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Some results on LP-Sasakian manifolds

by
A. A. Shaikh and Kanak Kanti Baishya

Abstract

LP-Sasakian manifolds are studied. Among others it is proved that in a non-semisymmetric Ricci-generalized pseudosymmetric LP-Sasakian manifold, the scalar curvature is constant if and only if the timelike vector field ξ is harmonic. LP-Sasakian manifolds admitting certain conditions on the Ricci tensor are studied and obtained several interesting results. Also ϕ-conformally flat LP-Sasakian manifolds are studied.

Key Words: LP-Sasakian manifolds, Ricci-generalized pseudosymmetric manifolds, η-parallel Ricci tensor, ϕ-conformally flat.
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1 Introduction

In 1989 K. Matsumoto [4] introduced the notion of LP-Sasakian manifolds. Then I. Mihai and R. Rosca [5] introduced the same notion independently and obtained many interesting results. LP-Sasakian manifolds are also studied by U. C. De, K. Matsumoto and A. A. Shaikh [2], A. A. Shaikh and S. Biswas [6] and others .

The object of the present paper is to study LP-Sasakian manifolds. Section 2 deals with preliminaries and fundamental results along with new examples of LP-Sasakian manifolds both in odd and even dimensions, and also we obtain that the Ricci operator Q commutes with the structure tensor ϕ. The notion of pseudosymmetric Riemannian manifolds were introduced and classified by R. Deszcz [1]. In section 3, we studied Ricci-generalized pseudosymmetric LP-Sasakian manifolds and it is proved that such a manifold is either a space of constant curvature 1 or an η-Einstein manifold. Section 4 is devoted to the study of LP-Sasakian manifolds satisfying some conditions on the Ricci tensor and introduce the notion that the Ricci tensor of an LP-Sasakian manifold to be η-recurrent, ϕ-parallel and ϕ-recurrent which generalizes the notion of η-parallel Ricci tensor. The notion of Ricci-η-parallelity was first introduced by M. Kon [3] for a Sasakian manifold.

Then we obtain some necessary and sufficient conditions for the Ricci tensor of an LP-Sasakian manifold to be η-recurrent and ϕ-parallel. In a Sasakian manifold with η-parallel Ricci tensor, the scalar curvature is always constant, but in an LP-Sasakian manifold with ϕ-parallel Ricci tensor, the scalar curvature is not constant, in general. However, in such a manifold the scalar curvature is constant if and only if $\psi = \frac{\mu}{n-1}$, where μ=Tr.$(Q\phi)$ and ψ=Tr.ϕ. Also it is proved that in an LP-Sasakian manifold with η-recurrent or ϕ-recurrent Ricci tensor, $\frac{1}{2}[r-(n-1)]$ is an eigenvalue of the Ricci tensor corresponding to the eigenvector $\phi\rho$, ρ being the associated vector field of the 1-form given by $A(X) = g(X, \rho)$.

The notion of ϕ-conformally flat K-contact manifold was introduced and studied by G. Zhen[7]. The last section is concerned with ϕ-conformally flat LP-Sasakian manifold and it is shown that such a manifold is either a space of constant curvature 1 or the timelike vector field ξ is harmonic. The conformally flat LP-Sasakian manifold has been studied in [2] and the notion of ϕ-conformally flat is much more weaker than that of conformally flat.

2 LP-Sasakian manifolds

An n-dimensional differentiable manifold M is said to be an LP-Sasakian manifold ([2], [4]) if it admits a (1, 1) tensor field ϕ, a unit timelike contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \quad g(X,\xi) = \eta(X), \quad \phi^2 X = X + \eta(X)\xi \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \qquad \nabla_X \xi = \phi X, \tag{2.2}$$

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.3}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. It can be easily seen that in an LP-Sasakian manifold, the following relations hold :

$$\phi\xi=0, \quad \eta(\phi X) = 0, \quad \text{rank } \phi = n-1. \tag{2.4}$$

Again, if we put

$$\Omega(X,Y) = g(X, \phi Y)$$

for any vector fields X, Y, then the tensor field $\Omega(X,Y)$ is a symmetric (0,2) tensor field [4]. Also, since the vector field η is closed in an LP-Sasakian manifold, we have ([2], [4])

$$(\nabla_X \eta)(Y) = \Omega(X,Y), \qquad \Omega(X,\xi) = 0 \tag{2.5}$$

for any vector fields X and Y.

