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Number of epimorphisms between finite Łukasiewicz algebras

by
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Abstract

In this note we provide an explicit construction of all epimorphisms between finite Łukasiewicz algebras and we determine its number.

Key Words: Boolean algebras, Łukasiewicz and Post algebras.
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1 Preliminaries

The notion of a three-valued Łukasiewicz algebra [5] was introduced by Gr. Moisil [6, 8, 9]. A. Monteiro [10, 11], (see also [12]) has shown that a three-valued Łukasiewicz algebra can be considered as an algebra $(L, 1, \sim, \nabla, \vee, \wedge)$ of type $(0, 1, 1, 2, 2)$ such that $(L, 1, \sim, \vee, \wedge)$ is a De Morgan algebra and ∇ fulfills the following conditions:

$$\sim x \vee \nabla x = 1, \quad \sim x \wedge x = \sim x \wedge \nabla x, \quad \nabla(x \wedge y) = \nabla x \wedge \nabla y.$$

For short, we shall say that L is a Łukasiewicz algebra. The necessity operator is defined by $\Delta x = \sim \nabla \sim x$. It is well known that $B(L) = \{x \in L : \nabla x = x\} = \{x \in L : \Delta x = x\}$ is a Boolean algebra. Gr. Moisil [6, 7] has proved that Łukasiewicz algebras satisfy the following *determination principle*: If $\nabla x = \nabla y$ and $\Delta x = \Delta y$ then $x = y$. (see also [14]). A Łukasiewicz algebra L is said to be axled provided there exists $e \in L$, called the axis of L ([7], page 88), if: $\Delta e = 0$ and $\nabla x \leq \Delta x \vee \nabla e$, for every $x \in L$. If the axis of L exists, it is unique, and using the determination principle, it is easy to see that $x = (\Delta x \vee e) \wedge \nabla x$, for every $x \in L$, [15].

Let B and B' be Boolean algebras, we denote by $Hom(B, B')$ ($Epi(B, B')$) the set of all boolean homomorphisms (epimorphisms) from B to B'. Denoting the Boolean algebra with n atoms by B_n, $n \in \mathbb{Z}$, $n \geq 0$, then if $m < n$ we have $Epi(B_m, B_n) = \emptyset$.

If $b \in B_m$ and $b' \in B_n$ let $Epi^{(b,b')}(B_m, B_n) = \{h \in Epi(B_m, B_n) : h(b) = b'\}$.

Let L and L' be Łukasiewicz algebras, we denote by $Epi(L, L')$ the set of all epimorphisms from L to L'.

Lemma 1.1. *If L and L' are Łukasiewicz algebras with axes e and e' respectively and $H \in Epi(L, L')$ then $H(e) = e'$. If $h = H_{|B(L)}$ then*

$$h \in Epi^{(\nabla e, \nabla e')}(B(L), B(L')).$$

Proof: (1) $\Delta H(e) = H(\Delta e) = H(0) = 0'$.
If $y \in L'$, then as H is a surjective map there exists $x \in L$ such that $H(x) = y$, thus (2) $\nabla y = \nabla H(x) = H(\nabla x) \leq H(\Delta x \vee \nabla e) = H(\Delta x) \vee H(\nabla e) = \Delta H(x) \vee \nabla H(e) = \Delta y \vee \nabla H(e)$.
From (1) and (2) we have that $H(e)$ is the axis of L' and as the axis is unique $H(e) = e'$.
Moreover $h(\nabla e) = H(\nabla e) = \nabla H(e) = \nabla e'$. □

A Łukasiewicz algebra L is said to be centered provided there exists $c \in L$, called the center of L, if $\sim c = c$ (Gr. Moisil, [6]). The centered Łukasiewcz algebras coincide with the three-valued Post algebras. For short, we shall say that L is a Post algebra. An element c of a Łukasiewicz algebra L is a center of L if and only if $\Delta c = 0$ and $\nabla c = 1$. All centered Łukasiewcz algebras are axled Łukasiewcz algebras. We denote by $\mathbf{T} = \{0, c, 1\}$ the ordered set where $0 < c < 1$, so as $\mathbf{T}$ is a finite chain, $\mathbf{T}$ is a bounded distributive lattice. If we define $\sim 0 = 0, \sim c = c, \sim 1 = 0, \nabla 0 = 0, \nabla c = \nabla 1 = 1$, then $\mathbf{T}$ is a Post algebra.

