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On the solution of a simple differential game with a singular focal line

by
T. Bouremani and Ş. Mirică

Abstract

We apply the ,,User's Guide" on Dynamic programming described recently by the first author to obtain the complete and theoretically justified solution of a differential game considered first by Breakwell and Bernhard ([4]). The optimal feedback strategies and the corresponding value function are constructed using a certain refinement of Cauchy's Method of characteristics for stratified Hamilton-Jacobi equations while the optimality is proved using a suitable ,,verification theorem" for locally-Lipschitz value function. In fact, the partial and rather ,,artisanal" solution in Breakwell and Bernhard ([4]) of this problem, is not only obtained by a general procedure but also ,,justified" in a rigorous theoretical setting of suitable concepts and results.

Key Words: Differential game, Dynamic Programming, feedback strategy, value function, verification theorem.
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1 Introduction

The aim of this paper is to use in a ,,step by step" manner the ,,theoretical algorithm" in Mirică ([12]), to obtain a rigorous solution of the differential game considered in Breakwell and Bernhard ([4]) as ,,*a simple game with a singular focal line*".

In fact, the study in [4] (see also [8]) concerns the ,,local behavior" (,,study in the small" in Isaacs' terminology) of trajectories of a non-smooth (in our case, ,,stratified") Hamiltonian system, around points situated on certain types of ,,discontinuity surfaces"; besides the fact that the Hamiltonian system it is not rigorously defined on the discontinuity surfaces, one may note also that in this approach there are neither clear criteria for choosing the ,,optimal trajectories" nor suitable ,,*verification theorems*" proving the optimality of the chosen trajectories, since the use of the theory of viscosity solutions it is not quite justified in this case (the fact that the value function is a viscosity solution it is proved only for fixed time-interval non-autonomous differential games in the case of non-anticipative open loop strategies and moreover, one needs to prove the uniqueness of the viscosity solution).

In this paper we use the theoretical approach in Mirică ([9]-[11]), summarized in Mirică ([12]) in the form of an ,,user's guide", to obtain a more complete and theoretically justified solution of this problem; more precisely, we use first a certain refinement of Cauchy's Method of characteristics for stratified Hamiltonian-Jacobi equations to identify a large set of ,,extremal" (possibly optimal) trajectories; as rigorous criterion for choosing the optimal trajectories we are using the associated ,,extremal" value functions, from which, only the ,,maximal" one proved to be admissible and associated to a certain pair of feedback strategies.

Finally, the optimality of the corresponding pair of feedback strategies is proved using the verification theorem for locally-Lipschitz value functions.

The paper is organized as follows: after the formulation of the problem and identification of data in *Section 2*, in the next sections are computed successively the Hamiltonian and the set of ,,transversality" terminal points, the generalized (,,stratified") Hamiltonian system, the ,,partial" Hamiltonian flows ending on each of the strata and the ,,partial" (smooth) value functions; finally, in the last sections we show that only the ,,maximal type" value function defines an admissible pair of feedback strategies and its optimality is proved using the verification theorem for locally-Lipschitz value functions.

2 Statement of the problem and identification of data

The differential game in Breakwell and Bernhard ([4]) in a rather vague formulation, is stated as follows:

Problem 2.1. Given $k > 0$, find:

$$\inf_{u(.)} \sup_{v(.)} [t_f + k|x(t_f)|],$$

subject to:

$$\begin{cases} x' = u(t) + 2v(t)e^{-y}, \ x(0) = x_0 \\ y' = |u(t)| - 2, \ y(0) = y_0, \end{cases}$$
$$u(t) \in U := [-1, 1], \ v(t) \in V := [-1, 1],$$
$$y(t) > 0 \ \forall \ t \in [0, t_f), \ y(t_1) = 0.$$

Therefore, this is a particular example of a ,,standard" autonomous differential game formulated as follows:

Problem 2.2. (DG_A-vague formulation). **Find**:

$$\inf_{u(.) \in \mathcal{U}_\alpha} \sup_{v(.) \in \mathcal{V}_\alpha} \mathcal{C}(y; u(.), v(.)) \ \forall \ y \in Y_0, \tag{2.1}$$

subject to

$$\mathcal{C}(y;u(.),v(.)) := g(x(t_1)) + \int_0^{t_1} f_0(x(t),u(t),v(t))dt,\ y \in Y_0, \tag{2.2}$$

$$x'(t) = f(x(t),u(t),v(t))\ a.e.\ (0,t_1),\ x(0) = y, \tag{2.3}$$

$$u(t) \in U(x(t)),\ v(t) \in V(x(t))\ a.e.\ (0,t_1),\ (u(.),v(.)) \in \mathcal{P}_\alpha, \tag{2.4}$$

$$\widehat{x} := (x(.),x_0(.)) \in \Omega_\alpha,\ x_0(t) := \int_0^t f_0(x(s),u(s),v(s)), \tag{2.5}$$

$$x(t) \in Y_0,\ \forall\ t \in [0,t_1),\ x(t_1) \in Y_1, \tag{2.6}$$

defined, in our case, by the following data:

$$\begin{array}{l} f(x,u,v) := (u + 2ve^{-x_2}, |u| - 2),\ f_0(x,u,v) \equiv 1, \\ u \in U(x) \equiv U := [-1,1],\ v \in V(x) \equiv V := [-1,1],\ g(\xi) = k|\xi_1|, \\ \mathcal{P}_\alpha = \mathcal{P}_1,\ \Omega_\alpha = \Omega_1,\ Y_0 := \mathbf{R} \times (0,+\infty),\ Y_1 := \mathbf{R} \times \{0\} \end{array} \tag{2.7}$$

where $\mathcal{P}_1 = \mathcal{U} \times \mathcal{V}$ is the (largest) class of measurable admissible control functions $(u(.),v(.))$ and Ω_1 is the corresponding class of absolutely continuous (hence Lipschitzian, in our case) admissible trajectories.

