c_{33}, c_{34}$ is 1. $\square$

Remark 1.9. The hypothesis that K is a field with $char(K) = 0$ is essential. Indeed, suppose $char(K) = p$ and $I = (x_1^p, x_2^p) \subset K[x_1, x_2]$. Then, simply using the definition of the generic initial ideal, we get $Gin(I) = I$ and, obviously, $I_p = \{x_1^p, x_2^p\}$ is not revlex.

Also, the hypothesis that $f_1, \ldots, f_n$ is a regular sequence of homogeneous polynomials is essential. Let $I = (f_1, f_2, f_3) \subset K[x_1, x_2, x_3]$, where $f_1 = x_1^2$, $f_2 = x_1x_2$ and $f_3 = x_1x_3$. In order to compute the generic initial ideal of I we can take a generic transformation of coordinates with an upper triangular matrix, i.e. $x_1 \mapsto x_1,\ x_2 \mapsto x_2 + c_{12}x_1,\ x_3 \mapsto x_3 + c_{23}x_2 + c_{13}x_1$, where $c_{ij} \in K$ for all i, j (see [5, §15.9]). We get

$$F_1(x_1, x_2, x_3) := f_1(x_1, x_2 + c_{12}x_1, x_3 + c_{23}x_2 + c_{13}x_1) = x_1^2,$$

$$F_2(x_1, x_2, x_3) := f_2(x_1, x_2 + c_{12}x_1, x_3 + c_{23}x_2 + c_{13}x_1) = c_{12}x_1^2 + x_1x_2,$$

$$F_3(x_1, x_2, x_3) := f_3(x_1, x_2 + c_{12}x_1, x_3 + c_{23}x_2 + c_{13}x_1) = c_{13}x_1^2 + c_{23}x_1x_2 + x_1x_3.$$

The generic initial ideal of I, $J = in(F_1, F_2, F_3)$ satisfies $J_2 = I_2$, but I_2 is not revlex.

2 Several examples of computation of the Gin.

Let $I = (f_1, \ldots, f_n) \subset S = K[x_1, \ldots, x_n]$ be an ideal generated by a regular sequence $f_1, \ldots, f_n \in S$ of homogeneous polynomials of degree d. Let $J = Gin(I)$ be the generic initial ideal of I, with respect to the revlex order.

In [2], the case $n = 3$ and $d \geq 2$ is treated completely, when $S/(f_1, f_2, f_3)$ has (SLP). More precisely, if d is odd, then

$$J = (x_1^{d-2}\{x_1, x_2\}^2, x_1^{d-2j-1}x_2^{3j+1}, x_1^{d-2j-2}x_2^{3j+2} \ for\ 1 \leq j \leq \frac{d-3}{2},\ x_2^{\frac{3d-1}{2}},$$

$$x_3x_2^{\frac{3d-3}{3}}, x_3^{2j+1}x_1^{2j}x_2^{\frac{3d-3}{2}-3j}, \ldots, x_3^{2j+1}x_2^{\frac{3d-3}{2}-j}, 1 \leq j \leq \frac{d-3}{2}$$

$$, x_3^{d-2+2j}\{x_1, x_2\}^{d-j}, 1 \leq j \leq d,$$

$$or \quad J = (x_1^{d-2}\{x_1, x_2\}^2, x_1^{d-2j-1}x_2^{3j+1}, x_1^{d-2j-2}x_2^{3j+2} \ for\ 1 \leq j \leq \frac{d-4}{2},$$

$$x_1x_2^{\frac{3d-4}{2}}, x_2^{\frac{3d-2}{2}},\ x_3^{2j}x_1^{2j-1}x_2^{\frac{3d}{2}-3j}, \ldots, x_3^{2j}x_2^{\frac{3d-2}{2}-j}, 1 \leq j \leq \frac{d-2}{2},$$

$$x_3^{d-2+2j}\{x_1, x_2\}^{d-j}, 1 \leq j \leq d,$$

if d is even (see [2, Proposition 3.3]).

In the following, we discuss some particular cases with $n \geq 4$.

The case $n = 4$, $d = 2$. We assume that S/I has (SLP). From Wiebe's Theorem, it follows that x_4 is a strong Lefschetz element for S/J. For a positive integer k, we denote $Shad(J_k) = \{x_1, \ldots, x_n\}J_k$. We have $H(S/J,t) = (1+t)^4 = 1 + 4t + 6t^2 + 4t^3 + t^4$.