We denote by $\mathcal{A}(B_n)$ the set of all atoms of B_n. If $b \in B_n \setminus \{0\}$ let $\mathcal{A}(b) = \{a \in \mathcal{A}(B_n) : a \leq b\}$.

If f is any map, $f : \mathcal{A}(B_n) \to \mathcal{A}(B_m)$ the map $h_f : B_m \to B_n$ defined by

$$h_f(x) = \bigvee \{a \in \mathcal{A}(B_n) : f(a) \leq x\},$$

verifies:

(A1) $h_f \in Hom(B_m, B_n)$,

(A2) If $a \in \mathcal{A}(B_m)$ then $h_f(a) = 0$ if and only if $a \notin f(\mathcal{A}(B_n))$,

(A3) h_f is an epimorphism if and only if f is an injective map, [19, 20],

(A4) h_f is one to one if and only if f is a surjective map [19, 20].

If $h \in Epi(B_m, B_n)$ and $b \in \mathcal{A}(B_n)$ then $[b)$ is a maximal filter of B_n and $h^{-1}([b))$ is a maximal filter of B_m then $h^{-1}([b)) = [a)$ where $a \in \mathcal{A}(B_m)$. Also $Nuc(h) \subseteq [a)$. Let $f : \mathcal{A}(B_n) \to \mathcal{A}(B_m)$ be the map defined by $f(a) = b$, then f is an injective map and $h_f = h$. It is well known that if we define $\Phi(f) = h_f$

then Φ is a bijection between the set $IM(\mathcal{A}(B_n), \mathcal{A}(B_m))$ of all injective maps from $\mathcal{A}(B_n)$ to $\mathcal{A}(B_m)$ and the set $Epi(B_m, B_n)$. The cardinality of a finite set X is denoted by $|X|$. If we set:

$$V_{m,n} = \begin{cases} \dfrac{m!}{(m-n)!} & \text{if } m \geq n \\ 0, & \text{if } m < n, \end{cases}$$

then:

$$|Epi(B_m, B_n)| = V_{m,n}.$$

Lemma 1.2. *If $m \geq n \geq 1$ and $h \in Epi(B_m, B_n)$ then:*

(A5) *If $a \in \mathcal{A}(B_m)$, then $h(a) = 0$ or $h(a) \in \mathcal{A}(B_n)$,*

(A6) *If $b \in \mathcal{A}(B_n)$, there exists a unique $a \in \mathcal{A}(B_m)$, such that $h(a) = b$,*

(A7) *If h is a monomorphism then for any $a \in \mathcal{A}(B_m)$, $h(a) \in \mathcal{A}(B_n)$.*
[2, 3], [18].

Let B be a Boolean algebra, F a filter of B, and consider the relation $x \equiv y$ (mod. F) if and only if there exists $f \in F$ such that $x \wedge f = y \wedge f$ is a congruence. Denote by B/F the quotient algebra. The following is a different way of determine $|Epi(B_m, B_n)|$, where $m \geq n \geq 1$. If $h \in Epi(B_m, B_n)$ then the quotient algebra $B_m/Nuc(h)$ is isomorphic to B_n, i.e. $B_m/Nuc(h) \cong B_n$. Since B_m is finite $Nuc(h) = [x)$, $x \in B_m$, and also $B_m/[x) \cong (x]$ then $|(x]| = |B_n| = 2^n$. Let B be a Boolean algebra and F a filter of B, then every congruence class module F has the same cardinality as F (A. Monteiro [17]). So $|(x]|.t = |B_m|$ i.e. $2^n.t = 2^m$ and then $t = 2^{m-n}$. Therefore if $b' \in B_n$ the set $\{b \in B_m : h(b) = b'\}$ has 2^{m-n} elements.