An LP-Sasakian manifold M is said to be η-Einstein if its Ricci tensor S of type (0,2) is of the form

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y) \tag{2.6}$$

for any vector fields X, Y where α, β are smooth functions on M. Let M be an n-dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g). Then the following relations hold ([2], [4]) :

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.7}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.8}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.9}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{2.10}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.11}$$

for any vector fields X, Y, Z where R is the Riemannian curvature tensor of the manifold.

Lemma 2.1. *Let $M^n(\phi,\xi,\eta,g)$ be an LP-Sasakian manifold. Then for any X, Y, Z on M^n, the following relation holds:*

$$\begin{aligned} R(X,Y)\phi Z - \phi R(X,Y)Z = {} & g(Y,Z)\phi X - g(X,Z)\phi Y + g(X,\phi Z)Y - \\ & -g(Y,\phi Z)X + 2\{g(X,\phi Z)\eta(Y) - \\ & -g(Y,\phi Z)\eta(X)\}\xi + 2\{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(Z). \end{aligned} \tag{2.12}$$

Proof: From (2.3), (2.5) and the Ricci identity, we can easily get (2.12). □

Lemma 2.2. *Let (M^n, g) be an LP-Sasakian manifold. Then*

$$\begin{aligned} & g(\phi R(\phi X,\phi Y)Z,\phi W) = \\ & = g(R(X,Y)Z,W) + g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ & +g(X,\phi Z)g(\phi Y,W) - g(X,\phi W)g(\phi Y,Z) + 2\{g(Y,Z)\eta(X)\eta(W) \\ & +g(X,W)\eta(Y)\eta(Z) - g(X,Z)\eta(W)\eta(Y) - g(Y,W)\eta(X)\eta(Z)\} \end{aligned} \tag{2.13}$$

for any vector fields X, Y, Z, W on M^n.

Proof: Using (2.2), (2.9), (2.12) and $\eta(\phi X) = 0$, we can calculate

$$\begin{aligned} & g(\phi R(\phi X,\phi Y)Z,\phi W) = g(R(\phi X,\phi Y)Z,W) = g(R(Z,W)\phi X,\phi Y) \\ & = g(\phi R(Z,W)X,\phi Y) + g(X,W)\{g(Y,Z) + \eta(Y)\eta(Z)\} \\ & -g(X,Z)\{g(Y,W) + \eta(Y)\eta(W)\} + g(X,\phi Z)g(\phi Y,W) \\ & -g(\phi W,X)g(Z,\phi Y) + 2[\{g(Z,Y) + \eta(Z)\eta(Y)\}\eta(W) \\ & -\eta(Z)\{g(Y,W) + \eta(W)\eta(Y)\}]\eta(X) \\ & = g(R(Z,W)X,Y) + \{g(X,W)\eta(Z) - g(X,Z)\eta(W)\}\eta(Y) \\ & +g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(X,W)\eta(Y)\eta(Z) \\ & -g(Z,X)\eta(W)\eta(Y) + 2[g(Z,Y)\eta(X)\eta(W) - g(W,Y)\eta(Z)\eta(X)] \\ & +g(\phi Z,X)g(W,\phi Y) - g(\phi W,X)g(Z,\phi Y). \end{aligned}$$

The relation (2.13) follows from this and $g(R(Z,W)X,Y) = g(R(X,Y)Z,W)$. □

Lemma 2.3. *Let (M^n, g) be an LP-Sasakian manifold. Then for any X, Y, Z on M^n, the following relation holds:*

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = g(R(X,Y)Z,W) + g(X,W)\eta(Y)\eta(Z) -g(X,Z)\eta(W)\eta(Y) + g(Y,Z)\eta(X)\eta(W) -g(Y,W)\eta(X)\eta(Z). \quad (2.14)$$

Proof: Replacing X, Y by $\phi X, \phi Y$ respectively in (2.12) and taking the inner product on both sides by ϕW and then using (2.1), (2.2) and (2.4) we get

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = g(\phi R(\phi X, \phi Y)Z, \phi W) + g(\phi Y, Z)g(X, \phi W) -g(\phi X, Z)g(Y, \phi W) + g(X,Z)g(Y,W) -g(X,W)g(Y,Z) + g(X,Z)\eta(W)\eta(Y) +g(Y,W)\eta(X)\eta(Z) - g(Y,Z)\eta(X)\eta(W) -g(X,W)\eta(Y)\eta(Z). \quad (2.15)$$

Using (2.13) in (2.15) we obtain (2.14). □

Lemma 2.4. *Let (M^n, g) be an LP-Sasakian manifold. If Q is the Ricci operator, i.e. , if $S(X,Y) = g(QX, Y)$ for all X, Y on M^n, then*

$$Q\phi = \phi Q. \quad (2.16)$$

Proof: From (2.14), it follows that

$$\phi R(\phi X, \phi Y \phi Z) = R(X,Y)Z + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi +\{\eta(Y)X - \eta(X)Y\}\eta(Z). \quad (2.17)$$

We now consider the following two cases :

(i) $\dim M = n =$ odd $= 2m+1$,

(ii) $\dim M = n =$ even$= 2m+2$.