Since this formulation in the framework of ,,open-loop controls" is rather vague and without any ,,practical utility", a more accurate and realistic formulation may be obtained only in the framework of ,,feedback strategies"; in fact the Algorithm in Mirică ([12]) we are going to apply in what follows, is trying to solve the following problem:

Problem 2.3. (DG_A-accurate formulation). *Given the data of Problem 2.2* **find the feedback strategies** $\widetilde{U}(x) \subset U(x),\ \widetilde{V}(x) \subset V(x),\ x \in \widetilde{Y}_0 \subseteq Y_0$ with the following properties:

(A) The pair $(\widetilde{U}(.),\widetilde{V}(.))$ is **admissible** in the sense that for any $y \in \widetilde{Y}_0$ the set $\widetilde{\Omega}_\alpha(y)$ of trajectories $x_y(.) \in \Omega_\alpha(y)$ of the differential inclusion:

$$x' \in f(x,\widetilde{U}(x),\widetilde{V}(x)),\ x(0) = y \in \widetilde{Y}_0, \tag{2.8}$$

that satisfy the constraints:

$$x_y(t) \in \widetilde{Y}_0\ \forall t \in [0,\widetilde{t}_1),\ x_y(\widetilde{t}_1) \in \widetilde{Y}_1 \subseteq Y_1,\ \widetilde{t}_1 = \widetilde{t}_1(x_y(.)), \tag{2.9}$$

is not empty; moreover, if $\widetilde{\mathcal{P}}_\alpha(y),\ y \in \widetilde{Y}_0$ are the corresponding sets of control mappings, $(u_y(.),v_y(.)) \in \mathcal{P}_\alpha$ that satisfy:

$$\begin{array}{l} x_y'(t) = f(x_y(t),u_y(t),v_y(t)),\ u_y(t) \in \widetilde{U}(x_y(t)), \\ v_y(t) \in \widetilde{V}(x_y(t))\ a.e.\ (0,\widetilde{t}_1), \end{array} \tag{2.10}$$

then there exists the associated value function defined by:

$$\widetilde{W}_0(y) := \mathcal{C}(y; u_y(.), v_y(.)) \ \forall (u_y(.), v_y(.)) \in \widetilde{\mathcal{P}}_\alpha(y), \ y \in \widetilde{Y}_0,$$
$$\widetilde{W}(y) := \begin{cases} \widetilde{W}_0(y) \ if \ y \in \widetilde{Y}_0, \\ g(y) \ if \ y \in \widetilde{Y}_1 := \{x_y(\widetilde{t}_1); x_y(.) \in \widetilde{\Omega}_\alpha(y), y \in \widetilde{Y}_0\}, \end{cases} \tag{2.11}$$

i.e. if $\widetilde{\mathcal{P}}_\alpha(y), \ y \in \widetilde{Y}_0$ contains more than one element, then $\mathcal{C}(y; u^1(.), v^1(.)) = \mathcal{C}(y; u^2(.), v^2(.)) \ \forall \ (u^j(.), v^j(.)) \in \widetilde{\mathcal{P}}_\alpha(y), \ j = 1, 2.$

(B) The feedback strategies $\widetilde{U}(.), \ \widetilde{V}(.)$ are **relatively optimal** for the restriction $DG_A|_{\widetilde{Y}_0}$ to mean that:

$$\widetilde{W}_0(y) = \min_{u(.), \overline{v}(.)} \mathcal{C}(y; u(.), \overline{v}(.)) \ \forall \ y \in \widetilde{Y}_0, \tag{2.12}$$

subject to:

$$x'(t) = f(x(t), u(t), \overline{v}(t)), \ u(t) \in U(x(t)), \ \overline{v}(t) \in \widetilde{V}(x(t)) \ a.e. \ (0, t_1), \tag{2.13}$$

$$x(0) = y, \ x(t) \in \widetilde{Y}_0 \ \forall \ t \in [0, t_1), \ x(t_1) \in \widetilde{Y}_1, \tag{2.14}$$

and also:

$$\widetilde{W}_0(y) = \max_{\overline{u}(.), v(.)} \mathcal{C}(y; \overline{u}(.), v(.)) \ \forall \ y \in \widetilde{Y}_0, \tag{2.15}$$

subject to (2.14) and to

$$x'(t) = f(x(t), \overline{u}(t), v(t)), \ \overline{u}(t) \in \widetilde{U}(x(t)), \ v(t) \in V(x(t)) \ a.e. \ (0, t_1). \tag{2.16}$$

(C) Either $\widetilde{Y}_0 = Y_0$ (hence $DG_A|_{\widetilde{Y}_0} = DG_A$) or the subset $\widetilde{Y}_0 \subset Y_0$ is **invariant with respect to the control system** in (2.3)-(2.4) to mean that for any $y \in \widetilde{Y}_0$ and for any two controls $(u(.), v(.)) \in \mathcal{U}_\alpha \times \mathcal{V}_\alpha$, the corresponding trajectory $x(.)$ satisfying (2.3) satisfies the state constraints $x(t) \in \widetilde{Y}_0 \ \forall \ t > 0$.

Remark 2.4. If condition (C) above is not satisfied, then $(\widetilde{U}(.), \widetilde{V}(.))$ cannot be considered a ,,satisfactory" solution of the problem DG_A since if $\widetilde{Y}_0 \subset Y_0$ is not invariant then at least one of the two players may choose strategies that ,,produce" trajectories $x(.)$ leaving the subset $\widetilde{Y}_0$ and for which the optimality conditions in statement (B) above are no longer satisfied. In this case one should use additional arguments to establish the real nature of the ,,restricted problem" $DG_A|_{\widetilde{Y}_0}$ solved by $(\widetilde{U}(.), \widetilde{V}(.), \widetilde{W}_0(.))$.

In what follows, we shall apply ,,step by step" the theoretical algorithm presented in [12] for the Problem 2.3.