If B_n is a homomorphic image of B_m then there exists $x \in B_m$ such that $(x] \cong B_n$ therefore x is the supremum of n atoms of B_m. There exist $\binom{m}{n}$ elements of B_m which are supremum of n atoms of B_m. Let $\mathcal{F}_n$ be the set of all filters $[x)$ where x is the supremum of n atoms of B_m. If we denote by $Aut(B_n)$ the set of all automorphisms of the Boolean algebra B_n, then if $\alpha \in Aut(B_n)$, α is in particular a bijection of $\mathcal{A}(B_n)$ then by (A7) of Lemma 1.2, $\alpha(a) \in \mathcal{A}(B_n)$ for any $a \in \mathcal{A}(B_n)$. Clearly there exist $n!$ bijections of $\mathcal{A}(B_n)$, so $|Aut(B_n)| = n!$.

If $h \in Epi(B_m, B_n)$ then $Nuc(h) \in \mathcal{F}_n$. Let $\beta : Epi(B_m, B_n) \to \mathcal{F}_n$ be the map defined by $\beta(h) = Nuc(h)$. If $\alpha \in Aut(B_n)$ then $\alpha \circ h \in Epi(B_m, B_n)$. Moreover if $[x) \in \mathcal{F}_n$ then $\beta^{-1}((x])$ is a filter of B_m and $|\varphi^{-1}((x])| = n!$ then $n!.\binom{m}{n} = V_{m,n} = |Epi(B_m, B_n)|$.

If $b \in B_m \setminus \{0\}$ and $h \in Epi(B_m, B_n)$ then $h(b) = 0$ or $h(b) \neq 0$. If $b' = h(b) = 0$ then $|Epi^{(b,0)}(B_m, B_n)|$ is equal to the number of injective maps from $\mathcal{A}(B_n)$ to $\mathcal{A}(B_m) \setminus \mathcal{A}(b)$ i.e.:

$$|Epi^{(b,0)}(B_m, B_n)| = V_{m-|\mathcal{A}(b)|,n},$$

and if $b' = h(b) \neq 0$, [15] then:

$$|Epi^{(b,b')}(B_m, B_n)| = V_{m,n} - V_{m-|\mathcal{A}(b)|,n}.$$

If $b \in B_m$ and $b' \in B_n \setminus \{0'\}$ then $h_f(b) = b'$ if and only if $f(\mathcal{A}(b')) \subseteq \mathcal{A}(b)$ and $f(\mathcal{A}(-b')) \subseteq \mathcal{A}(-b)$, so

$$|Epi^{(b,b')}(B_m, B_n)| = V_{|\mathcal{A}(b)|,|\mathcal{A}(b')|}.V_{m-|\mathcal{A}(b)|,n-|\mathcal{A}(b')|}.$$

If $b \neq 0$ then $h(b) = 0$ if and only if $f(\mathcal{A}(B_n)) \subseteq \mathcal{A}(B_m) \setminus \mathcal{A}(b)$.

Lemma 1.3. *If $m \geq n$, $h \in Epi^{(b,b')}(B_m, B_n)$, where $b \in B_m \setminus \{0\}$ and $b' \in B_n \setminus \{0'\}$, then*

(A8) *If $a \in \mathcal{A}(b)$ and $h(a) \neq 0'$ then $h(a) \in \mathcal{A}(b')$.*

(A9) *If $a \notin \mathcal{A}(b)$ and $h(a) \neq 0'$ then $h(a) \notin \mathcal{A}(b')$.*

Proof: If $a \in \mathcal{A}(b)$ i.e. $a \leq b$ then $h(a) \leq h(b) = b'$ so if $h(a) \neq 0'$, $h(a) \in \mathcal{A}(B_n)$ thus $h(a) \in \mathcal{A}(b')$.
If $h(a) \leq b' = h(b)$ then $h(a) = h(a) \wedge h(b) = h(a \wedge b)$, so $a \equiv a \wedge b$ (mod. $Nuc(h)$). Since B_m is finite $Nuc(h) = [f)$ where $f \in B_m$, then (1) $a \wedge f = a \wedge b \wedge f$. From $0 \leq a \wedge f \leq a$ and $a \in \mathcal{A}(B_m)$ it follows (2) $a \wedge f = a$ or (3) $a \wedge f = 0$. In the case (2) we have by (1) that $a = a \wedge b \wedge f \leq b$, contradiction. In the case (3), Since $h(f) = 1$ we have $0 = h(0) = h(a \wedge f) = h(a) \wedge h(f) = a \wedge 1 = a$, contradiction. So $h(a) \leq b'$. □

From (A3) and Lemma 1.3, if $h \in Epi^{(b,b')}(B_m, B_n)$, $b \neq 0$ $b' \neq 0'$ then $f = \Phi^{-1}(h)$ verifies:

(C1) $f \in IM(\mathcal{A}(B_n), \mathcal{A}(B_m))$,

(C2) $f(\mathcal{A}(b')) \subseteq \mathcal{A}(b)$,

(C3) $f(\mathcal{A}(-b')) \subseteq \mathcal{A}(-b)$.