We have $|J_2| = 4$. From Proposition 1.8, J_2 is revlex, therefore

$$J_2 = \{x_1^2, x_1x_2, x_2^2, x_1x_3\} = \{\{x_1, x_2\}^2, x_1x_3\}.$$

We have $|Shad(J_2)| = 12$. On the other hand, $|J_3| = 16$, so we need to add 4 new generators at $Shad(J_2)$ to get J_3. If we add a new monomial which is divisible by x_4^2, then the map $(S/J)_1 \xrightarrow{\cdot x_4^2} (S/J)_3$, will be no longer injective. Since $|(S/J)_1| = |(S/J)_3|$, we get a contradiction with the fact that x_4 is a strong Lefschetz element for S/J. But there exists only 16 monomials in S which are not multiple of x_4^2. Thus

$$J_3 = \{\{x_1, x_2, x_3\}^3, x_4\{x_1, x_2, x_3\}^2\}, \textit{ and therefore}$$

$$Shad(J_3) = \{\{x_1, x_2, x_3\}^4, x_4\{x_1, x_2, x_3\}^3, x_4^2\{x_1, x_2, x_3\}^2\}.$$

Since $|Shad(J_3)| = 31$ and $|J_4| = |S_4| - |(S/J)_4| = 35 - 1 = 34$ we have to add 3 new generators at $Shad(J_3)$ in order to get J_4. Since J is strongly stable, these new generators are $x_4^3x_1$, $x_4^3x_2$ and $x_4^3x_3$. So

$$J_4 = \{x_1, x_2, x_3, x_4\}^4 \setminus \{x_4^4\}. \textit{ We get } Shad(J_4) = \{x_1, x_2, x_3, x_4\}^5 \setminus \{x_4^5\}$$

and since $J_5 = S_5$ it follows that we must add x_4^5 at $Shad(J_4)$ to obtain J_5. From now one, we cannot add any new monomial. J is the ideal generated by all monomials added at some step k to $Shad(J_k)$, thus we proved the following proposition:

Proposition 2.1. *If $I = (f_1, f_2, f_3, f_4)$ is an ideal generated by a regular sequence of homogeneous polynomials $f_1, f_2, f_3, f_4 \in S = k[x_1, x_2, x_3, x_4]$ of degree 2 such that the algebra S/I has (SLP) then the generic initial ideal of I with respect to the revlex order is*

$$J = (x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3,\ x_2x_3^3,\ x_3^3,\ x_3^2x_4,\ x_3^2x_4,\ x_4^3x_1,\ x_4^3x_2,\ x_4^3x_3,\ x_4^5).$$

In particular, this assertion holds for a generic sequence of homogeneous polynomials $f_1, f_2, f_3, f_4 \in S$ or if $f_i \in k[x_i, \ldots, x_4]$, $1 \leq i \leq 4$.

The case $n = 5$, $d = 2$. In the following, we suppose that S/I has (SLP), so x_5 is a strong Lefschetz element for S/J. Also, we suppose that J_2 is revlex. We have $H(S/J,t) = (1+t)^5 = 1 + 5t + 10t^2 + 10t^3 + 5t^4 + t^5$. We have $|J_2| = 5$. Since J_2 is revlex from the assumption, we have $J_2 = \{\{x_1, x_2\}^2, x_3\{x_1, x_2\}\}$. So

$$Shad(J_2) = \{\{x_1, x_2\}^3, \{x_1, x_2\}^2\{x_3, x_4, x_5\}, x_3\{x_1, x_2\}\{x_3, x_4, x_5\}\}.$$

We have $|Shad(J_2)| = 19$. On the other hand $|J_3| = |S_3| - |(S/J)_3| = 35-10 = 25$, so we must add 6 new generators, from a list of 16 monomials, at $Shad(J_2)$ to get J_3.