3 The Hamiltonian and the set of ,,transversality" terminal points

The ,,pseudo-Hamiltonian" $\mathcal{H}(x, p, u, v) :=< p, f(x, u, v) > +f_0(x, u, v)$ is given in our case by:

$$\mathcal{H}(x, p, u, v) = p_1 u + p_2 |u| + 2p_1 v e^{-x_2} - 2p_2 + 1. \tag{3.1}$$

Using the well known fact that: $\sup_{v \in V}[p_1.v] = |p_1|$ and the fact that:

$$\min_{u \in U}[p_1 u + p_2|u|] = \min\{0, p_1 + p_2, p_2 - p_1\} = \begin{cases} 0 \; if \; p_2 \geq |p_1| \\ p_2 - |p_1| \; if \; p_2 < |p_1|, \end{cases}$$

the Isaacs' Hamiltonian:

$$H(x,p) := \min_{u \in U} \max_{v \in V} \mathcal{H}(x,p,u,v) = \max_{v \in V} \min_{u \in U} \mathcal{H}(x,p,u,v)$$

as well as the corresponding ,,extremal value" of the control parameters turn out to be defined on $Z := Y_0 \times \mathbf{R^2}$ by:

$$H(x,p) = \begin{cases} 2|p_1|e^{-x_2} - 2p_2 + 1, \; p_2 \geq |p_1|, \; (x,p) \in Z \\ |p_1|(2e^{-x_2} - 1) - p_2 + 1, \; p_2 < |p_1|, \end{cases} \tag{3.2}$$

$$\widehat{U}(x,p) = \begin{cases} \{0\} \; if \; p_2 > |p_1| \\ \{1\} \; if \; p_2 < -p_1, \; p_1 < 0 \\ U_1 = [0,1] \; if \; p_2 = -p_1 \\ \{-1\} \; if \; p_2 < p_1, \; p_1 > 0 \\ U_2 = [-1,0] \; if \; p_2 = p_1 \\ \{-1,1\} \; if \; p_2 < 0, \; p_1 = 0, \end{cases}$$
$$\widehat{V}(x,p) = \begin{cases} \{1\} \; if \; p_1 > 0 \\ \{-1\} \; if \; p_1 < 0 \\ V := [-1,1] \; if \; p_1 = 0. \end{cases} \tag{3.3}$$

Next, we need to compute the set of terminal ,,transversality" values defined in the general case by:

$$Z_1^* = \{(\xi, q) \in Y_1 \times \mathbf{R^2}, \; H(\xi,q) = 0, \; < q, \overline{\xi} >= Dg(\xi)\overline{\xi} \; \forall \; \overline{\xi} \in T_\xi Y_1\}.$$

Since $g(.) : Y_1 \to \mathbf{R}$ is stratified by $\mathcal{S}_g = \{Y_1^+, Y_1^-, Y_1^0\}$ where $Y_1^\pm := \{\xi = (\xi_1, 0); \xi_1 \in R_\pm^*\}$, $Y_1^0 := \{(0,0)\}$ and the ,,stratified" derivative of $g(.)$ is given by:

$$Dg(\xi) = \begin{cases} (\pm k, 0) \; if \; \xi \in Y_1^\pm \\ (0,0) \; if \; \xi \in Y_1^0, \end{cases}$$

a point $(\xi, q) \in Z_1^*$ is completely characterized by:

$$q_2 = |q_1| + \frac{1}{2}, \; q_1 = \pm k \; if \; \xi \in Y_1^\pm, \; q_1 \in \mathbf{R} \; if \; \xi \in Y_1^0,$$

hence the set Z_1^* above is the union of the following subsets:

$$\begin{aligned} Z_{1,+}^* &:= \{((\xi_1, 0), (k, k + \tfrac{1}{2})); \; \xi_1 > 0\}, \\ Z_{1,-}^* &:= \{((\xi_1, 0), (-k, k + \tfrac{1}{2})); \; \xi_1 < 0\}, \\ Z_{1,0}^* &:= \{((0,0), (q_1, Q(q_1))); \; Q(q_1) := |q_1| + \tfrac{1}{2}, \; q_1 \in \mathbf{R}\}. \end{aligned} \tag{3.4}$$

4 The generalized Hamiltonian and characteristic flows

The first main computational operation consists in the „backward integration" (for $t \leq 0$), of the Hamiltonian inclusion:

$$(x', p') \in d_S^{\#} H(x, p), \ (x(0), p(0)) = z = (\xi, q) \in Z_1^*, \tag{4.1}$$

defined by the generalized Hamiltonian orientor field $d_S^{\#} H(.,.)$:

$$\begin{aligned} d_S^{\#} H(x, p) &:= \{(x', p') \in T_{(x,p)} Z; \ x' \in f(x, \widehat{U}(x, p), \widehat{V}(x, p)), \\ &< x', \overline{p} > - < p', \overline{x} >= DH(x, p).(\overline{x}, \overline{p}) \ \forall \ (\overline{x}, \overline{p}) \in T_{(x,p)} Z\}. \end{aligned} \tag{4.2}$$

As it is specified in the algorithm in [12], for each „terminal point" $z = (\xi, q) \in Z_1^*$ one should identify the maximal solutions: $X^*(.) := (X(.), P(.)) : I(z) := (t^-(z), 0] \to Z$, of the Hamiltonian inclusion in (4.1) that satisfy the following conditions:

$$\begin{aligned} & X(t) = (X_1(t), X_2(t)) \in Y_0 \ \forall \ t \in I_0(z) := (t^-(z), 0) \\ & H(X(t), P(t)) = 0 \ \forall \ t \in I(z) \\ & X'(t) = f(X(t), u_X(t), v_X(t)) \ a.e. \ I_0(z), \ (u_X(.), v_X(.)) \in \mathcal{P}_1 \\ & u_X(t) \in \widehat{U}(X^*(t)), \ v_X(t) \in \widehat{V}(X^*(t)) \ a.e. \ I_0(z). \end{aligned} \tag{4.3}$$

In the case in which there exist more such solutions for the same terminal point $z = (\xi, q) \in Z_1^*$, one should parameterize by $\lambda \in \Lambda(z)$ the set of these solutions to obtain a generalized Hamiltonian flow: $X^*(.,.) := (X(.,.), P(.,.)) : B := \{(t, a); \ t \in I(a), \ a \in A\} \to Z; \ A := graph(\Lambda(.)), \ a = (z, \lambda)$. We recall also the fact that for each $(t, a) \in B_0 := \{(t, a) \in B; \ t \neq 0\}$ the Hamiltonian flow $X^*(.,.)$ defines the controls and, respectively, trajectories:

$$\begin{aligned} & u_{t,a}(s) := u_a(t+s), \ v_{t,a}(s) := v_a(t+s), \ s \in [0, -t] \\ & x_{t,a}(s) := X(t+s, a), \end{aligned} \tag{4.4}$$

which are admissible with respect to the initial point $y = X(t, a) \in Y_0$, and for which the value of the cost functional in (2.2) is given by the function $V(.,.)$ defined by:

$$V(t, a) := g(\xi) + \int_0^t < P(\sigma, a), X'(\sigma, a) > d\sigma, \ if \ a = (\xi, q, \lambda) \tag{4.5}$$

and which, together with the Hamiltonian flow $X^*(.,.)$ defines the generalized characteristic flow $C^*(.,.) := (X^*(.,.), V(.,.))$; using the definition of the Hamiltonian $H(.,.)$ and the second condition in (4.3) one has $< P(\sigma, \xi_1), X'(\sigma, \xi_1) >= -f_0(X(\sigma, \xi_1), \widehat{u}(X^*(\sigma, \xi_1)), \widehat{v}(X^*(\sigma, \xi_1))) \equiv -1$; it follows from (2.7) that in our case the function $V(.,.)$ is given by:

$$V(t, a) = k|\xi_1| - t \ \forall \ (t, a) \in B, \ a = (\xi, q, \lambda), \ \xi = (\xi_1, 0). \tag{4.6}$$