If $h \in Hom(B_m, B_n)$ then (1) $S = h(B_m)$ is a subalgebra of B_n so (2) $h(B_m) \cong B_t$ where $1 \leq t \leq n$ and $t \leq m$. Let $\mathcal{A}(S) = \{s_1, s_2, \ldots, s_t\}$.

From (1) and (2), $h \in Epi(B_m, S)$ then $g = \Phi^{-1}(h) \in IM(\mathcal{A}(S), \mathcal{A}(B_m))$ and (3) $h_g = h$. Since $\{\mathcal{A}(s_1), \mathcal{A}(s_2), \ldots, \mathcal{A}(s_t)\}$ is a partition of the set $\mathcal{A}(B_n)$, if $b \in \mathcal{A}(B_n)$ there exists an index i, $1 \leq i \leq t$ such that $b \in \mathcal{A}(s_i)$, i.e. $b \leq s_i$.

Let us define a map $f : \mathcal{A}(B_n) \to \mathcal{A}(B_m)$ by: $f(b) = g(s_i)$ if and only if $b \in \mathcal{A}(s_i)$. If there exists i, $1 \leq i \leq t$ such that $|\mathcal{A}(s_i)| > 1$ then f is not an injective map. Note that by definition of the map f we have $f(\mathcal{A}(B_n)) = g(\mathcal{A}(S))$. By (A1) $h_f = \Phi(f) \in Hom(B_m, B_n)$. Let us prove that $h_f = h$. In view of (3), it suffices to prove that $h_f = h_g$ which is equivalent to prove that $h_f(a) = h_g(a)$

for every $a \in \mathcal{A}(B_m)$. If $a \in f(\mathcal{A}(S))$ i.e. $a = f(b)$, $b \in \mathcal{A}(S)$, as by definition $f(b) = g(s_i)$ where $b \in \mathcal{A}(s_i)$, then $h_f(a) = s_i$ and $h_g(a) = s_i$. If $a \notin f(\mathcal{A}(S))$ then $h_f(a) = 0$ and $h_g(a) = 0$.

If $m < n$ then $|Epi(B_m, B_n)| = 0$, and if $h \in Hom(B_m, B_n)$ then $h(B_m)$ is a boolean subalgebra of B_n such that $1 \leq |\mathcal{A}(h(B_m))| \leq m < n$.

It is well known that there exists a bijective map from boolean subalgebras of B_n and the partitions of the set $\mathcal{A}(B_n)$, moreover the number of boolean subalgebras of B_n with t atoms $1 \leq t \leq n$, is:

$$P(n,t) = \frac{\sum_{i=0}^{t-1} (-1)^i \cdot \binom{t}{i} \cdot (t-i)^n}{t!}.$$

So if S is a boolean subalgebra of B_n with t atoms, where $1 \leq t \leq m < n$ then there exist $V_{m,t}$ epimorphisms from B_m to S, so if $m < n$

$$|Hom(B_m, B_n)| = \sum_{t=1}^{m} P(n,t) \cdot V_{m,t}.$$

Note that if $m \geq n$ then

$$|Hom(B_m, B_n)| = |Epi(B_m, B_n)| + \sum_{t=1}^{n-1} P(n,t) \cdot |Epi(B_m, B_t)| =$$

$$V_{m,n} + \sum_{t=1}^{n-1} P(n,t) \cdot V_{m,t}.$$

Let us denote $\mathbf{B}$ the Boolean algebra B_1. Let L be a finite Łukasiewicz algebra, then

$$L \cong \mathbf{B}^j \times \mathbf{T}^k, \text{ where } j,k \in \mathbb{Z},\ j \geq 0,\ k \geq 0.$$

(T) If $j = k = 0$ then L is trivial, i.e. L have a unique element,

(B) If $j \geq 1,\ k = 0$ then L is a Boolean algebra with j atoms,

(P) Ii $j = 0,\ k \geq 1$ then L is a Post algebra and $B(L)$ is a Boolean algebra with k atoms,