If $n = 2m+1$, let $\{e_i, \phi e_i, \xi\}, i = 1, 2, ..., m$ be a local ϕ-basis. Then putting $Y = Z = e_i$ in (2.17) and taking summation over i and using $\eta(e_i) = 0$, we get

$$\sum_{i=1}^{m} \epsilon_i \phi R(\phi X, \phi e_i)\phi e_i = \sum_{i=1}^{m} \epsilon_i R(X, e_i)e_i + m\eta(X)\xi, \quad (2.18)$$

where $\epsilon_i = g(e_i, e_i)$.

Again putting $Y = Z = e_i$ in (2.17) and taking summation over i and using $\eta \circ \phi = 0$, (2.1) and (2.2) we have

$$\sum_{i=1}^{m} \epsilon_i \phi R(\phi X, e_i)e_i = \sum_{i=1}^{m} \epsilon_i R(X, \phi e_i)\phi e_i + m\eta(X)\xi. \quad (2.19)$$

Adding (2.18) and (2.19) and using the definition of the Ricci operator we obtain

$$\phi(Q\phi X - R(\phi X, \xi)\xi) = QX - R(X,\xi)\xi + 2m\eta(X)\xi.$$

Using (2.9) and $\phi\xi = 0$ in the above relation we have

$$\phi(Q\phi X) = QX + 2m\eta(X)\xi.$$

Operating both sides by ϕ and using (2.1), Q is symmetric, $\phi\xi = 0$ and (2.10) we get (2.16).
Next, if $n = 2m + 2$, let $\{e_i, \phi e_i\}$, $i = 1, 2, ..., m + 1$ be a local ϕ-basis such that each e_i is orthogonal to ξ, i.e., $\eta(e_i) = 0$ for i=1, 2, ... $m + 1$. Then putting $Y = Z = e_i$ in (2.17) and taking summation over i and using $\eta(e_i) = 0$, we get

$$\sum_{i=1}^{m+1} \epsilon_i \phi R(\phi X, \phi e_i)\phi e_i = \sum_{i=1}^{m+1} \epsilon_i R(X, e_i)e_i + (m+1)\eta(X)\xi, \tag{2.20}$$

where $\epsilon_i = g(e_i, e_i)$.
Again replacing Y and Z by ϕe_i in (2.17) and taking summation over i and using $\eta(e_i) = 0$, (2.1) and (2.2), it follows that

$$\sum_{i=1}^{m+1} \epsilon_i \phi R(\phi X, e_i)e_i = \sum_{i=1}^{m+1} \epsilon_i R(X, \phi e_i)\phi e_i + (m+1)\eta(X)\xi. \tag{2.21}$$

Adding (2.20) and (2.21) and then proceeding similarly as in the privious case we can easily obtain (2.16).
This proves the lemma. □

The above results will be needed in the next sections.
We now give some new examples of LP-Sasakian manifolds both in odd and even dimensions.

Example 1. Let us consider a 3-dimensional manifold $M = \{(x_1, x_2, x_3) \in R^3\}$ where x_1, x_2, x_3 are the standard coordinates in R^3. In R^3 we define

$$\eta = dx_3 - x_2 dx_1, \qquad \xi = \frac{\partial}{\partial x_3},$$
$$g = (e^{2x_3} + x_2^2 - 1)dx_1^2 + e^{2x_3}dx_2^2 - (x_2 - 1)dx_1 \otimes dx_3 - (x_2 - 1)dx_3 \otimes dx_1 - \eta \otimes \eta,$$
$$\phi(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3},$$
$$\phi(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_2}, \qquad \phi(\frac{\partial}{\partial x_3}) = 0.$$

Then it can be easily seen that (ϕ, ξ, η, g) forms an LP-Sasakian structure in R^3.