First, we remark that the Hamiltonian $H(.,.)$ in (3.2) as well as its domain $Z \subset R^2 \times R^2$ are $\mathcal{C}^1$-stratified by the stratification $\mathcal{S}_H = \{Z_1^{\pm}, Z_2^{\pm}, Z_0^{\pm}, Z_0^{0,\pm}\}$ defined by:

$$\begin{array}{l} Z_1^{\pm} := \{(x,p) \in Z;\ p_1 \in R_{\pm}^*,\ p_2 > \pm p_1\} \\ Z_2^{\pm} := \{(x,p) \in Z;\ p_1 \in R_{\pm}^*,\ p_2 < \pm p_1\} \\ Z_0^{\pm} := \{(x,p) \in Z;\ p_1 \in R_{\pm}^*,\ p_2 = \pm p_1\} \\ Z_0^{0,\pm} := \{(x,p) \in Z;\ p_1 = 0,\ p_2 \in R_{\pm}^*\}. \end{array} \tag{4.7}$$

If we denote by: $H_j^{\pm}(.,.) := H(.,.)|_{Z_j^{\pm}},\ j = 1,2,\ H_0^{\pm}(.,.) := H(.,.)|_{Z_0^{\pm}}$, $H_0^{0,\pm}(.,.) := H(.,.)|_{Z_0^{0,\pm}}$, then from (3.2) it follows:

$$\begin{array}{l} H_1^{\pm}(x,p) = \pm 2p_1 e^{-x_2} - 2p_2 + 1,\ (x,p) \in Z_1^{\pm} \\ H_2^{\pm}(x,p) = \pm p_1(2e^{-x_2} - 1) - p_2 + 1,\ (x,p) \in Z_2^{\pm} \\ H_0^{\pm}(x,p) = \pm 2p_1(2e^{-x_2} - 1) + 1,\ (x,p) \in Z_0^{\pm} \\ H_0^{0,+}(x,p) = 1 - 2p_2,\ (x,p) \in Z_0^{0,+} \\ H_0^{0,-}(x,p) = 1 - p_2,\ (x,p) \in Z_0^{0,-} \\ d_S^{\#}H(x,p) := \left\{ \begin{array}{l} d_S^{\#}H_j^{\pm}(x,p),\ (x,p) \in Z_j^{\pm},\ j = 1,2 \\ d_S^{\#}H_0^{\pm}(x,p),\ (x,p) \in Z_0^{\pm} \\ d_S^{\#}H_0^{0,\pm}(x,p),\ (x,p) \in Z_0^{0,\pm}. \end{array} \right. \end{array} \tag{4.8}$$

Since the manifolds $Z_j^{\pm} \subset Z,\ j = 1,2$ are open subsets, the Hamiltonians orientor fields $d_S^{\#}H_j^{\pm}(.,.),\ j = 1,2$ in (4.2) coincide with classical Hamiltonian vector fields:

$$d_S^{\#}H_j^{\pm}(x,p) := \{(\frac{\partial H_j^{\pm}}{\partial p}(x,p), -\frac{\partial H_j^{\pm}}{\partial x}(x,p))\},\ (x,p) \in Z_j^{\pm},\ j = 1,2, \tag{4.9}$$

which are easy to calculate and will be described and studied later, while on the 3-dimensional singular strata $Z_0^{\pm},\ Z_0^{0,\pm} \subset Z$ the corresponding Hamiltonian fields are more difficult to compute.

The Hamiltonian system on the singular stratum $Z_0^{0,+}$.

In order to compute the generalized Hamiltonian field $d_S^{\#}H_0^{0,+}(.,.)$, we note first that, according to certain classical results, the tangent space to the 3-dimensional manifolds $Z_0^{0,\pm}$ are given by:

$$T_{(x,p)}Z_0^{0,\pm} = \{(\overline{x},\overline{p}) \in R^2 \times R^2;\ \overline{p}_1 = 0\} \tag{4.10}$$

and $DH_0^{0,+}(x,p).(\overline{x},\overline{p}) = -2\overline{p}_2$; therefore a vector $(x',p') \in d_S^{\#}H_0^{0,+}(x,p)$ is fully characterized by the properties:

$$\begin{array}{l} x_1' \in R, x' \in f(x, \widehat{U}(x,p), \widehat{V}(x,p)),\ \widehat{U}(x,p) = \{0\},\ \widehat{V}(x,p) = [-1,1] \\ (x_2' + 2)\overline{p}_2 - p_1'\overline{x}_1 - p_2'\overline{x}_2 = 0\ \forall\ \overline{x}_1, \overline{x}_2, \overline{p}_2 \in R. \end{array}$$

It follows that at each point (x,p) in the singular stratum $Z_0^{0,+}$ one has:

$$d_S^{\#}H_0^{0,+}(x,p) = \{((2ve^{-x_2}, -2), (0,0)); v \in [-1,1]\} \tag{4.11}$$

and it defines the differential inclusion:

$$\begin{cases} x' \in \{(2ve^{-x_2}, -2),\ v \in [-1,1]\},\ (x(0), p(0)) = z = (\xi, q) \in Z_1^* \\ p' = (0,0), \end{cases} \tag{4.12}$$

which, for each terminal point $(\xi, q) \in Z_0^{0,+}$, has an infinite number of solutions described by the set $\mathcal{V}$ of measurable functions $v(.) : (-\infty, 0] \to V$ for which $\frac{x_1'(t)}{2e^{-x_2(t)}} = v(t) \in [-1,1]$ *a.e.* $(-\infty, 0)$; as we shall see later, the same domain is covered by „the constant control mappings" $v(t) \equiv \lambda \in [-1,1]\ \forall\ t \in (-\infty, 0]$; we obtain a set of trajectories of the inclusion (4.12) in the form of the maximal flow $X_{0,+}^{0,*}(.,.)$ described by the formulas:

$$\begin{cases} X_0^{0,+}(t; \xi, \lambda) = (\lambda e^{-\xi_2}(e^{2t} - 1) + \xi_1, -2t + \xi_2),\ z = (\xi, q) \in Z_1^* \\ P_0^{0,+}(t; \xi, \lambda) = q = (q_1, q_2),\ \lambda \in [-1,1]. \end{cases} \tag{4.13}$$

The Hamiltonian system on the singular stratum $Z_0^{0,-}$.

