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On a certain ring construction

by
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Abstract

Let D be an integral domain, K a subset of D and $(P_k)_{k\in K}$ a family of prime ideals of D such that $j-k$ is invertible modulo P_k for all $j,k\in K, j\neq k$. Beginning with this data, we construct an overring E of the polynomial ring $D[x]$ such that every ideal P_k is contained in a principal prime ideal of E.

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In [3], the following ring construction has been given. Let A be a domain containing a field L. Assume that the set of maximal ideals of A can be indexed by some subset K of L, say $Max(A)=(M_k)_{k\in K}$. Consider the domain

$$B = A[x, \frac{f}{(x-k)^n},\ f\in M_k, k\in K, n\geq 1]$$

where x is an indeterminate. In [3], this construction is iterated to produce a domain in which every proper finitely generated ideal I is contained in $\cap_{n\geq 1} J^n$ for some proper finitely generated ideal J.

The aim of this paper is to study the construction above in a slightly more general setting. Starting with a domain A and a suitably indexed family of prime ideals $\mathcal{P}$, we construct an overring $A^{\mathcal{P}}$ of the polynomial ring $A[x]$ such that every member of $\mathcal{P}$ is contained in a principal prime ideal of $A^{\mathcal{P}}$. We obtain a

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description of the prime ideals of $A^{\mathcal{P}}$. When $\mathcal{P}$ consists of maximal ideals and A is integrally closed, we show that $A^{\mathcal{P}}$ is integrally closed as well.

As an application, some interesting domains are obtained. For instance, the ring $\mathbf{C}[x, y, (y-a)/(x-a)^n,\ a \in \mathbf{C}, n \geq 1]$ where x, y are indeterminates, is a two-dimensional strongly discrete Prüfer domain. On the other hand, the ring $\mathbf{Z}[x, p/(x-p)^n,\ p$ prime number, $n \geq 1]$ is a two-dimensional integrally closed domain whose localizations are either strongly discrete valuation domains of rank ≤ 2 or two-dimensional regular local rings. In particular, it is not a Prüfer domain.

Throughout this paper, all rings are commutative and unitary. Undefined terminology and notation is standard, as is [5] or [8]. A local ring is always Noetherian.

Let A be a domain, K a nonempty subset of A and $\mathcal{P} = (P_k)_{k \in K}$ a family of prime ideals of A. Assume that

$$(\sharp) \qquad P_k + (j-k)A = A \quad \text{for all } \ j, k \in K, j \neq k.$$

Note that this condition is satisfied if K is contained in some subfield of A, or if every P_k is a maximal ideal and $j \in P_k$ if and only if $j = k$. Let x be an indeterminate over A. For every $a \in A$, we denote

$$(x-a)^{-\infty} = \{\frac{1}{(x-a)^n} \mid n \geq 0\}.$$

The aim of this paper is to study the domain

$$(\sharp\sharp) \qquad A^{\mathcal{P}} := A[x, P_k(x-k)^{-\infty},\ k \in K].$$

Recall that a domain D is said to be *Archimedean* if $\cap_{n \geq 1} a^n D = 0$ for each nonunit $a \in D$. It is well-known that a completely integrally closed domain or a domain satisfying the ascending chain condition for the principal ideals is Archimedean. Hence a Noetherian domain is Archimedean. The next result collects some general properties of the construction $A^{\mathcal{P}}$.

Theorem 1. *Let A be a domain, K a nonempty subset of A, $\mathcal{P} = (P_k)_{k \in K}$ a family of prime ideals of A satisfying condition ($\sharp$) and $A^{\mathcal{P}}$ the ring defined in ($\sharp\sharp$). Fix an element $k \in K$. Then the following hold*

(*a*) *$A^{\mathcal{P}}/(x-k)A^{\mathcal{P}}$ is canonically isomorphic to A/P_k.*

(*b*) *$A^{\mathcal{P}}_{(x-k)} = A[x, P_k(x-k)^{-\infty}]_{(x-k)}$.*

(*c*) *$(A^{\mathcal{P}}/P_k(x-k)^{-\infty}A^{\mathcal{P}})_{(x-k)} \simeq F_k[x, 1/(x-j),\ j \in K \setminus \{k\}]$, where F_k is the quotient field of A/P_k.*

(*d*) *$\bigcap_{n \geq 1}(x-k)^n A^{\mathcal{P}} = P_k(x-k)^{-\infty}A^{\mathcal{P}}_{(x-k)} \cap A^{\mathcal{P}}$.*

(*e*) *The domain $A^{\mathcal{P}}$ is not Archimedean. So it is neither Noetherian nor completely integrally closed.*

(*f*) *The units of $A^{\mathcal{P}}$ are exactly the units of A.*

Proof: Without lose of generality, we may assume that $0 \in K$ and $k = 0$. Set $B = A^{\mathcal{P}}$ and $P = P_0$.