Example 2. Let R^4 be the 4-dimensional real number space with the standard coordinates x, y, z, t. In R^4 we define

$$\eta = dt - ydz - dx, \qquad \xi = \frac{\partial}{\partial t},$$
$$g = e^{2t}dx^2 + e^{2t}dy^2 + (e^{2t} + y^2)dz^2 + ydz \otimes dx + ydx \otimes dz - ydz \otimes dt - ydt \otimes dz - \eta \otimes \eta,$$
$$\phi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \qquad \phi(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y},$$
$$\phi(\frac{\partial}{\partial z}) = \frac{\partial}{\partial z}, \qquad \phi(\frac{\partial}{\partial t}) = 0,$$

Then it can be easily seen that (ϕ, ξ, η, g) forms an LP-Sasakian structure in R^4.
The matrix g can be expressed by

$$g = \begin{pmatrix} e^{2t}-1 & 0 & 0 & 1 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

3 Ricci-generalized pseudosymmetric LP-Sasakian manifolds

Definition 3.1. *An LP-Sasakian manifold (M^n, g) is said to be Ricci-generalized pseudosymmetric if and only if the relation*

$$R \cdot R = f(p)Q(S, R) \tag{3.1}$$

holds on the set $A = \{x \in M : Q(S, R) \neq 0 \text{ at } x\}$, where $f \in C^\infty(A)$ for $p \in A$, $R \cdot R$ and $Q(S, R)$ are respectively defined by

$$(R(X,Y) \cdot R)(U,V)W = R(X,Y)R(U,V)W - R(R(X,Y)U,V)W$$
$$-R(U,R(X,Y)V)W - R(U,V)R(X,Y)W, \tag{3.2}$$

$$Q(S,R) = (S(X,Y) \cdot R)(U,V)W = ((X \wedge_S Y) \cdot R)(U,V)W \tag{3.3}$$

for all $X, Y, U, V, W \in \chi(M), \chi(M)$ being the Lie algebra of all differentiable vector fields on M. Here the endomorphism $(X \wedge_S Y)$ is defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y. \tag{3.4}$$

It is clear that any semisymmetric as well as any Ricci flat manifold is Ricci-generalized pseudosymmetric.

Let us consider a Ricci-generalized pseudosymmetric LP-Sasakian manifold. Then we have the relation (3.1), which can be written by virtue of (3.3)

$$(R(X,Y) \cdot R)(U,V)W = f(p)[(S(X,Y) \cdot R)(U,V)W]$$

for all $X, Y, U, V, W \in \chi(M)$. From the above relation, it follows that

$$(R(X,\xi) \cdot R)(U,V)W = f(p)[(S(X,\xi) \cdot R)(U,V)W]. \tag{3.5}$$

By virtue of (2.8) we obtain from (3.2) that

$$(R(X,Y) \cdot R)(U,V)W = \eta(R(U,V)W)X - \tilde{R}(U,V,W,X)\xi$$
$$-\eta(U)R(X,V)W + g(X,U)R(\xi,V)W - \eta(V)R(U,X)W$$
$$+g(X,V)R(U,\xi)W - \eta(W)R(U,V)X + g(X,W)R(U,V)\xi, \tag{3.6}$$

where $\tilde{R}(U,V,W,X) = g(R(U,V)W,X)$.
Also from the definition we have

$$(S(X,\xi) \cdot R)(U,V)W = ((X \wedge_S \xi) \cdot R)(U,V)W = (X \wedge_S \xi)R(U,V)W$$
$$+R((X \wedge_S \xi)U,V)W + R(U,(X \wedge_S \xi)V)W + R(U,V)(X \wedge_S \xi)W.$$

Using (3.4) in the above relation we get

$$(S(X,\xi) \cdot R)(U,V)W = (n-1)[\eta(R(U,V)W)X + \eta(U)R(X,V)W$$
$$+\eta(V)R(U,X)W + \eta(W)R(U,V)X] - S(X,R(U,V)W)\xi$$

$$-S(X,U)R(\xi,V)W - S(X,V)R(U,\xi)W - S(X,W)R(U,V)\xi. \tag{3.7}$$

Applying (3.6) and (3.7) in (3.5) we obtain

$$\begin{aligned}
&\eta(R(U,V)W)X - \tilde{R}(U,V,W,X)\xi \\
&-\eta(U)R(X,V)W + g(X,U)R(\xi,V)W - \eta(V)R(U,X)W \\
&+g(X,V)R(U,\xi)W - \eta(W)R(U,V)X + g(X,W)R(U,V)\xi \\
&= f(p)\{(n-1)[\eta(R(U,V)W)X + \eta(U)R(X,V)W \\
&+\eta(V)R(U,X)W + \eta(W)R(U,V)X] - S(X,R(U,V)W)\xi \\
&-S(X,U)R(\xi,V)W - S(X,V)R(U,\xi)W - S(X,W)R(U,V)\xi\}.
\end{aligned} \tag{3.8}$$