Symmetrically, elementary computations and arguments show that a vector $(x', p') \in d_S^\# H_0^{0,-}(x,p)$ is fully characterized by the properties:

$$\begin{aligned} &x_1' \in R,\ x' \in f(x, \widehat{U}(x,p), \widehat{V}(x,p)),\ \widehat{U}(x,p) = \{+1, -1\},\ \widehat{V}(x,p) = [-1,1] \\ &(x_2' + 1)\overline{p}_2 - p_1'\overline{x}_1 - p_2'\overline{x}_2 = 0\ \forall\ \overline{x}_1, \overline{x}_2, \overline{p}_2 \in R. \end{aligned}$$

It follows that at each point (x,p) in the singular stratum $Z_0^{0,-}$ one has:

$$d_S^\# H_0^{0,-}(x,p) = \{((u + 2ve^{-x_2}, -1), (0,0)); u \in \{-1,1\}, v \in [-1,1]\} \tag{4.14}$$

and it defines the differential inclusion:

$$\begin{cases} x' \in \{(\{1,-1\} + 2ve^{-x_2}, -1),\ v \in [-1,1]\} \\ p' = (0,0),\ (x(0), p(0)) = z = (\xi, q) \in Z_1^*. \end{cases} \tag{4.15}$$

The set of all the trajectories of this inclusion is described by the set of measurable mappings $u(.) : (-\infty, 0] \to \{-1, 1\},\ v(.) : (-\infty, 0] \to [-1,1]$ which define the differential systems:

$$\begin{cases} x' = (u(t) + 2v(t)e^{-x_2}, -1),\ (x(0), p(0)) = (\xi, q) \in Z_1^* \\ p' = (0,0), \end{cases} \tag{4.16}$$

whose solutions $X_{0,-}^{0,*}(.,.)$ are given by:

$$\begin{cases} X_{1,0}^{0,-}(t, \xi, u(.), v(.)) = \xi_1 + \int_0^t [u(s) + 2v(s)e^{s - \xi_2}] ds \\ X_{2,0}^{0,-}(t, \xi, u(.), v(.)) = -t + \xi_2,\ P_0^{0,-}(t,q) = q,\ z = (\xi, q) \in Z_1^*. \end{cases} \tag{4.17}$$

The Hamiltonian system on the singular strata $Z_0^{\pm}$.

In order to compute the generalized Hamiltonian field $d_S^\# H_0^{\pm}(.,.)$, we note first that, according some classical results, the tangent space to the 3-dimensional manifolds $Z_0^{\pm}$ for which $p_2 = \pm p_1,\ p_1 \in R_{\pm}^*$ are given by:

$$T_{(x,p)} Z_0^{\pm} = \{(\overline{x}, \overline{p}) \in R^2 \times R^2;\ \overline{p}_2 = \pm \overline{p}_1\}; \tag{4.18}$$

on the stratum Z_0^+ one has $DH_0^+(x,p).(\overline{x},\overline{p}) = 2[(e^{-x_2}-1)\overline{p}_1 - p_1\overline{x}_2 e^{-x_2}]$ hence a vector $(x',p') \in d_S^{\#}H_0^+(x,p)$ is fully characterized by the properties:

$$p_1' = p_2', [x_1' + x_2' - 2(e^{-x_2}-1)]\overline{p}_1 - p_1'\overline{x}_1 - (p_2' - 2p_1 e^{-x_2})\overline{x}_2 = 0 \;\forall\; \overline{p}_1, \overline{x}_1, \overline{x}_2.$$

It follows that at each point (x,p) in the stratum Z_0^+ one has:

$$p_1' = 0,\ p_2' = 2p_1 e^{-x_2},\ x_1' + x_2' = 2(e^{-x_2}-1). \tag{4.19}$$

Using the fact that if $(x',p') \in T_{(x,p)}Z_0^+$ then $p_1' = p_2'$, while from (4.19) it follows that $p_1' = 0 = p_2' = 2p_1 e^{-x_2}$ hence $p_1 = 0$; this contradicts the fact that $(x,p) \in Z_0^+$ and therefore $d_S^{\#}H_0^+(x,p) = \emptyset$.

Symmetrically, on the singular stratum Z_0^-, using the same type of computations and arguments as above we obtain:

$$p_1' = 0,\ p_2' = -2p_1 e^{-x_2},\ x_1' - x_2' = 2(1 - e^{-x_2}); \tag{4.20}$$

since $(x',p') \in T_{(x,p)}Z_0^+ = T_{(x,p)}Z_0^-$ one has $p_1' = -p_2' = 0 = 2p_1 e^{-x_2}$, and therefore, we obtain the same contradiction as above, hence $d_S^{\#}H_0^-(x,p) = \emptyset$.

Summarizing, the Hamiltonian field in (4.2) is given by the formulas:

$$d_S^{\#}H(x,p) = \begin{cases} d_S^{\#}H_j^{\pm}(x,p)\ if\ (x,p) \in Z_j^{\pm},\ j = 1,2 \\ d_S^{\#}H_0^{0,\pm}(x,p)\ if\ (x,p) \in Z_0^{0,\pm} \\ \emptyset\ if\ (x,p) \in Z_0^{\pm}, \end{cases} \tag{4.21}$$

where $d_S^{\#}H_0^{0,\pm}(.,.)$ are the Hamiltonian fields in the formulas (4.11), (4.14) and $d_S^{\#}H_j^{\pm}(.,.),\ j = 1,2$ will be described and studied in what follows.