(E) If $j \geq 1,\ k \geq 1$ then L is an axled Łukasiewicz algebra. Then L is neither a Boolean algebra, nor a Post algebra. The axis of L is the $(j+k)$-tuple

$$e = (\underbrace{0,0,\ldots,0}_{j},\underbrace{c,\ldots,c}_{k}), \quad \text{then} \quad \nabla e = (\underbrace{0,0,\ldots,0}_{j},\underbrace{1,\ldots,1}_{k}).$$

$B(L)$ is a Boolean algebra with $(j+k)$ atoms, whose elements are

$$(b_1, b_2, \ldots, b_j, b_{j+1}, \ldots, b_{j+k}),$$

where $b_i = 0 \in \mathbf{B} = \{0,1\}$ for $1 \leq i \leq j$ and $b_i \in \{0,1\} \subset \mathbf{T}$ for $j+1 \leq i \leq j+k$. If $b \in B(L)$ let

$$J(b) = \{i : b_i = 1, 1 \leq i \leq j\}, \quad \text{and} \quad K(b) = \{i : b_i = 1, j+1 \leq i \leq j+k\},$$

then $0 \leq |J(b)| \leq j$ and $0 \leq |K(b)| \leq k$.

It is well known that if L is a finite non trivial Łukasiewicz algebra then all homomorphic images of L are determined by the filters $[b)$ where $b \in B(L)$, and the quotient algebra $L/[b)$ is isomorphic to the Łukasiewicz algebra $(b] = \{x \in L : x \leq b\}$. Since $B(L)$ has 2^{j+k} elements then: (C) there exist 2^{j+k} homomorphic images of L.

(B) If $j \geq 1,\ k = 0$, then L has 2^j homomorphic images, which are Boolean algebras.

(P) If $j = 0,\ k \geq 1$, then L has 2^k homomorphic images, which are Post algebras.

(E) If $j \geq 1,\ k \geq 1$, then L has 2^{j+k} homomorphic images, which are axled algebras.

(E1) If $|K(b)| = 0$, then $(b] \cong B^{|J(b)|}$ so there exist $\binom{j}{j_1}$, $0 \leq j_1 \leq j$ homomorphic images of L which are Boolean algebras with j_1 atoms, and therefore there exist 2^j homomorphic images which are Boolean algebras. Note that if $|J(b)| = 0$, then $L/[b)$ is a trivial algebra.

(E2) If $|J(b)| = 0$, then $(b] \cong \mathbf{T}^{|K(b)|}$ so there exist $\binom{k}{k_1}$, $0 \leq k_1 \leq k$ homomorphic images L' of L which are Post algebras, such that $B(L')$ is a Boolean algebra with k_1 atoms, and therefore there exist 2^k homomorphic images which are Post algebras. If $|K(b)| = 0$, then $L/[b)$ is a trivial algebra trivial, which coincides with the trivial algebra of (E1).

(E3) If $1 \leq j_1 = |J(b)| \leq j$ and $1 \leq k_1 = |K(b)| \leq k$, then $L/[b) \cong (b] \cong \mathbf{B}^{|J(b)|} \times \mathbf{T}^{|H(b)|}$ is an axled homomorphic image.
Then the number of axled homomorphic images of L is:

$$\sum_{i=1}^{j} \binom{j}{i} \cdot \left(\sum_{i=1}^{k} \binom{k}{i} \right) = \sum_{i=1}^{j} \binom{j}{i} \cdot (2^k - 1) =$$

$$(2^k - 1) \cdot \sum_{i=1}^{k} \binom{k}{i} = (2^k - 1) \cdot (2^j - 1).$$

Since in the cases (E1) and (E2) we obtain the same homomorphic image, the trivial algebra, the number of homomorphic images is:

$$2^j + (2^k - 1) + (2^k - 1) \cdot (2^j - 1) = 2^j + (2^k - 1) \cdot (1 + (2^j - 1)) =$$

$$= 2^j + (2^k - 1) \cdot 2^j = 2^j \cdot (1 + 2^k - 1) = 2^j \cdot 2^k = 2^{j+k},$$

which coincides with (C).