(a). We have to show that the canonical morphism $\alpha : A \to B/xB$ is surjective and its kernel is P. Let $f \in P$ and $n \geq 1$. Since $f/x^n = x(f/x^{n+1})$, the image of f/x^n in B/xB is zero. Now, let $0 \neq j \in K$, $f \in P_j$ and $n \geq 0$ and set $g = f/(x-j)^n$. As $P \subseteq xB$, condition ($\sharp$) implies that j is invertible modulo xB. So from $(x-j)^n g = f$, we deduce that the image of g in B/xB is $\alpha((-j)^{-n} f)$. It is now clear that α is surjective.

Clearly, $P \subseteq xB \cap A = ker(\alpha)$. Conversely, let $a \in ker(\alpha)$. Then $a = xg$ for some $g \in B$. Consider the subring E of B, $E = A[x] + P[x^{-1}]$. There exist $j_1, ..., j_t \in K \setminus \{0\}$, not necessarily distinct, and $h \in E$ such that $(x - j_1) \cdots (x - j_t)a = xh$. Equating the degree zero terms we get $j_1 \cdots j_t a \in P$. By condition ($\sharp$), the element $j_1 \cdots j_t$ is invertible modulo P, so $a \in P$. Thus $ker(\alpha) = P$.

$(b), (c)$. Set $Q = Px^{-\infty}B$ and let E be as in the proof of (a). Let W be the multiplicative set of B generated by $\{x - j |\ j \in K \setminus \{0\}\}$. By condition ($\sharp$), W is disjoint from xB and xE. So $B_{(x)} = (B_W)_{(x)} = E_{(x)}$. Consequently,

$$(B/Q)_{(x)} \simeq (E/Q)_{(x)} \simeq (A/P)[x]_{(x)} = F[x]_{(x)}$$

where F is the quotient field of A/P.

(d). Set $Q = Px^{-\infty}B$. $\cap_{n\geq 1} x^n B$ contains Q and it is the unique prime ideal directly below xB. By (c), it follows that $\cap_{n \geq 1} x^n B = QB_{xB} \cap B$.

(e) follows from (d).

(f). Let W be the multiplicative set of $A[x]$ generated by $\{x - j |\ j \in K\}$. Then $B \subseteq A[x]_W$ and the saturation of W is $U(A)W$. Since each $x - j$ is a nonunit of B, it follows that $U(B) = U(A)$. □

Let W be the multiplicative set of $A^{\mathcal{P}}$ generated by $\{x - k |\ k \in K\}$. The prime ideals of $A^{\mathcal{P}}$ which are disjoint from W are in a one to one correspondence with the prime ideals of $A^{\mathcal{P}}_W$. It is easy to see that $A^{\mathcal{P}}_W = A[x, 1/(x-k),\ k \in K]$. So we have the following consequence of Theorem 1.

Corollary 2. *Let A, $\mathcal{P}$ and $A^{\mathcal{P}}$ be as in Theorem 1.*

(*a*) *The prime ideals of $A^{\mathcal{P}}$ have one of the following two forms:*

(*i*) *$(Q, x-k)A^{\mathcal{P}}$ where Q is a prime ideal of A containing P_k,*

(*ii*) *$N \cap A^{\mathcal{P}}$ where N is a prime ideal of $A[x, 1/(x-k),\ k \in K]$.*

(*b*) *If every ideal P_k is maximal, then a localization of $A^{\mathcal{P}}$ at some maximal ideal is either of the form $A^{\mathcal{P}}_{(x-k)}$ or a localization of $A[x, 1/(x-k),\ k \in K]$, so*

$$A^{\mathcal{P}} = \cap_{k \in K} A^{\mathcal{P}}_{(x-k)} \cap A[x, 1/(x-k),\ k \in K].$$

(c) If every ideal P_k is maximal and A is integrally closed, then $A^{\mathcal{P}}$ is integrally closed.