Since x_5 is a strong Lefschetz element for S/J it follows that we cannot add any monomial of the form $x_5 \cdot m$, where m is nonzero in $(S/J)_2$ because, in that case, the map $(S/J)_2 \stackrel{\cdot x_5}{\to} (S/J)_3$ will be no longer injective. But there are $|(S/J)_2| = 10$ such monomials m. Therefore, we must add the remaining 6 monomials, $x_3^3, x_3^2x_4, x_1x_4^2, x_2x_4^2, x_3x_4^2, x_4^3$. Thus

$$J_3 = \{\{x_1, x_2, x_3, x_4\}^3, x_5(\{x_1, x_2, x_3\}^2 \setminus \{x_3^2\})\}. Therefore:$$

$$Shad(J_3) = \{\{x_1, x_2, x_3, x_4\}^4, x_5\{x_1, x_2, x_3, x_4\}^3, x_5^2(\{x_1, x_2, x_3\}^2 \setminus \{x_3^2\})\}.$$

We have $|Shad(J_3)| = 60$ and $|J_4| = |S_4| - |(S/J)_4| = 70 - 5 = 65$. So we need to add 5 new generators at $Shad(J_3)$ to get J_4. If we add a monomial which is divisible by x_5^3 we obtain a contradiction from the fact that the map $(S/J)_1 \stackrel{\cdot x_5^3}{\to} (S/J)_4$ is no longer injective. Therefore, we must add:

$$x_3^2x_5^2, x_1x_4x_5^2, x_2x_4x_5^2, x_3x_4x_5^2, x_4^2x_5^2,$$

and so

$$J_4 = \{\{x_1, x_2, x_3, x_4\}^4, x_5\{x_1, x_2, x_3, x_4\}^3, x_5^2\{x_1, x_2, x_3, x_4\}^2\}.$$

$$So\ \ Shad(J_4) = \{\{x_1, x_2, x_3, x_4\}^5, \cdots, x_5^3\{x_1, x_2, x_3, x_4\}^2\}.$$

We have $|J_5| - |Shad(J_4)| = 4$, so we must add 4 new generators at $Shad(J_4)$ to get J_5. Since J is strongly stable, these new generators are:

$$x_5^4x_1, x_5^4x_2, x_5^4x_3, x_5^4x_4.$$

Therefore $J_5 = \{\{x_1, \dots, x_5\}^5 \setminus \{x_5^5\}\}$. Finally, we must add x_5^6 to $Shad(J_5)$ in order to obtain J_6. We proved the following proposition, with the help of [4, Theorem 1.2] and Theorem 1.5.

Proposition 2.2. *If $I = (f_1, f_2, \dots, f_5) \subset K[x_1, \dots, x_5]$ is an ideal generated by a generic (regular) sequence of homogeneous polynomials of degree 2 or if $f_1, f_2, \dots, f_5$ is a regular sequence of homogeneous polynomials of degree 2 with $f_i \in K[x_i, \dots, x_5]$ for $i = 1, \dots, 5$ then $J = Gin(I)$ the generic initial ideal of I with respect to the revlex order is:*

$$J = (x_1^2,\ x_1x_2,\ x_2^2,\ x_1x_3,\ x_2x_3,\ x_3^3,\ x_3^2x_4,\ x_1x_4^2,\ x_2x_4^2,\ x_3x_4^2,\ x_4^3,$$

$$x_3^2x_5^2,\ x_1x_4x_5^2,\ x_2x_4x_5^2,\ x_3x_4x_5^2,\ x_4^2x_5^2,\ x_5^4x_1,\ x_5^4x_2,\ x_5^4x_3,\ x_5^4x_4,\ x_5^6)$$

The case $n = 4$, $d = 3$. We suppose that S/I has (SLP), so x_4 is a strong Lefschetz element for S/J. Also, we suppose that J_3 is revlex.

We have $H(S/J,t) = (1+t+t^2)^4 = (1+2t+3t^2+2t3+t^4)^2 =$

$$= 1 + 4t + 10t^2 + 16t^3 + 19t^4 + 16t^5 + 10t^6 + 4t^7 + t^8.$$

Since $|J_3| = 4$ and J_3 is revlex, it follows that $J_3 = \{x_1,x_2\}^3$. Therefore, we have $Shad(J_3) = \{\{x_1,x_2\}^4, \{x_1,x_2\}^3\{x_3,x_4\}\}$. Since $|J_4| - |Shad(J_3)| = 4$, we must add 4 new generators to $Shad(J_3)$ to obtain J_4. Since x_4 is a strong Lefschetz element for S/J we cannot add any monomial of the form $x_4 \cdot m$, where $m \neq 0$ in J_3. Therefore, since J is strongly stable, we have to choose 3 monomials from the list $x_3^2\{x_1,x_2\}^2, x_3^3\{x_1,x_2\}, x_3^4$. There are two different chooses: either we add (I) $x_3^2\{x_1,x_2\}^2$, either (II) $x_3^2x_1\{x_1,x_2,x_3\}$.