2 Main text

The following result generalizes those obtained by us in [13, 15] and lemma 4.1 of [16].

Theorem 2.1. *If L and L' are axled Łukasiewiz algebras with axis e and e' respectively and $h \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$ then the map $H : L \to L'$ defined by*

$$H(x) = (h(\Delta x) \vee e') \wedge h(\nabla x)$$

verifies (I) *H is an extension of h,* (II) *$H \in Epi(L, L')$, and* (III) *H is the unique extension of h.*

Proof: (I) Indeed, if $b \in B(L)$ then $\Delta b = \nabla b = b$, thus $H(b) = (h(\Delta b) \vee e') \wedge h(\nabla b) = (h(b) \vee e') \wedge h(b) = h(b)$.

(IIa) $H(x \wedge y) = (h(\Delta(x \wedge y)) \vee e') \wedge h(\nabla(x \wedge y)) = (h(\Delta x \wedge \Delta y) \vee e') \wedge h(\nabla x \wedge \nabla y) = (h(\Delta x) \wedge h(\Delta y)) \vee e') \wedge h(\nabla x) \wedge h(\nabla y) = ((h(\Delta x) \vee e') \wedge (h(\Delta y) \vee e') \wedge h(\nabla x) \wedge h(\nabla y) = (h(\Delta x) \vee e') \wedge h(\nabla x) \wedge (h(\Delta y) \vee e') \wedge h(\nabla y) = H(x) \wedge H(y)$.

(IIb) Since H is an extension of h and $\nabla x \in B(L)$ then (1) $H(\nabla x) = h(\nabla x)$.
(2) $\nabla H(x) = \nabla((h(\Delta x) \vee e') \wedge h(\nabla x)) = (\nabla h(\Delta x) \vee \nabla e') \wedge \nabla h(\nabla x) = (h(\Delta x) \vee \nabla e') \wedge h(\nabla x)$.
Since $\nabla x \leq \Delta x \vee \nabla e$, $\nabla x, \Delta x, \nabla e, \Delta x \vee \nabla e \in B(L)$, $\nabla x \leq \Delta x \vee \nabla e$ and h is a boolean homomorphism verifying $h(\nabla e) = \nabla e'$ we have (3) $h(\nabla x) \leq h(\Delta x \vee \nabla e) = h(\Delta x) \vee h(\nabla e) = h(\Delta x) \vee \nabla e'$. From (2) and (3) we have (4) $\nabla H(x) = h(\nabla x)$. From (1) and (4) we conclude $\nabla H(x) = H(\nabla x)$.

(IIc) Moreover $H(\sim x) = (h(\Delta \sim x) \vee e') \wedge h(\nabla \sim x)$, so
(5) $\Delta H(\sim x) = (h(\Delta \sim x) \vee \Delta e') \wedge h(\nabla \sim x) = (h(\Delta \sim x) \vee 0') \wedge h(\nabla \sim x) = h(\Delta \sim x) \wedge h(\nabla \sim x) = h(\Delta \sim x)$, and
$\nabla H(\sim x) = (h(\Delta \sim x) \vee \nabla e') \wedge h(\nabla \sim x) = (h(\Delta \sim x) \vee h(\nabla e)) \wedge h(\nabla \sim x) = (h(\Delta \sim x \vee \nabla e) \wedge \nabla \sim x))$ and since $\nabla \sim x \leq \Delta \sim x \vee \nabla e$ we have:
(6) $\nabla H(\sim x) = h(\nabla \sim x)$. On the other hand, $\sim H(x) = (\sim h(\Delta x) \wedge \sim e') \vee \sim h(\nabla x)$.
Since h is a boolean epimorphism, if $b \in B(L)$ then (7) $h(\sim b) = \sim h(b)$,
so $\sim H(x) = (\sim h(\Delta x) \wedge \sim e') \vee \sim h(\nabla x) = (h(\sim \Delta x) \wedge \sim e') \vee h(\sim \nabla x)$.
Therefore (8) $\Delta \sim H(x) = (\Delta h(\sim \Delta x) \wedge \Delta \sim e') \vee \Delta h(\sim \nabla x) =$
$(h(\sim \Delta x) \wedge \sim \nabla e') \vee h(\sim \nabla x) = (\sim h(\Delta x) \wedge \sim h(\nabla e)) \vee \sim h(\nabla x) =$
$\sim [h(\Delta x) \vee h(\nabla e)) \wedge h(\nabla x)] = \sim h((\Delta x \vee \nabla e) \wedge \nabla x) = \sim h(\nabla x) = \sim h(\nabla x) =$
$h(\Delta \sim x)$, and (9) $\nabla \sim H(x) = (\nabla h(\sim \Delta x) \wedge \nabla \sim e') \vee \nabla h(\sim \nabla x) =$
$(h(\sim \Delta x) \wedge \sim \Delta e') \vee h(\sim \nabla x) = (h(\sim \Delta x) \wedge \sim 0) \vee h(\sim \nabla x) = h(\sim \Delta x) \vee h(\sim \nabla x) = h(\sim \Delta x) = h(\nabla \sim x)$.