The Hamiltonian system on the stratum Z_1^+.

On the open stratum Z_1^+ in (4.7) for which $h(x,p) := p_2 - p_1 > 0,\ p_1 > 0$, the differential inclusion in (4.1) coincides with the ,,smooth Hamiltonian system":

$$\begin{cases} x' = (2e^{-x_2}, -2) \\ p' = (0, 2p_1 e^{-x_2}). \end{cases} \tag{4.22}$$

Standard results from differential equations theory show that the general solution of the system in (4.22) is given by the formulas:

$$\begin{cases} x^+(t) = (e^{2t-c_2} + c_1, -2t + c_2),\ c_i \in R,\ i = 1,4,\ t < 0 \\ p^+(t) = (c_3,\ c_3 e^{2t-c_2} + c_4). \end{cases} \tag{4.23}$$

The Hamiltonian system on the stratum Z_1^-.

On the open stratum Z_1^- in (4.7), for which $h(x,p) := p_2 + p_1 > 0,\ p_1 < 0$, the differential inclusion in (4.1) coincides with the ,,smooth Hamiltonian system":

$$\begin{cases} x' = (-2e^{-x_2}, -2) \\ p' = (0, -2p_1 e^{-x_2}) \end{cases} \tag{4.24}$$

whose general solution is immediately obtained using the fact that if $(x^+(.), p^+(.))$ is a solution of the differential system (4.22) then $(x^-(.), p^-(.))$ defined by:

$$\begin{array}{l} x_1^-(t) := -x_1^+(t),\ x_2^-(t) := x_2^+(t),\ t \le 0 \\ p_1^-(t) := -p_1^+(t),\ p_2^-(t) := p_2^+(t), \end{array} \tag{4.25}$$

is a solution of the differential system in (4.24).

The Hamiltonian system on the stratum Z_2^+.

On the open stratum Z_2^+ in (4.7), for which $h(x,p) := p_2 - p_1 < 0,\ p_1 > 0$, the differential inclusion in (4.1) coincides with the ,,smooth Hamiltonian system":

$$\begin{cases} x' = (2e^{-x_2} - 1,\ -1) \\ p' = (0,\ 2p_1 e^{-x_2}), \end{cases} \tag{4.26}$$

whose general solution is given by the formulas:

$$\begin{cases} x^{+,+}(t) = (2e^{t-c_2} - t + c_1,\ -t + c_2),\ c_i \in R,\ i = 1,4,\ t \le 0 \\ p^{+,+}(t) = (c_3,\ 2c_3 e^{t-c_2} + c_4). \end{cases} \tag{4.27}$$

The Hamiltonian system on the stratum Z_2^-.

On the open stratum Z_2^- in (4.7), for which $h(x,p) := p_2 + p_1 < 0,\ p_1 < 0$, the differential inclusion in (4.1) coincides with the ,,smooth Hamiltonian system":

$$\begin{cases} x' = (1 - 2e^{-x_2}, -1) \\ p' = (0, -2p_1 e^{-x_2}). \end{cases} \tag{4.28}$$

whose general solution is immediately obtained using the fact that if $(x^{+,+}(.), p^{+,+}(.))$ is a solution of the differential system (4.26) then $(x^{-,-}(.), p^{-,-}(.))$ defined by:

$$\begin{array}{l} x_1^{-,-}(t) := -x_1^{+,+}(t),\ x_2^{-,-}(t) := x_2^{+,+}(t),\ t \le 0 \\ p_1^{-,-}(t) := -p_1^{+,+}(t),\ p_2^{-,-}(t) := p_2^{+,+}(t), \end{array} \tag{4.29}$$

is a solution of the differential system in (4.28).

5 The Hamiltonian flow ending on the stratum Z_1^+

In this section we describe the ,,partial" Hamiltonian flow whose trajectories have terminal segments on the stratum Z_1^+. Considering the general solution in (4.23), an admissible trajectory $X_+^*(., z) = (X^+(., z), P^+(., z)),\ z \in Z_1^*$ of the system (4.22) should satisfy the terminal conditions from the set of transversality terminal points Z_1^* in (3.4) and also the fact that $X_+^*(t, z) \in Z_1^+\ \forall\, t < 0$; we note first that from the condition $P_1(t, z) > 0\ \forall\, t < 0$, it follows that $P_1(0, z) = q_1 \ge 0$ hence only the terminal points $z = (\xi, q) \in Z_{1,+}^* \cup Z_{1,0}^*$ are admissible (since for $\xi_1 < 0$ one has $q_1 = -k < 0$). From the terminal condition in (3.4) it follows that $c_1 = \xi_1 - 1,\ c_2 = 0,\ c_3 = k,\ c_4 = \frac{1}{2}$.

Therefore, we obtain the solution of the differential system in (4.22) in the form of a ,,maximal flow" whose components are given by the formulas:

$$\begin{cases} X^+(t,\xi_1) = (e^{2t}+\xi_1-1,-2t),\ \xi_1 \geq 0 \\ P^+(t,\xi_1) = (k, ke^{2t}+\frac{1}{2}) \end{cases} \tag{5.1}$$

From the dynamic programming algorithm in [12] it follows that we must retain only the trajectories $X_+^*(.,z),\ z=(\xi,q)\in Z_1^*$, that satisfy the conditions in (4.3). We note first that the second condition in (4.3) is ,,automatically" satisfied since $H_1^+(.,.)$ defined in (4.8) is a first integral of the differential system (4.22) hence:

$$\begin{array}{l} h_+^*(t,\xi_1) := h_+(X_+^*(t,\xi_1)) = P_2^+(t,\xi_1) - P_1^+(t,\xi_1) > 0, P_1^+(t,\xi_1) > 0 \\ X_2^+(t,\xi_1) > 0,\ H_1^+(X_+^*(t,\xi_1)) = 0\ \forall\ t<0. \end{array} \tag{5.2}$$

