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Properties of a sequence generated by positive integers

by
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Abstract

For every positive integer m let b_m be the least positive integer such that mb_m is a square. We show that $\limsup_{n\to\infty}(b_{n+1}-b_n)=+\infty$, $\liminf_{n\to\infty}(b_{n+1}-b_n)=-\infty$ and $\sum_{i=1}^{n}\frac{1}{b_i}=\sqrt{n}\frac{\zeta(3/2)}{\zeta(3)}+O(\ln n)$.

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For every positive integer m we shall denote by $b_m \geq 1$ the least positive integer such that mb_m is a square.

For instance $b_1 = 1$, $b_2 = 2$, $b_3 = 3, b_4 = 1$, $b_{12} = 3$, $b_{60} = 15$.

It is very easy to prove that for every squarefree integer f there are infinitely many n (for instance take $n = k^2 f$) such that $b_n = f$.

In this paper we use the following results:

Denote by $P(k)$ the greatest prime that divides k. For instance $P(12) = 3$, $P(30) = 5$. Deshouillers J.M. and Iwaniec H proved in [1] that

Theorem A

$$P(k^2+1) > k^{\frac{6}{5}} \text{ for infinitely many positive integers } k.$$

Denote by $Q(x)$ the number of squarefree positive integers less or equal than x. In [2] Gegenbauer proved

Theorem B

$$Q(x) = \frac{6}{\pi^2}x + O(\sqrt{x}).$$

We will also use a special case of a classical formula:

Theorem C

If $h \in C^1, g$ continous and $G(x) = \sum_{\substack{f \leq x \\ f \text{ squarefree}}} g(f)$, then

$$\sum_{\substack{f \leq x \\ f \text{ squarefree}}} h(f)g(f) = h(x)G(x) - \int_1^x h'(t)G(t)dt.$$

We prove now some properties of the sequence $(b_n)_{n\geq 1}$.

Theorem 1.

$$\limsup_{n\to\infty}(b_{n+1} - b_n) = +\infty \tag{1}$$

$$\liminf_{n\to\infty}(b_{n+1} - b_n) = -\infty \tag{2}$$

Proof: Let k be a positive integer such that $k^2 + 1$ verifies **Theorem A** and

$$p = P(k^2 + 1) > k^{\frac{6}{5}}.$$

Then

$$p^2 > k^{\frac{12}{5}} > k^2 + 1, \text{ so } p^2 \nmid k^2 + 1.$$

Let $n = k^2$ be such that $n + 1 = k^2 + 1$ be as stated before. Then there exists $p \geq k^{\frac{6}{5}}$ such that $P(n + 1) = p$, so $b_n = 1$ and $b_{n+1} \geq p$.

Since there are infinitely many numbers $n = k^2$ as above we have:

$$\limsup_{n\to\infty}(b_{n+1} - b_n) = +\infty.$$

In order to prove (2), take $n = k^4 - 1$ such that there exists a prime p so $p|k^2 + 1$, $p > k^{\frac{6}{5}}$ leading to $p^2 \nmid k^2 + 1$. We have $n = (k^2 + 1)(k^2 - 1)$, so $b_n \geq p$ and $b_{n+1} = 1$, hence

$$\liminf_{n\to\infty}(b_{n+1} - b_n) = -\infty$$

because there are infinitely many k with that property. □

Now $b_1, \ldots, b_n$ are squarefree and do not exceed n. Given $f \leq n$ squarefree we have

$$1^2 f \leq 2^2 f \leq \ldots \leq k_f^2 f \leq n < (k_f + 1)^2 f,$$

where $k_f = \left[\sqrt{\frac{n}{f}}\right]$. Therefore, the value f appears in the sequence $b_1, \ldots, b_n$ exactly $\left[\sqrt{\frac{n}{f}}\right]$ times. Hence,

$$B_n = b_1 b_2 \cdot \ldots \cdot b_n = \prod_{\substack{1 \leq f \leq n \\ f \text{squarefree}}} f^{\left[\sqrt{\frac{n}{f}}\right]}.$$

Theorem 2. *We have the relation*

$$\ln B_n = n \ln n + O(n).$$

Proof: We have

$$\ln B_n = \sum_{\substack{1\leq f\leq n\\ f\text{squarefree}}} \left[\sqrt{\frac{n}{f}}\right] \ln f.$$

In order to compute this sum we will evaluate $\left[\sqrt{\frac{n}{f}}\right]$. We have $\left[\sqrt{\frac{n}{f}}\right] = k$ if $\frac{n}{(k+1)^2} < f \leq \frac{n}{k^2}$ for any given positive integer k between 1 and $[\sqrt{n}]$. Then

$$\ln B_n = \sum_{\frac{n}{4}<f\leq n} \ln f + \sum_{\frac{n}{9}<f\leq\frac{n}{4}} 2\ln f + \ldots + \sum_{\frac{n}{[\sqrt{n}]^2}<f\leq\frac{n}{([\sqrt{n}]-1)^2}} [\sqrt{n}] \ln f$$

so

$$\ln B_n = S(n) + S\left(\frac{n}{2^2}\right) + \ldots + S\left(\frac{n}{[\sqrt{n}]^2}\right) \tag{3}$$

where $S(n) = \sum_{\substack{1\leq f\leq n\\ f\text{squarefree}}} \ln f$.