Taking the inner product on both sides of (3.8) by ξ and then using (2.1), (2.10) and the property of the curvature tensor, we get

$$\begin{aligned}
&\eta(R(U,V)W)\eta(X) + \tilde{R}(U,V,W,X) \\
&-\eta(U)\eta(R(X,V)W) + g(X,U)\eta(R(\xi,V)W) - \eta(V)\eta(R(U,X)W) \\
&+g(X,V)\eta(R(U,\xi)W) - \eta(W)\eta(R(U,V)X) \\
&= f(p)\{(n-1)[\eta(R(U,V)W)\eta(X) + \eta(U)\eta(R(X,V)W) \\
&+\eta(V)\eta(R(U,X)W) + \eta(W)\eta(R(U,V)X)] + S(X,R(U,V)W) \\
&-S(X,U)\eta(R(\xi,V)W) - S(X,V)\eta(R(U,\xi)W).
\end{aligned} \tag{3.9}$$

Putting $W = \xi$ in (3.9) and noting that $\eta(R(U,V)\xi) = 0$, for all U, V we obtain

$$f(p)\{-(n-1)\eta(R(U,V)X) + S(X,R(U,V)\xi)\} = 0,$$

from which it follows that

$$\text{either } f(p) = 0, \text{ or } S(X,R(U,V)\xi) - (n-1)\eta(R(U,V)W) = 0. \tag{3.10}$$

If $f(p) = 0$, then from (3.5) we have

$$(R(X,\xi)\cdot R)(U,V)W = 0,$$

and hence the right hand side of (3.6)is equal to zero, which yields

$$\begin{aligned}
&\eta(R(U,V)W)\eta(X) + \tilde{R}(U,V,W,X) \\
&-\eta(U)\eta(R(X,V)W) + g(X,U)\eta(R(\xi,V)W) - \eta(V)\eta(R(U,X)W) \\
&+g(X,V)\eta(R(U,\xi)W) - \eta(W)\eta(R(U,V)X) = 0.
\end{aligned} \tag{3.11}$$

Using (2.7)-(2.8) in (3.11) we obtain

$$\tilde{R}(U,V,W,X) = g(X,U)g(V,W) - g(X,V)g(U,W)$$

and hence

$$R(U,V)W = g(V,W)U - g(U,W)V, \tag{3.12}$$

which means that the manifold is a space of constant curvature 1.

Again replacing U by ξ in (3.10) and applying (2.1) and (2.10) we get

$$S(X,V) = (n-1)g(X,V) - 2(n-1)\eta(X)\eta(V). \tag{3.13}$$

for all X, V. This implies that the manifold is η-Einstein. Hence in view of (3.12) and (3.13) we can state the following :

Theorem 3.1. *A Ricci-generalized pseudosymmetric LP-Sasakian manifold $(M^n, g)(n > 3)$ is either a space of constant curvature 1 or an η-Einstein manifold.*

Differentiating (3.13) covariantly along Y and using (2.5)we get

$$(\nabla_Y S)(X,V) = 2(1-n)[\Omega(X,Y)\eta(V) + \Omega(Y,V)\eta(X)]. \tag{3.14}$$

Taking an orthonormal frame field and contracting (3.14) over Y and V we obtain

$$dr(X) = 4(1-n)\psi\eta(X),$$

where $\psi = Tr.\phi$ and r is the scalar curvature of the manifold. From the above relation, it follows that $dr(X) = 0$ if and only if $\psi = 0$, which implies that ξ is harmonic.
Thus we can state the following:

Theorem 3.2. *Let $(M^n, g)(n > 3)$ be a non-semisymmetric Ricci-generalized pseudosymmetric LP-Sasakian manifold . Then the scalar curvature of the manifold is constant if and only if the timelike vector field ξ is harmonic.*

Corollary 3.1. [6] *Let $(M^n, g)(n > 3)$ be an LP-Sasakian manifold satisfying the condition $S(X,\xi)\cdot R = 0$ Then the scalar curvature of the manifold is constant if and only if the timelike vector field ξ is harmonic.*

4 LP-Sasakian manifold satisfying some conditions on the Ricci tensor

Definition 4.1. *The Ricci tensor of an LP-Sasakian manifold is said to be η-recurrent if its Ricci tensor satisfies the following :*

$$(\nabla_X S)(\phi Y, \phi Z) = A(X)S(\phi Y, \phi Z) \tag{4.1}$$

for all X, Y, Z where $A(X) = g(X, \rho)$, ρ is the associated vector field of the 1-form A. In particular, if the 1-form A vanishes then the Ricci tensor of the LP-Sasakian manifold is said to be η-parallel and this notion for Sasakian manifolds was first introduced by Kon [3].