Proof: (a) was proved in the paragraph before the corollary and (b) follows from (a) and part (a) of Theorem 1.

(c). Since A is integrally closed, $A[x, 1/(x-j),\ j \in K]$ is integrally closed, because it is a fraction ring of $A[x]$. By (b), it suffices to show that $A^{\mathcal{P}}_{(x-k)}$ is integrally closed for each $k \in K$. By part (b) of Theorem 1, $A^{\mathcal{P}}_{(x-k)} = A[x, P_k(x-k)^{-\infty}]_{(x-k)}$. The following lemma applies. □

Lemma 3. *Let D be an integrally closed domain, P a prime ideal of D and x an indeterminate. Then the domain $D^P := D[x, Px^{-\infty}]$ is integrally closed.*

Proof: Since D is integrally closed, so is $D[x, x^{-1}]$. Hence the integral closure of D^P is contained in $D[x, x^{-1}]$. Note that D^P is a graded subring of $D[x, x^{-1}]$. By [10, Theorem 11, page 157], it suffices to see that whenever a monomial ax^{-n} is integral over D^P, where $a \in D$ and $n > 0$, it follows that $a \in P$. Let $(ax^{-n})^k + f_{k-1}(ax^{-n})^{k-1} + \cdots + f_0 = 0$ be an integral dependence relation of ax^{-n} over D^P. Computing the coefficient of x^{-nk}, we see that $a^k \in P$, so $a \in P$. □

In the second part of this paper, we consider some particular cases. Let A be the polynomial ring $\mathbf{C}[y]$ and $\mathcal{P} = (P_a)_{a \in \mathbf{C}}$ where P_a is the prime ideal $(y-a)\mathbf{C}[y]$. Then

$$\mathbf{C}[y]^{\mathcal{P}} = \mathbf{C}[x, y, (y-a)(x-a)^{-\infty},\ a \in \mathbf{C}].$$

Recall that D is said to be a *Prüfer domain* if D_M is a valuation domain for every maximal ideal M of D. A Prüfer domain D is *strongly discrete* if each nonzero prime ideal of D is not idempotent. For basic facts about Prüfer domains the reader may consult [4] or [5].

Theorem 4. *In the setup above, let $a \in \mathbf{C}$. Then*

(a) $\mathbf{C}[y]^{\mathcal{P}}/(x-a) \simeq \mathbf{C}$.

(b) $\mathbf{C}[y]^{\mathcal{P}}_{(x-a)}$ is a rank two valuation domain.

(c) $\mathbf{C}[y]^{\mathcal{P}}/((y-a)(x-a)^{-\infty}) \simeq \mathbf{C}[x]_{(x)}$.

(d) The nonzero prime ideals of $\mathbf{C}[y]^{\mathcal{P}}$ have one of the following three forms, the maximal ideals being those in (a) and (c),

(i) $(x-b)\mathbf{C}[y]^{\mathcal{P}}$ with $b \in \mathbf{C}$,

(ii) $(y-b)(x-b)^{-\infty}\mathbf{C}[y]^{\mathcal{P}}$ with $b \in \mathbf{C}$,

(iii) $f\mathbf{C}(x)[y] \cap \mathbf{C}[y]^{\mathcal{P}}$ where $f \in \mathbf{C}(x)[y] \setminus \mathbf{C}[y]$ is a monic irreducible polynomial.

(e) If $P = f\mathbf{C}(x)[y] \cap \mathbf{C}[y]^{\mathcal{P}}$ where $f \in \mathbf{C}(x)[y] \setminus \mathbf{C}[y]$ is a monic irreducible polynomial, then $\mathbf{C}[y]^{\mathcal{P}}_P = \mathbf{C}(x)[y]_{(f)}$.

(f) $\mathbf{C}[y]^{\mathcal{P}}$ is a two-dimensional strongly discrete Prüfer domain.

Proof: Set $B = \mathbf{C}[y]^{\mathcal{P}}$. For (a)-(c) it suffices to settle the case when $a = 0$. Let W be the multiplicative set of B generated by $\{x - b |\ b \in \mathbf{C} \setminus \{0\}\}$.