In order to evaluate $S(n)$ we use **Theorem C** with

$$h(t) = \ln t \text{ and } g(t) = 1,$$

and obtain

$$G(n) = \sum_{\substack{f\leq n\\ f\text{ squarefree}}} 1 = Q(n) = \frac{6}{\pi^2}n + \alpha(n),$$

where $\alpha(n) = O(\sqrt{n})$.

Therefore,

$$S(n) = \sum_{f\leq n} \ln f = \ln n \left(\frac{6}{\pi^2}n + \alpha(n)\right) - \int_1^n \frac{\frac{6}{\pi^2}t + \alpha(t)}{t} dt =$$

$$= \frac{6}{\pi^2}n(\ln n - 1) + O(\sqrt{n}\ln n) + O(\sqrt{n}) = \frac{6}{\pi^2}n\ln n + O(n).$$

Thus we obtain

$$S(n) = \frac{6}{\pi^2}n\ln n + x_n,$$

where $x_n = O(n)$. We use this relation in (3) and obtain

$$\ln B_n = \frac{6}{\pi^2}\left(n\ln n + \frac{n}{2^2}\ln\frac{n}{2^2} + \ldots + \frac{n}{[\sqrt{n}]^2}\ln\frac{n}{[\sqrt{n}]^2}\right) + \sum_{k=1}^{[\sqrt{n}]} x_{\frac{n}{k^2}}.$$

Therefore

$$\ln B_n = \frac{6n\ln n}{\pi^2}\left(1+\frac{1}{2^2}+\ldots+\frac{1}{[\sqrt{n}]^2}\right)+$$

$$+\frac{6n}{\pi^2}\left(-\frac{1}{2^2}\ln 2^2-\frac{1}{3^2}\ln 3^2-\ldots-\frac{1}{[\sqrt{n}]^2}\ln[\sqrt{n}]^2\right)+\sum_{k=1}^{[\sqrt{n}]} x_{\frac{n}{k^2}} \tag{4}$$

We evaluate every part of this sum.
For the first part we have

$$\frac{\pi^2}{6}-\frac{1}{1^2}-\ldots-\frac{1}{[\sqrt{n}]^2}=\frac{1}{([\sqrt{n}]+1)^2}+\ldots<\frac{1}{[\sqrt{n}]}-\frac{1}{[\sqrt{n}]+1}+\ldots<\frac{1}{[\sqrt{n}]}.$$

Therefore,

$$\frac{1}{1^2}+\ldots+\frac{1}{[\sqrt{n}]^2}=\frac{\pi^2}{6}+O\left(\frac{1}{\sqrt{n}}\right). \tag{5}$$

Secondly we have

$$\frac{1}{2^2}\ln 2+\frac{1}{3^2}\ln 3+\ldots+\frac{1}{[\sqrt{n}]^2}\ln[\sqrt{n}]=\sum_{k=1}^{[\sqrt{n}]}\frac{\ln k}{k^2}<\sum_{k=1}^{[\sqrt{n}]}\frac{\sqrt{k}}{k^2}<\sum\frac{1}{k^{3/2}}$$

which converges, so

$$\frac{6n}{\pi^2}\left(\frac{1}{2^2}\ln 2^2+\frac{1}{3^2}\ln 3^2+\ldots+\frac{1}{[\sqrt{n}]^2}\ln[\sqrt{n}]^2\right)=O(n). \tag{6}$$

Thirdly, because $x_n = O(n)$ we have

$$kn < x_n < hn,$$

$$k\frac{n}{2^2} < x_{\frac{n}{2^2}} < h\frac{n}{2^2},$$

$$\ldots$$

We sum all these relations and obtain

$$knM < kn\left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{[\sqrt{n}]^2}\right)<\sum_{k=1}^{[\sqrt{n}]}x_{\frac{n}{k^2}}<hn\left(1+\frac{1}{2^2}+\ldots\right)<hn\frac{\pi^2}{6}.$$