In view of (2.3), (2.4), (2.10) and (2.11), it can be easily seen that

$$(\nabla_X S)(\phi Y, \phi Z) = (\nabla_X S)(Y,Z) - S(X,\phi Z)\eta(Y) - S(X,\phi Y)\eta(Z)$$
$$+(n-1)[\Omega(X,Y)\eta(Z) + \Omega(X,Z)\eta(Y)]. \tag{4.2}$$

Using (2.11) and (4.2) in (4.1) we obtain

$$(\nabla_X S)(Y,Z) = S(X,\phi Z)\eta(Y) + S(X,\phi Y)\eta(Z) - (n-1)[\Omega(X,Y)\eta(Z)+$$

$$+\Omega(X,Z)\eta(Y)] + A(X)[S(Y,Z) + (n-1)\eta(Y)\eta(Z)]. \tag{4.3}$$

Hence we can state the following :

Theorem 4.1. *In an LP-Sasakian manifold $(M^n, g)(n > 3)$, the Ricci tensor is η-recurrent if and only if (4.3) holds.*

Let $\{e_i; i = 1, 2, ..., n\}$ be an orthonormal frame field at any point of the manifold. Then contracting over Y and Z in (4.3) we get

$$dr(X) = A(X)[r - (n-1)]. \tag{4.4}$$

Again contracting over X and Z in (2.4) we obtain

$$\frac{1}{2}dr(X) = [\mu - (n-1)\psi]\eta(Y) + S(Y,\rho) + (n-1)\eta(Y)\eta(\rho), \tag{4.5}$$

where $\mu = Tr.(Q\phi) = \sum_{i=1}^{n} \epsilon_i S(\phi e_i, e_i)$, $\psi = Tr.\phi = \sum_{i=1}^{n} \epsilon_i g(\phi e_i, e_i)$ and $\epsilon_i = g(e_i, e_i)$.
By virtue of (4.4) and (4.5) we get

$$\frac{1}{2}[r - (n-1)]A(Y) = [\mu - (n-1)\psi]\eta(Y) + S(Y,\rho) + (n-1)\eta(Y)\eta(\rho). \quad (4.6)$$

The relation (4.6) yields for $Y = \xi$

$$\frac{1}{2}[r - (n-1)]\eta(\rho) = [\mu - (n-1)\psi]. \quad (4.6)$$

In view of (4.6) and (4.7) we obtain

$$S(Y,\rho) = \frac{1}{2}[r - (n-1)]\{g(Y,\rho) + \eta(Y)\eta(\rho)\}. \quad (4.7)$$

This leads to the following:

Theorem 4.2. *If the Ricci tensor of an LP-Sasakian manifold (M^n, g) $(n > 3)$ is η-recurrent, then its Ricci tensor along the associated vector field of the 1-form is given by* (4.8).

Substituting Y by ϕY in (4.8) we obtain by virtue of (2.4) that

$$S(\phi Y,\rho) = \frac{1}{2}[r - (n-1)]g(\phi Y,\rho). \tag{4.9}$$

By virtue of Lemma 2.4 and the symmetry of ϕ we get from (4.9)

$$S(Y,L) = \alpha g(Y,L), \tag{4.10}$$

where $L = \phi\rho$ and $\alpha = \frac{1}{2}[r - (n-1)]$. From (4.10) we can state the following:

Theorem 4.3. *If the Ricci tensor of an LP-Sasakian manifold $(M^n, g)(n > 3)$ is η-recurrent, then $\alpha = \frac{1}{2}[r - (n-1)]$ is an eigenvalue of the Ricci tensor corresponding to the eigenvector $\phi\rho$ defined by $g(X, L) = T(X) = g(X, \phi\rho)$.*

Definition 4.2. *The Ricci tensor of an LP-Sasakian manifold is said to be ϕ-parallel if its Ricci tensor satisfies the following :*

$$(\nabla_{\phi X} S)(\phi Y, \phi Z) = 0 \tag{4.11}$$

for allX, Y, Z.