The admissible trajectories must satisfy also the conditions:

$$\begin{array}{l} X_+^*(t,\xi_1) = (X^+(t,\xi_1), P^+(t,\xi_1)) \in Z_1^+\ \forall\ t\in(\tau^+(\xi_1),0) \\ X^+(t,\xi_1)\in Y_0 := R\times(0,+\infty),\ \xi_1\geq 0 \end{array} \tag{5.3}$$

on the maximal intervals $I^+(\xi_1) := (\tau^+(\xi_1),0),\ \xi_1 \geq 0$, hence the extremity $\tau^+(\xi_1)<0$ is defined by:

$$\tau^+(\xi_1) := \inf\{\tau<0; h_+^*(t,\xi_1)>0, P_1^+(t,\xi_1)>0, X_2^+(t,\xi_1)>0\ \forall\ t\in(\tau,0)\} \tag{5.4}$$

where the functions $h_+^*(.,\xi_1),\ \xi_1\geq 0$ are given by the formula:

$$h_+^*(t,\xi_1) = k(e^{2t}-1)+\frac{1}{2}. \tag{5.5}$$

We remark that, the second condition in (5.3) is satisfied since $X_2^+(t,\xi_1) = -2t>0\ \forall\ t<0$ and also $P_1^+(t,\xi_1)=k>0$; on the other hand, the expression in (5.5) allows an explicit formula for the extremity $\tau^+(.)$:

$$\tau^+(k) = \tau^+(\xi_1) = \begin{cases} -\infty\ if\ k\in(0,\frac{1}{2}] \\ \frac{1}{2}\ln(1-\frac{1}{2k})\ if\ k>\frac{1}{2}; \end{cases} \tag{5.6}$$

one may note here that ,,geometrically" the trajectories $X^+(.,\xi_1),\ \xi_1\geq 0$ are the curves in Fig. 1, 2 described by the equations:

$$x_2 = a^+(x_1,\xi_1) := -\ln(x_1-\xi_1+1),\ \begin{cases} x_1\in(\xi_1-1,\xi_1), k\in(0,\frac{1}{2}] \\ x_1\in(\xi_1-\frac{1}{2k},\xi_1), k>\frac{1}{2} \end{cases} \tag{5.7}$$

and ,,cover" the domain $Y^+ = Y_0^+\cup Y_1^+,\ Y_0^+ := X^+(B^+)$ defined by:

$$\begin{array}{l} B^+ = \begin{cases} (-\infty,0)\times[0,+\infty),\ k\in(0,\frac{1}{2}] \\ (\frac{1}{2}\ln(1-\frac{1}{2k}),0)\times[0,+\infty),\ k>\frac{1}{2} \end{cases} \\ Y_0^+ = \begin{cases} \{x\in Y_0; x_1>-1, x_2\geq -\ln(x_1+1)\}, k\in(0,\frac{1}{2}] \\ \{x\in Y_0; x_1>-1, x_2\in[-\ln(x_1+1), -\ln(1-\frac{1}{2k}))\}, k>\frac{1}{2} \end{cases} \\ Y_1^+ = (0,+\infty)\times\{0\}. \end{array} \tag{5.8}$$

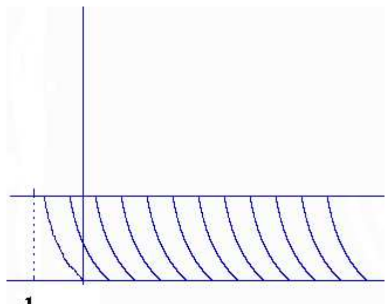

Fig. 1. $X^+(.,\xi_1), k > \frac{1}{2}$

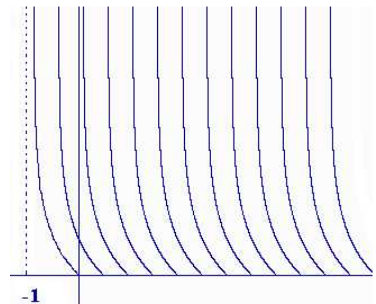

Fig. 2. $X^+(.,\xi_1), k \in (0, \frac{1}{2}]$.

Continuation of trajectories on the stratum Z_2^+ in the case $k > \frac{1}{2}$.

The continuation for $t < \tau^+(k)$ of the trajectories in (5.1) is possible only in the case $k > \frac{1}{2}$; since the point:

$$z^+(\xi_1) := X_+^*(\tau^+(k), \xi_1) = ((\xi_1 - \frac{1}{2k}, -\ln(1 - \frac{1}{2k})), (k, k)) \tag{5.9}$$

belongs to the stratum $Z_1^+ \subset Z$ in (4.7) but also to the boundary of the open stratum Z_2^+, the continuation for $t < \tau^+(k),\ k > \frac{1}{2}$ of the trajectories $X_+^*(.,\xi_1),\ \xi_1 \geq 0$ is possible only on the stratum Z_2^+.

Since $\frac{d}{dt}h_+^*(X_+^*(\tau^+(k),\xi_1)) = 2P_1^+(\tau^+(k),\xi_1)e^{-X_2^+(\tau^+(k),\xi_1)} = 2ke^{\ln(1-\frac{1}{2k})} = 2k - 1 > 0$ (for $k > \frac{1}{2}$), the trajectories in (5.1) may be continuated on the stratum Z_2^+.

In this case the trajectories in (5.1) may be continuated by the trajectories $X_{+,+}^*(.,\xi_1) = (X^{+,+}(.,\xi_1), P^{+,+}(.,\xi_1)),\ \xi_1 \geq 0$, which are solutions of the Hamiltonian system in (4.26), that satisfy $X_{+,+}^*(\tau^+(k),\xi_1) = z^+(\xi_1),\ \xi_1 \geq 0,\ k > \frac{1}{2}$, and for which, there exist the numbers $\tau^{+,+}(k) < \tau^+(k),\ k > \frac{1}{2}$ such that:

$$\begin{array}{l} X^{+,+}(t,\xi_1) \in Y_0\ \forall\ t \in (\tau^{+,+}(k), \tau^+(k)),\ P_1^{+,+}(t,\xi_1) > 0 \\ h_{+,+}^*(t,\xi_1) := h(X_{+,+}^*(t,\xi_1)) = P_2^{+,+}(t,\xi_1) - P_1^{+,+}(t,\xi_1) < 0. \end{array} \tag{5.10}$$