From (5) and (8) we conclude (10) $\Delta H(\sim x) = \Delta \sim H(x)$ and from (6) and (9) we have (11) $\nabla H(\sim x) = \nabla \sim H(x)$. Finally from (10) and (11) we have by the determination principle of Moisil: $\sim H(x) = H(\sim x)$.

(IId) Let $y \in L'$ then $y = (\Delta y \vee e') \wedge \nabla y$. Since $\nabla y, \Delta y \in B(L')$ and h is a boolean epimorphism there exist $b_1, b_2 \in B(L)$ such that $h(b_1) = \Delta y$ and $h(b_2) = \nabla y$. Let $b_3 = b_1 \vee b_2 \in B(L)$, and $x = (b_1 \vee e) \wedge b_3 \in L$. Then $\Delta x = (\Delta b_1 \vee \Delta e) \wedge \Delta b_3 = (b_1 \vee 0) \wedge b_3 = b_1 \wedge b_3 = b_1$, and $\nabla x = (\nabla b_1 \vee \nabla e) \wedge \nabla b_3$. Then $h(\Delta x) = h(b_1) = \Delta y$ and as $h(b_3) = h(b_1) \vee h(b_2) = \Delta y \vee \nabla y = \nabla y$ then $h(\nabla x) = h((\nabla b_1 \vee \nabla e) \wedge \nabla b_3) = (h(\nabla b_1) \vee h(\nabla e)) \wedge h(\nabla b_3) = (\Delta y \vee \nabla e') \wedge \nabla y$. As $\nabla y \leq \Delta y \vee \nabla e'$ then $h(\nabla x) = \nabla y$. Therefore $H(x) = (h(\Delta x) \vee e') \wedge h(\nabla x) = (\Delta y \vee e') \wedge \nabla y = y$. So H is a surjective map.

(III) If $H' \in Epi(L, L')$ verifies $H'(b) = h(b)$ for every $b \in B(L)$ then

$$H'(x) = (\Delta H'(x) \vee e') \wedge \nabla H'(x) = (H'(\Delta x) \vee e') \wedge H'(\nabla x) =$$

$$= (h(\Delta x) \vee e') \wedge h(\nabla x) = H(x).$$

□

Corollary 2.1. *If L and L' are centered Łukasiewicz algebras, with centers c and c', and $h \in Epi(B(L), B(L'))$ then the map $H : L \to L'$ defined by $H(x) = (h(\Delta x) \vee c') \wedge h(\nabla x)$ is the unique epimorphism from L to L' such that $H(b) = h(b)$ for every $b \in B(L)$.*

Proof: It suffices to note that every centered algebra is an axled algebra and $h(\nabla c) = h(1) = 1' = \nabla c'$. □

Theorem 2.2. *If L and L' are finite axled Łukasiewicz algebras, with axes e and e' respectively, then: $|Epi(L, L')| = |Epi^{(\nabla e, \nabla e')}(B(L), B(L'))|$.*

Proof: If $h \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$, then by Theorem 2.1, the map:

$$H(x) = (h(\Delta x) \vee e') \wedge h(\nabla x)$$

verifies $H \in Epi(L, L')$. If we denote $\delta(h) = H$, then by Theorem 2.1, (III) δ is a map.
If $H \in Epi(L, L')$ then by Lemma 1.1, $h = H_{|B(L)} \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$ and by Theorem 2.1 the extension of h to L is the epimorphism H, then $\delta(h) = H$, so δ is a surjective map.
If $h, h' \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$ are such that $h \neq h'$ there exists $b \in B(L)$ such that $h(b) \neq h'(b)$. If H and H' are the homomorphisms extension of h and h' respectively then $H(b) = h(b) \neq h'(b) = H'(b)$, so δ is an injective map. □