We note that the condition of ϕ-parallelity is more weaker than η-parallelity. Then proceeding in the same manner as before we can state the following:

Theorem 4.4. *The Ricci tensor of an LP-Sasakian manifold $(M^n, g)(n > 3)$ is ϕ-parallel if and only if the following relation holds:*

$$(\nabla_{\phi X} S)(Y, Z) = S(X, Y)\eta(Z) + S(X, Z)\eta(Y)$$
$$-(n-1)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y)] \tag{4.12}$$

for all X, Y, Z.

Let us now consider an LP-Sasakian manifold $(M^n, g)(n > 3)$, whose Ricci tensor is ϕ-parallel. Then (4.12) holds good. Putting $X = \phi X$ in (4.12) and then using (2.1) we get

$$(\nabla_X S)(Y, Z) + \eta(X)(\nabla_\xi S)(Y, Z) = S(\phi X, Y)\eta(Z) + S(\phi X, Z)\eta(Y)$$
$$-(n-1)[\Omega(X, Y)\eta(Z) + \Omega(X, Z)\eta(Y)]. \tag{4.13}$$

Taking an orthonormal frame field at any point of the manifold and contracting over X and Z in (4.13) we obtain

$$dr(Y) = 2[\mu - (n-1)\psi]\eta(Y), \tag{4.14}$$

where $\mu = Tr.(Q\phi)$
From (4.14) we can state the following:

Theorem 4.5. *Let the Ricci tensor of an LP-Sasakian manifold $(M^n, g)(n > 3)$ be ϕ-parallel. Then the scalar curvature of the manifold is constant if and only if $\psi = \frac{\mu}{n-1}$, where $\mu = Tr.(Q\phi)$ and $\psi = Tr.(\phi)$.*

Corollary 4.1. *Let $(M^n, g)(n > 3)$ be an LP-Sasakian manifold with η-parallel Ricci tensor. Then the scalar curvature of the manifold is constant.*

Definition 4.3. *The Ricci tensor S of an LP-Sasakian manifold is said to be ϕ-recurrent if it satisfies*

$$(\nabla_{\phi X} S)(\phi Y, \phi Z) = A(\phi X)S(\phi Y, \phi Z) \tag{4.15}$$

for all X, Y, Z.

Let the Ricci tensor of an LP-Sasakian manifold $(M^n, g)(n > 3)$ be ϕ-recurrent. Then (4.15) holds. Hence proceeding similarly as before it can be easily seen that

$$S(X, \rho) = \tfrac{1}{2}[r - (n-1)]g(X, \rho) + \tfrac{1}{2}[r - 3(n-1)]\eta(Y)\eta(\rho). \tag{4.16}$$

By virtue of (4.16) we can state the following:

Theorem 4.6. *Let $(M^n, g)(n > 3)$ be an LP-Sasakian manifold with ϕ-recurrent Ricci tensor. Then its Ricci tensor along the associated vector field of the 1-form is given by (4.16) and also $\frac{1}{2}[r - (n-1)]$ is the eigenvalue of the Ricci tensor corresponding to the eigenvector $\phi\rho$, where ρ is the associated vector field of the 1-form A.*

5 ϕ-conformally flat LP-Sasakian manifolds

Definition 5.1. *An LP-Sasakian manifold* $(M^n, g)(n > 3)$ *is said to be* ϕ-*conformally flat if it satisfies*

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0 \tag{5.1}$$

for any vector fields X, Y, Z *in* T_pM.

The notion of ϕ-conformally flat for K-contact manifolds was first introduced by G. Zhen [7].

Let us consider an LP-Sasakian manifold $(M^n, g)(n > 3)$ which is ϕ-conformally flat. Then (5.1) holds. By virtue of (2.1) and (2.4), (5.1) yields for any $W \in T_pM$

$$g(C(\phi X, \phi Y))\phi Z, \phi W) = 0.$$

Hence using the definition of conformal curvature tensor, the above relation implies

$$\begin{aligned}&\tilde{R}(\phi X, \phi Y, \phi Z, \phi W) = \tfrac{1}{n-2}[S(\phi Y, \phi Z)g(\phi X, \phi W)\\ &-S(\phi X, \phi Z)g(\phi Y, \phi W) + g(\phi Y, \phi Z)S(\phi X, \phi W)\\ &-g(\phi X, \phi Z)S(\phi Y, \phi W)] - \tfrac{r}{(n-1)(n-2)}[g(\phi Y, \phi Z)g(\phi X, \phi W)\\ &-g(\phi X, \phi Z)g(\phi Y, \phi W)],\end{aligned} \tag{5.2}$$

where $\tilde{R}(\phi X, \phi Y, \phi Z, \phi W)= g(R(\phi X, \phi Y)\phi Z, \phi W)$.