In the case (I), we get $J_4 = \{\{x_1,x_2\}^4, \{x_1,x_2\}^3\{x_3,x_4\}, x_3^2\{x_1,x_2\}^2\}$, so

$$Shad(J_4) =$$

$$\{\{x_1,x_2\}^5, \{x_1,x_2\}^4\{x_3,x_4\}, \{x_1,x_2\}^3\{x_3,x_4\}^2, x_3^2\{x_3,x_4\}\{x_1,x_2\}^2\}.$$

Since $|J_5| - |Shad(J_4)| = 40 - 34 = 6$, we need to add 6 new generators at $Shad(J_4)$ to get J_5. Since x_4 is a strong Lefschetz element for S/J we cannot add any monomial of the form $x_4^2 m$, where m is a nonzero monomial in J_3. So, we must add: $x_3^4\{x_1,x_2,x_3\}, x_4x_3^3\{x_1,x_2,x_3\}$. Thus

$$J_5 = \{\{x_1,x_2,x_3\}^5, x_4\{x_1,x_2,x_3\}^4, x_4^2\{x_1,x_2\}^3\}.$$

In the case (II), we have $J_4 = \{\{x_1,x_2\}^4, \{x_1,x_2\}^3\{x_3,x_4\}, x_1x_3^2\{x_1,x_2,x_3\}\}$, so $Shad(J_4)$ is the set $\{\{x_1,x_2\}^5, \{x_1,x_2\}^4\{x_3,x_4\},$ $\{x_1,x_2\}^3\{x_3,x_4\}^2, x_3^2x_1\{x_3,x_4\}\{x_1,x_2\}, x_3^3x_1\{x_3,x_4\}\}$. Since $|J_5| - |Shad(J_4)| = 40 - 34 = 6$, we must add 6 new generators at $Shad(J_4)$ to get J_5. Since x_4 is a strong-Lefschetz element for S/J, we cannot add any monomial of the form $x_4^2 m$, where $m \neq 0$ in J_3. So, we must add: $x_3^3x_2^2, x_3^4x_2, x_3^5, x_4x_3^2x_2^2, x_4x_3^3x_2, x_4x_3^4$. Thus

$$J_5 = \{\{x_1,x_2,x_3\}^5, x_4\{x_1,x_2,x_3\}^4, x_4^2\{x_1,x_2\}^3\},$$

the same as in the case (I). Thus, in both cases (I) and (II), we get:

$$Shad(J_5) = \{\{x_1,x_2,x_3\}^6, x_4\{x_1,x_2,x_3\}^5, x_4^2\{x_1,x_2,x_3\}^4, x_4^3\{x_1,x_2\}^3\}.$$

Since $|Shad(J_5)| = |S_6| - 16$ and $|J_6| = |S_6| - 10$, we must add 6 new generators to $Shad(J_5)$ in order to obtain J_6. Since x_4 is a strong-Lefschetz element for S/J, these new generators are not divisible by x_4^4. So, we add

$$x_4^3x_3\{x_1,x_2\}^2, x_4^3x_3^2\{x_1,x_2\}, x_4^3x_3^3$$

and thus,

$$J_6 = \{\{x_1,x_2,x_3\}^6, x_4\{x_1,x_2,x_3\}^5, x_4^2\{x_1,x_2,x_3\}^4, x_4^3\{x_1,x_2,x_3\}^3\}. \textit{So}$$

$$Shad(J_6) = \{\{x_1, x_2, x_3\}^7, x_4\{x_1, x_2, x_3\}^6, \ldots, x_4^4\{x_1, x_2, x_3\}^3\}.$$

$|S_7| - |Shad(J_6)| = 6+4 = 10$ and $|S_7| - |J_7| = 4$, so we must add 6 new generators at $Shad(J_6)$ to get J_7. Using the same argument, these new generators must be $x_4^5\{x_1, x_2, x_3\}^2$ and therefore

$$J_7 = \{\{x_1, x_2, x_3\}^7, x_4\{x_1, x_2, x_3\}^6, \ldots, x_4^5\{x_1, x_2, x_3\}^2\}.$$

We get

$$Shad(J_7) = \{\{x_1, x_2, x_3\}^8, x_4\{x_1, x_2, x_3\}^7, \ldots, x_4^6\{x_1, x_2, x_3\}^2\}.$$

Since $|S_8| - |Shad(J_7)| = 4$ and $|S_8| - |J_8| = 1$, we must add 3 new generators at $Shad(J_7)$ in order to get J_8. Since x_4 is strong-Lefschetz, these new generators are $x_4^7\{x_1, x_2, x_3\}$, so $J_8 = \{x_1, x_2, x_3, x_4\}^8 \setminus \{x_4^8\}$. Finally, we must add x_4^9 to $Shad(J_8)$ in order to obtain J_9. We proved the following proposition, with the help of [4, Theorem 1.2] and Theorem 1.5.