Theorem 2.3. *Let L and L' finite non trivial Łukasiewicz algebras, $L \cong \mathbf{B}^j \times \mathbf{T}^k$ and $L' \cong \mathbf{B}^{j'} \times \mathbf{T}^{k'}$, where $j, k, j', k' \geq 1$. Then*

$$|Epi(L, L')| = |Epi^{(\nabla e, \nabla e')}(B(L), B(L'))| = V_{k,k'} \cdot V_{j,j'}.$$

Proof: We have proved, see Lemma 1.1 and Theorem 2.1, that:

$$H \in Epi(L, L') \text{ if and only if } h = H_{|B(L)} \in Epi^{(\nabla e, \nabla e')}(B(L), B(L'))$$

so $f = \Phi^{-1}(h) \in IM(\mathcal{A}(B(L')), \mathcal{A}(B(L)))$ verifies conditions (C2) and (C3), i.e.

$$f(\mathcal{A}(\nabla e')) \subseteq \mathcal{A}(\nabla e) \text{ and } f(\mathcal{A}(\sim \nabla e')) \subseteq \mathcal{A}(\sim \nabla e). \tag{1}$$

Since

$$|\mathcal{A}(\nabla e)| = k, \quad |\mathcal{A}(\nabla e')| = k',$$
$$|\mathcal{A}(B(L)) \setminus \mathcal{A}(\nabla e)| = j \text{ and } |\mathcal{A}(B(L')) \setminus \mathcal{A}(\nabla e')| = j'$$

then the set of all injective maps from $\mathcal{A}(B(L'))$ to $\mathcal{A}(B(L))$ verifying (1) is not empty if and only if $k \geq k'$ and $j \geq j'$. Then by the preceding results:

$$|Epi(L, L')| = |Epi^{(\nabla e, \nabla e')}(B(L), B(L'))| = V_{k,k'} \cdot V_{j,j'}. \tag{2}$$

□

If L is a Łukasiewicz algebra we denote by $Aut(L)$ the set of all automorphism of L.

Corollary 2.2. *If $L \cong B^j \times T^k$, $j, k \geq 1$, then $|Aut(L)| = k! \cdot j!$.*

Note that:

- If L and L' are finite centered algebras, i.e. $L \cong \mathbf{T}^k$ and $L' \cong \mathbf{T}^{k'}$ then:

$$|Epi(L, L')| = |Epi^{(1,1')}(B(L), B(L'))| = |Epi(B(L), B(L'))| =$$
$$V_{|\mathcal{A}(B(L))|, |\mathcal{A}(B(L'))|} = V_{k,k'}.$$

- If L and L' are finite Boolean algebras, i.e. $L \cong \mathbf{B}^j$ and $L' \cong \mathbf{B}^{j'}$ then:

$$|Epi(L, L')| = |Epi^{(0,0')}(B(L), B(L'))| = |Epi(B(L), B(L'))| =$$
$$V_{|\mathcal{A}(B(L))|, |\mathcal{A}(B(L'))|} = V_{j,j'}.$$

Let L be a finite non trivial Łukasiewicz algebra, $P(L)$ the set of join-irreducible elements of L and $\varphi : P(L) \to P(L)$ the Birula-Rasiowa map [4]. If L' is a finite non trivial Łukasiewicz algebra, an injective map $f : P(L') \to P(L)$ is said to be a H-function [1], if f verifies $f(\nabla p') = \nabla f(p')$, $f(\varphi(p')) = \varphi(f(p'))$. M. Abad and A. Figallo [1] have proved that there exists a bijective map from the set of all H-functions from $P(L')$ to $P(L)$ and the set $Epi(L, L')$, where L and L' are axled Łukasiewicz algebras which are neither Boolean algebras nor Post algebras. The number of elements of $Epi(L, L')$ determined by these authors coincides with the number indicated by us in (2).

Clearly is much more difficult determine H-functions, than injective functions from $\mathcal{A}(B_n)$ to $\mathcal{A}(B_m)$ verifying conditions (1).

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