Using Lemma 2.3, (2.11) and (2.2) in (5.2) we obtain

$$\begin{aligned}&\tilde{R}(X, Y, Z, W) + g(Y, Z)\eta(X)\eta(W) + g(X, W)\eta(Y)\eta(Z)\\ &-g(X, Z)\eta(Y)\eta(W) - g(Y, W)\eta(X)\eta(Z)\\ &= \tfrac{1}{n-2}[\{S(Y, Z) + (n-1)\eta(Y)\eta(Z)\}\{g(X, W) + \eta(X)\eta(W)\}\\ &-\{S(X, Z) + (n-1)\eta(X)\eta(Z)\}\{g(Y, W) + \eta(Y)\eta(W)\}\\ &+\{g(Y, Z) + \eta(Y)\eta(Z)\}\{S(X, W) + (n-1)\eta(X)\eta(W)\}\\ &-\{g(X, Z) + \eta(X)\eta(Z)\}\{S(Y, W) + (n-1)\eta(Y)\eta(W)\}]\\ &-\tfrac{r}{(n-1)(n-2)}[\{g(Y, Z) + \eta(Y)\eta(Z)\}\{g(X, W) + \eta(X)\eta(W)\}\\ &-[\{g(X, Z) + \eta(X)\eta(Z)\}\{g(Y, W) + \eta(Y)\eta(W)\}].\end{aligned} \tag{5.3}$$

Taking an orthonormal frame field and contracting over X and W in (5.3), it follows that

$$S(Y, Z) = (\tfrac{r}{n-1} - 1)g(Y, Z) + (\tfrac{r}{n-1} - n)\eta(Y)\eta(Z). \tag{5.4}$$

This leads to the following :

Theorem 5.1. *A* ϕ-*conformally flat LP-Sasakian manifold* $(M^n, g)(n > 3)$ *is an* η-*Einstein manifold.*

Again from (5.4), it follows by virtue of (2.5) that

$$(\nabla_X S)(Y, Z) = \tfrac{dr(X)}{n-1}[g(Y, Z) + \eta(Y)\eta(Z)]$$

$$+(\tfrac{r}{n-1}-n)[\Omega(X,Y)\eta(Z)+\Omega(X,Z)\eta(Y)]. \quad (5.5)$$

Taking an orthonormal frame field at any point of the manifold and contracting over X and Z in (5.5) we have

$$\tfrac{n-3}{2}dr(Y)=dr(\xi)\eta(Y)+[r-n(n-1)]\psi\eta(Y). \quad (5.6)$$

Replacing Y by ξ in (5.6) we get

$$dr(\xi)=\tfrac{-2}{n-1}[r-n(n-1)]\psi. \quad (5.7)$$

Using (5.7) in (5.6), it follows that (since $n>3$)

$$\tfrac{1}{2}dr(Y)=\tfrac{1}{n-1}[r-n(n-1)]\psi\eta(Y). \quad (5.8)$$

Now, if r is constant, then (5.8) implies that either $r=n(n-1)$ or $\psi=0$. If $r=n(n-1)$, then (5.4) takes the form

$$S(Y,Z)=(n-1)g(Y,Z) \text{ for all } Y,Z. \quad (5.9)$$

By virtue of (5.9), the relation (5.3) reduces to

$$\tilde{R}(X,Y,Z,W)=g(Y,Z)g(X,W)-g(X,Z)g(Y,W),$$

which implies that the manifold is a space of constant curvature 1. Hence we can state the following:

Theorem 5.2. *Let $(M^n,g)(n>3)$ be a ϕ-conformally flat LP-Sasakian manifold. If the scalar curvature of the manifold is constant, then either the manifold is a space of constant curvature 1 or the timelike vector field ξ is harmonic.*

Corollary 5.1. [2] *A conformally flat LP-Sasakian manifold $(M^n,g)(n>3)$ is a space of constant curvature 1.*

Again, if $r\neq n(n-1)$, then (5.8) implies that r is constant if and only if $\psi=0$. Thus we can state the following:

Theorem 5.3. *Let $M^n(\phi,\xi,\eta,g)(n>3)$ be a ϕ-conformally flat LP-Sasakian manifold which is not an Einstein one. Then the scalar curvature of M^n is constant if and only if the timelike vector field ξ is harmonic.*

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Department of Mathematics,
University of Burdwan,
Golapbag, Burdwan-713 104,
West Bengal,
INDIA.
E-mail: `aask@epatra.com`
E-mail: `aask2003@yahoo.co.in`