(a). By part (a) of Theorem 1, $B/xB \simeq \mathbf{C}[y]/(y) \simeq \mathbf{C}$. So xB is a maximal ideal of B.

(b). Note that $B_W = \mathbf{C}[x, yx^{-\infty}]_W$. Since W is disjoint from xB, we obtain $B_{xB} = (B_W)_{(x)}$. Hence $B_{xB} = \mathbf{C}[x, yx^{-\infty}]_{(x)}$. We remark that B_{xB} can be written as a directed union of rings $\cup_{n\geq 0}\mathbf{C}[x, y/x^n]_{(x,y/x^n)}$. Note that every term $\mathbf{C}[x, y/x^n]_{(x,y/x^n)}$ of this union is isomorphic to $\mathbf{C}[x, y]_{(x,y)}$, thus it is a two-dimensional regular local ring. Let $0 \neq f \in \mathbf{C}[x, y]_{(x,y)}$. Then f can be written as $f = y^k(\alpha x^m + yg)$ for some $0 \neq \alpha \in \mathbf{C}$, $g \in \mathbf{C}[x, y]$ and $k, m \geq 0$. Then $f = y^k x^m(\alpha + yx^{-m}g)$ and $\alpha + yx^{-m}g$ is a unit of B_{xB}. Now, it follows easily that B_{xB} is a rank two valuation domain with nonzero prime ideals xB_{xB} and $yx^{-\infty}B_{xB}$.

(c). Set $Q = yx^{-\infty}B$. For $b \in \mathbf{C} \setminus \{0\}$, $B = (y, y - b) \subseteq (Q, x - b)$. So W is disjoint from Q, hence

$$B/Q \simeq B_W/QB_W \simeq \mathbf{C}[x, yx^{-\infty}]_W/(yx^{-\infty}) \simeq \mathbf{C}[x]_W \simeq \mathbf{C}[x]_{(x)}.$$

In particular, Q is a nonmaximal prime ideal. Note that $Q = y\mathbf{C}(x)[y] \cap B$. By (b), Q is the unique nonzero prime ideal contained in xB.

(d). Note that the fraction ring of B with respect to the multiplicative V set generated by $\{x - a |\ a \in \mathbf{C}\}$ is $\mathbf{C}(x)[y]$. Let $P = f\mathbf{C}(x)[y] \cap \mathbf{C}[y]^{\mathcal{P}}$ where $f \in \mathbf{C}(x)[y] \setminus \mathbf{C}[y]$ is a monic irreducible polynomial. By (a), every element of V is invertible modulo P. So $B/P \simeq B_V/PB_V \simeq \mathbf{C}(x)[y]/(f)$, hence P is a maximal ideal. Now, Corollary 2 applies.

(e). Let $P = f\mathbf{C}(x)[y] \cap \mathbf{C}[y]^{\mathcal{P}}$ where $f \in \mathbf{C}(x)[y] \setminus \mathbf{C}[y]$ is a monic irreducible polynomial and let V be as in (d). Then $\mathbf{C}[y]^{\mathcal{P}}_P = (\mathbf{C}[y]^{\mathcal{P}}_V)_P = \mathbf{C}(x)[y]_{(f)}$.

(f). By (b), (d) and (e), the localizations of B at its maximal ideals are strongly discrete valuation domains of rank ≤ 2. So B is a strongly discrete two-dimensional Prüfer domain, cf. [4, Proposition 5.3.5]. □

Recall that a *generalized Dedekind domain* is a strongly discrete Prüfer domain such that each principal ideal of D has finitely many minimal prime ideals.

Remark 5. The ring $B = \mathbf{C}[y]^{\mathcal{P}}$ considered above is not a generalized Dedekind domain. Indeed, let $a \in \mathbf{C}$. Then $x - y = (x - a) - (y - a)$ is divisible in B by $x - a$. Moreover, $(x - y)/(x - a) = 1 - (y - a)/(x - a)$ is invertible in $B_{(x-a)}$, so $(x - y)B_{(x-a)} = (x - a)B_{(x-a)}$. Hence $(x - a)B$ is a minimal prime ideal over $(x - y)B$. Thus $(x - y)B$ has infinitely many minimal primes, so B is not a generalized Dedekind domain.