Proposition 2.3. *If $I = (f_1, f_2, f_3, f_4) \subset K[x_1, x_2, x_3, x_4]$ is an ideal generated by a generic (regular) sequence of homogeneous polynomials of degree 3 or if f_1, f_2, f_3, f_4 is a regular sequence of homogeneous polynomials of degree 3 with $f_i \in k[x_i, \ldots, x_4]$, for $i = 1, \ldots, 4$, then $J = Gin(I)$ the generic initial ideal of I with respect to the revlex order has one of the following forms:*

$$(I) \quad J = (\{x_1, x_2\}^3,\ x_3^2\{x_1, x_2\}^2,\ x_3^4\{x_1, x_2, x_3\},\ x_4x_3^3\{x_1, x_2, x_3\},$$
$$x_4^3x_3\{x_1, x_2\}^2, x_4^3x_3^2\{x_1, x_2\}, x_4^3x_3^3,\ x_4^5\{x_1, x_2, x_3\}^2,\ x_4^7\{x_1, x_2, x_3\},\ x_4^9)$$

$$(II)\ J = (\{x_1, x_2\}^3,\ x_3^2x_1\{x_1, x_2, x_3\},\ x_3^3x_2^2,\ x_3^4x_2,\ x_3^5,\ x_4x_3^2x_2^2,\ x_4x_3^3x_2,\ x_4x_3^4,$$
$$x_4^3x_3\{x_1, x_2\}^2, x_4^3x_3^2\{x_1, x_2\}, x_4^3x_3^3,\ x_4^5\{x_1, x_2, x_3\}^2,\ x_4^7\{x_1, x_2, x_3\},\ x_4^9)$$

Remark 2.4. *It seems Conca-Herzog-Hibi noticed in [3], page 838, that, if f_1, f_2, f_3, f_4 is a generic sequence of homogeneous polynomials of degree 3 then the generic initial ideal J has the form (I), and $J = Gin(x_1^3, x_2^3, x_3^3, x_4^3)$ has the form (II).*

References

[1] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Revised Edition, Cambridge, 1996.

[2] M. Cimpoeaş, Generic initial ideal for complete intersections of embedding dimension three with strong Lefschetz property, Bull.Math Soc.Sc.Math.Roumanie, 50(98),no 1,(2007), 33-66.

[3] A. Conca, J. Herzog, T. Hibi, Rigid resolutions and big Betti numbers, Comment. Math. Helv. 79 (2004), 826-839.

[4] A. Conca, J. Sidman, Generic initial ideals of points and curves, Preprint 2005, arXiv:math.AC/0402418.

[5] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer 1994.

[6] T. Harima, Migliore, Nagel, Watanabe, The weak and strong Lefschetz properties for Artinian K-algebras, J. Algebra 262 (2003), no. 1, 99-126.

[7] T. Harima, J. Watanabe, The finite free extension of K-algebras with the strong Lefschetz property, Rend. Sem. Mat. Univ. Padova, 110 (2003), 119-146.

[8] J. Herzog, D. Popescu, The strong Lefschetz property and simple extensions, Preprint.

[9] K. Pardue, Generic polynomials, Preprint 1999.

[10] D. Popescu, The strong Lefschetz property and certain complete intersection extensions, Bull.Math Soc.Sc.Math.Roumanie, 48(96),no 4,(2005),421-431.

[11] D. Popescu, M. Vladoiu, Strong Lefschetz property on algebras of embedding dimension three, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 49(97) (2006), no. 1, 75-86.

[12] A. Wiebe, The Lefschetz property for componentwise linear ideals and Gotzmann ideals, Comm. Algebra 32 (2004), no. 12, 4601-4611.

Received: 12.02.2007.

Institute of Mathematics
of the Romanian Academy
Bucharest, Romania
E-mail: `mircea.cimpoeas@imar.ro`