Therefore, we have

$$\sum_{k=1}^{[\sqrt{n}]}x_{\frac{n}{k^2}}=O(n). \tag{7}$$

Now we take (5), (6) and (7) back in (4) and obtain

$$\ln B_n=\frac{6}{\pi^2}n\ln n\left(\frac{\pi^2}{6}+O\left(\frac{1}{\sqrt{n}}\right)\right)+O(n),$$

so

$$\ln B_n = n \ln n + O(\sqrt{n} \ln n) + O(n) = n \ln n + O(n).$$

□

Next we prove a result concerning the sum of the inverses of b_i:

Theorem 3. *We have the relation*

$$\sum_{i=1}^{n} \frac{1}{b_i} = \frac{\zeta(3/2)}{\zeta(3)} \sqrt{n} + O(\ln n).$$

Proof: We have

$$\sum_{i=1}^{n} \frac{1}{b_i} = \sum_{f \text{ squarefree}} \frac{1}{f} \left[\sqrt{\frac{n}{f}} \right] = \sum \frac{1}{f} \left(\sqrt{\frac{n}{f}} + O(1) \right),$$

also written as

$$\sum_{i=1}^{n} \frac{1}{b_i} = \sqrt{n} \sum_{\substack{f \le n \\ f \text{ squarefree}}} \frac{1}{f^{\frac{3}{2}}} + O\left(\sum_{\substack{f \le n \\ f \text{ squarefree}}} \frac{1}{f} \right). \tag{8}$$

Noticing that

$$0 < \sum_{\substack{f \le n \\ f \text{ squarefree}}} \frac{1}{f} < \sum_{i=1}^{n} \frac{1}{i} = O(\ln n),$$

we obtain

$$\sum_{\substack{f \le n \\ f \text{ squarefree}}} \frac{1}{f} = O(\ln n). \tag{9}$$

Let $f_1 < f_2 < \ldots$ be the sequence of all squarefree numbers. We have

$$S = \sum_{k=1}^{\infty} \frac{1}{f_k^{3/2}} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^{3/2}} \right) = \frac{\prod_{p \text{ prime}} \left(1 - \frac{1}{p^3} \right)}{\prod_{p \text{ prime}} \left(1 - \frac{1}{p^{3/2}} \right)}.$$

As for $\operatorname{Re} z > 1$ we have

$$\zeta(z) = \frac{1}{\prod_{p \text{ prime}} \left(1 - \frac{1}{p^z} \right)},$$

it follows that

$$S = \frac{\zeta(3/2)}{\zeta(3)}.$$

Now we evaluate

$$\left| \frac{\zeta(3/2)}{\zeta(3)} - \sum_{\substack{f\le n\\ f \text{ squarefree}}} \frac{1}{f^{3/2}} \right| = \sum_{\substack{f\ge n+1\\ f\text{squarefree}}} \frac{1}{f^{3/2}} < \frac{1}{(n+1)^{3/2}} + \frac{1}{(n+2)^{3/2}} + \dots .$$

Because

$$\frac{1}{(k+1)^{3/2}} < 2\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) \quad \text{for large enough } k, \text{ we obtain:}$$

$$\sum_{k=n}^{\infty} \frac{1}{(k+1)^{3/2}} < \frac{2}{\sqrt{n}}\,.$$

Therefore,

$$\left| \frac{\zeta(3/2)}{\zeta(3)} - \sum_{\substack{f\le n\\ f \text{ squarefree}}} \frac{1}{f^{3/2}} \right| < \frac{2}{\sqrt{n}},$$

$$\text{so} \sum_{\substack{f\le n\\ f \text{ squarefree}}} \frac{1}{f^{3/2}} = \frac{\zeta(3/2)}{\zeta(3)} + O\left(\frac{1}{\sqrt{n}}\right).$$

Back in (8) and taking (9) into account we obtain

$$\sum_{i=1}^{n} \frac{1}{b_i} = \sqrt{n}\left(\frac{\zeta(3/2)}{\zeta(3)} + O\left(\frac{1}{\sqrt{n}}\right)\right) + O(\ln n),$$

$$\text{so} \sum_{i=1}^{n} \frac{1}{b_i} = \sqrt{n}\,\frac{\zeta(3/2)}{\zeta(3)} + O(\ln n).$$

□

Remark 1 We have $\dfrac{\zeta(3/2)}{\zeta(3)} = 2,1732543...$

Remark 2 The series

$$\sum_{n=1}^{\infty} \frac{1}{b_n^{\alpha}}$$

diverges for any α because there exist infinitely many $b_n = 1$ corresponding to every square $n = k^2$.

Remark 3 Using Mirsky's theorem [3], it is proved in [4] that for the sequence $(f_n)_n$ there exist infinitely many numbers m for which

$$f_{m+1} - f_m = 1 \text{ and } f_m - f_{m-1} = 1.$$

For these m, $f_m = b_n$ so there exist infinitely many numbers n for which

$$b_{n+1} - b_n = 1 \text{ and } b_n - b_{n-1} = 1.$$

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