Remark 6. We may consider the proof of part (b) of Theorem 4 from the following point of view. Let (R, M) be a Nagata regular local ring of dimension two and let (V, N) be a valuation domain that birationally dominates R (that is, V is an overring of R and $N \cap R = M$). The first local quadratic transform of R

along V is defined to be $R_1 = R[M/a]_{N\cap R[M/a]}$ where $a \in M$ is such $MV = aV$, cf. [9, page 141] (see also [6, page 38]). Inductively, the $(i+1)$th local quadratic transform of R along V, R_{i+1}, is defined as the first local quadratic transform of R_i along V. A classical result of Zariski and Abhyankar asserts that $V = \cup_{i\geq 1} R_i$, cf. [1, Lemma 12].

In the setup of the proof of part (b) of Theorem 4, let $R = \mathbf{C}[x,y]_{(x,y)}$ and $V = B_{xB}$. Then $\mathbf{C}[x, y/x^n]_{(x,y/x^n)}$ is exactly the nth local quadratic transform of R along V. So the representation $V = \cup_{n\geq 1}\mathbf{C}[x, y/x^n]_{(x,y/x^n)}$ is a particular case of theorem of Zariski and Abhyankar.

We consider another particular case. Let $A = \mathbf{Z}$ and $\mathcal{P} = (p\mathbf{Z})_{p\in Y}$, where Y is the set of all prime numbers. Then

$$\mathbf{Z}^{\mathcal{P}} = \mathbf{Z}[x, p(x-p)^{-\infty},\ p \in Y].$$

The next result collects some properties of this ring.

Theorem 7. *Let $\mathbf{Z}^{\mathcal{P}}$ as above and p a prime number.*

(a) *The prime ideals of $\mathbf{Z}^{\mathcal{P}}$ have one of the following two forms*

(i) $(x-q)\mathbf{Z}^{\mathcal{P}}$, $q \in Y$,

(ii) $N \cap \mathbf{Z}^{\mathcal{P}}$ *where N is a prime ideal of* $\mathbf{Z}[x, 1/(x-q), q \in Y]$.

(b) $\mathbf{Z}^{\mathcal{P}}/(x-p) \simeq \mathbf{Z}_p$.

(c) $\mathbf{Z}^{\mathcal{P}}/(p(x-p)^{-\infty}) \simeq \mathbf{Z}_p[x, 1/(x^{p-1}-1)]$.

(d) $\mathbf{Z}^{\mathcal{P}}_{(x-p)}$ *is a rank two valuation domain.*

(e) $\mathbf{Z}^{\mathcal{P}}$ *is a two-dimensional integrally closed domain.*

(f) $\mathbf{Z}^{\mathcal{P}}/(x-1) \simeq \mathbf{Q}$.

(g) $\mathbf{Z}^{\mathcal{P}}_{(x-1)} \simeq \mathbf{Q}[x]_{(x-1)}$ *is a discrete valuation domain.*

(h) $\mathbf{Z}^{\mathcal{P}}(3, x^2+1)$ *is a maximal ideal of* $\mathbf{Z}^{\mathcal{P}}$ *and* $\mathbf{Z}^{\mathcal{P}}_{(3,x^2+1)} \simeq \mathbf{Z}[x]_{(3,x^2+1)}$. *So* $\mathbf{Z}^{\mathcal{P}}$ *is not a Prüfer domain.*

(i) *Let* $R = x\mathbf{Z}[x, 1/(x-q), q \in Y] \cap \mathbf{Z}^{\mathcal{P}}$ *an let* $\{p_1, p_2, \cdots\}$ *be the sequence of prime numbers. Then* $R = \cup_{n\geq 1}(x(x-p_1)^{-1}\cdots(x-p_n)^{-1})$ *and* $\mathbf{Z}^{\mathcal{P}}/R \simeq \mathbf{Q}$.

Proof: (a) follows from Corollary 2. Set $B = \mathbf{Z}^{\mathcal{P}}$ and let W be the multiplicative set of B generated by $\{x-q \mid q \in Y \setminus \{p\}\}$.

(b). By part (a) of Theorem 1, $B/(x-p) \simeq \mathbf{Z}_p$. So xB is a maximal ideal of B.

(c). Set $Q = p(x-p)^{-\infty}B$. For $q \in Y \setminus \{p\}$, $B = (p,q) \subseteq (Q, x-q)$. So W is disjoint from Q, hence

$$B/Q \simeq B_W/QB_W \simeq \mathbf{Z}[x, p(x-p)^{-\infty}]_W/(p(x-p)^{-\infty}) \simeq$$

$$\simeq \mathbf{Z}_p[x]_W \simeq \mathbf{Z}_p[x, 1/(x-q), q \in Y \setminus \{p\}] \simeq \mathbf{Z}_p[x, 1/(x^{p-1}-1)]$$

by Dirichlet's Theorem on primes in an arithmetic progression (see [7, Theorem 1, page 251]). In particular, Q is a nonmaximal prime ideal. By (e), Q is the unique nonzero prime ideal contained in $(x-p)B$.

(d). Note that $B_W = \mathbf{Z}[x, p(x-p)^{-\infty}]_W$. Since W is disjoint from $(x-p)B$, we get $B_{(x-p)} = (B_W)_{(x-p)}$. Hence $B_{(x-p)} = \mathbf{Z}[x, p(x-p)^{-\infty}]_{(x-p)}$. So $B_{(x-p)}$ can be written as a directed union of rings $\cup_{n\geq 1}\mathbf{Z}[x, p(x-p)^{-n}]_{(x,p(x-p)^{-n})}$. It can be shown as in the proof of Theorem 4 (b) that $B_{(x-p)}$ is a rank two valuation domain with nonzero prime ideals $(x-p)B_{(x-p)}$ and $p(x-p)^{-\infty}B_{(x-p)}$. Note that every term $\mathbf{Z}[x, p(x-p)^{-n}]_{(x,p(x-p)^{-n})}$ of the union above is isomorphic to $\mathbf{Z}[x,y]_{(p,x,y)}/(x^n y - p)$, thus it is a two-dimensional regular local ring. Note that Remark 6 also applies here.

(e). By (a), a maximal ideal M of B has the form $(x-a)B$ or $N \cap B$ where N is a prime ideal of $\mathbf{Z}[x, 1/(x-q), q \in Y]$. It follows that B_M equals $B_{(x-a)}$ or a localization of $\mathbf{Z}[x, 1/(x-q), q \in Y]$. Hence B is a two-dimensional integrally closed domain. Note that we can also apply part (c) Corollary 2.

(f). Let V be the multiplicative set of B generated by $\{x-q \mid q \in Y\}$. Since every element of V is invertible modulo $(x-1)B$, we get $B/(x-1) = (B_V)/(x-1)$. Note that $B_V = \mathbf{Z}[x]_V$. So $B/(x-1) \simeq \mathbf{Z}[1/(1-q),\ q \in Y] = \mathbf{Q}$, by Dirichlet's Theorem on primes in an arithmetic progression (see [7, Theorem 1, page 251]).

(g). Let V be as above. Since $(x-1)B$ is disjoint from V, $B_{(x-1)} = (B_V)_{(x-1)} = (\mathbf{Z}[x]_V)_{(x-1)} = (\mathbf{Q}[x]_V)_{(x-1)}$.

(h) can be proved as (f) and (g).

(i) Let $q \in Y$. In B, $x-q$ divides x and $(x-q)^2$ does not divide x. Indeed, $x = x - q + q$ and $(x-q)^n$ divides q for each $n \geq 1$. Now it easily follows that $R = \cup_{n\geq 1}(x(x-p_1)^{-1}\cdots(x-p_n)^{-1})$. Let V be as in (f). Note that $(Q, x-q) = B$ for each $q \in Y$. So

$$B/R \simeq \mathbf{Z}[x, 1/(x-q),\ q \in Y]/(x) \simeq \mathbf{Z}[1/q,\ q \in Y] = \mathbf{Q}.$$

□

Remark 8. It is easy to see that $\mathbf{Z}[x, 1/(x-q), q \in Y]$ is a two-dimensional regular ring. So the proof of part (e) of Theorem 7 shows that the localizations of $\mathbf{Z}^{\mathcal{P}}$ at its maximal ideals are either strongly discrete valuation domains of rank ≤ 2 or two-dimensional regular local rings. So $\mathbf{Z}^{\mathcal{P}}$ is a locally GCD domain. It would be interesting to know if $\mathbf{Z}^{\mathcal{P}}$ is a generalized GCD domain. Recall that D is a *generalized GCD domain* if $aD \cap bD$ is an invertible ideal for each nonzero $a, b \in D$, cf. [2].

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