First, starting from the general solution of the system (4.26) on the stratum Z_2^+, given by the formulas in (4.27) and taking the terminal conditions $z^+(\xi_1),\ \xi_1 \geq 0$ in (5.9), the components of the Hamiltonian flow $X_{+,+}^*(.,.)$, are given by:

$$\left\{ \begin{array}{l} X_1^{+,+}(t,\xi_1) = 2e^t\sqrt{1 - \frac{1}{2k}} - t + \frac{1}{2}\ln(1 - \frac{1}{2k}) + \xi_1 + \frac{1}{2k} - 2 \\ X_2^{+,+}(t,\xi_1) = -t - \frac{1}{2}\ln(1 - \frac{1}{2k}) = -(t + \tau^+(k)),\ \xi_1 > 0 \\ P_1^{+,+}(t,\xi_1) = k,\ t \in I^{+,+}(k) := (\tau^{+,+}(k), \tau^+(k)), k > \frac{1}{2} \\ P_2^{+,+}(t,\xi_1) = 2ke^t\sqrt{1 - \frac{1}{2k}} - k + 1, \end{array} \right. \tag{5.11}$$

where the extremity $\tau^{+,+}(.)$ is defined by:

$$\tau^{+,+}(k) = \inf\{\tau < \tau^{+}(k); h^{*}_{+,+}(t,\xi_1) < 0, P_1^{+,+}(t,\xi_1) > 0 \forall t \in (\tau, \tau^{+}(k))\}. \quad (5.12)$$

The function $h^{*}_{+,+}(.,.)$in (5.10) is given by the formula:

$$h^{*}_{+,+}(t,\xi_1) = 2k(e^{t}\sqrt{1-\frac{1}{2k}} - 1) + 1,\ t \in (-\infty, \tau^{+}(k)), k > \frac{1}{2}, \quad (5.13)$$

and elementary arguments show that $h^{*}_{+,+}(t,\xi_1) < 0\ \forall\ t \in (-\infty, \tau^{+}(k))$, hence the extremity $\tau^{+,+}(.)$ of the maximal interval $I^{+,+}(.)$ is given by $\tau^{+,+}(k) = -\infty$; as in the other case, ,,geometrically" (see Fig. 5), the trajectories $X^{+,+}(.,\xi_1),\ \xi_1 \geq 0$ are the curves described by the equations:

$$\begin{array}{l} x_1 = a^{+,+}(x_2,\xi_1) := 2e^{-x_2} + x_2 + \ln(1-\frac{1}{2k}) + \xi_1 + \frac{1}{2k} - 2 \\ \alpha(x_2) = a^{+,+}(x_2, 0),\ x_2 > -\ln(1-\frac{1}{2k}) \end{array} \quad (5.14)$$

and ,,cover" the domain $Y^{+,+} := Y_0^{+,+}$ defined by:

$$\begin{array}{l} Y_0^{+,+} = X^{+,+}(B^{+,+}) = \{x \in Y_0; x_1 > -1, x_2 \in (-\ln(1-\frac{1}{2k}), \alpha^{-1}(x_1)]\} \\ B^{+,+} = (-\infty, \tau^{+}(k)) \times [0, +\infty). \end{array} \quad (5.15)$$

Finally, the trajectories $X^{*}_{+}(.,\xi_1),\ \xi_1 \geq 0$, in (5.1) together with $X^{*}_{+,+}(.,\xi_1)$, $\xi_1 \geq 0$ in (5.11) may be ,,concatenated" to obtain a new ,,extended" Hamiltonian flow, described by the formula:

$$X^{*}_{\oplus,\oplus}(t,\xi_1) := \left\{ \begin{array}{l} X^{*}_{+}(t,\xi_1),\ t \in [\tau^{+}(k), 0),\ \xi_1 \geq 0,\ k > \frac{1}{2} \\ X^{*}_{+,+}(t,\xi_1),\ t < \tau^{+}(k), \end{array} \right. \quad (5.16)$$

whose trajectories are illustrated below in Fig. 5.

6 The Hamiltonian flow ending on the stratum Z_1^{-}

Symmetrically, we shall describe a ,,partial" Hamiltonian flow whose trajectories are ending on the stratum Z_1^{-}.

Starting from the general solution in (4.25) of the system (4,24) and formulas which describe the ,,partial" Hamiltonian flow in (5.1), we infer that the components of the ,,partial" Hamiltonian flow $X^{*}_{-}(.,.)$ associated with the Hamiltonian system in (4.24) are given by formulas:

$$\left\{ \begin{array}{l} X^{-}(t,\xi_1) = (-X_1^{+}(t,-\xi_1), X_2^{+}(t,-\xi_1)) = (-e^{2t} + \xi_1 + 1, -2t),\ \xi_1 \leq 0 \\ P^{-}(t,\xi_1) = (-P_1^{+}(t,-\xi_1), P_2^{+}(t,-\xi_1)) = (-k, ke^{2t} + \frac{1}{2}),\ t \in I^{+}(k), \end{array} \right. \quad (6.1)$$

where the maximal interval $I^{-}(.)$ is of the same form as in (5.4), where the function $\tau^{-}(.)$ is defined in this case as:

$$\begin{array}{l} \tau^{-}(\xi_1) := \inf\{\tau < 0;\ h^{*}_{-}(t,\xi_1) > 0,\ P_1^{-}(t,\xi_1) < 0\ \forall\ t \in (\tau, 0)\} \\ h^{*}_{-}(t,\xi_1) := h(X^{*}_{-}(t,\xi_1)) = h^{*}_{+}(t,-\xi_1) = k(e^{2t} - 1) + \frac{1}{2},\ \xi_1 \leq 0. \end{array} \quad (6.2)$$

From (5.5) and (6.2) it follows that the extremity $\tau^-(.)$ is given by:

$$\tau^-(k) = \tau^-(\xi_1) = \tau^+(k) = \begin{cases} -\infty \ if \ k \in (0, \frac{1}{2}] \\ \frac{1}{2}\ln(1 - \frac{1}{2k}) \ if \ k > \frac{1}{2}; \end{cases} \tag{6.3}$$

as in the other case, ,,geometrically", the the trajectories $X^-(., \xi_1), \ \xi_1 \leq 0$ are the curves in Fig. 3, 4 described by the equations:

$$x_2 = a^-(x_1, \xi_1) := -\ln(\xi_1 + 1 - x_1), \ \begin{cases} x_1 \in (\xi_1, \xi_1 + 1), k \in (0, \frac{1}{2}] \\ x_1 \in (\xi_1, \xi_1 + \frac{1}{2k}), k > \frac{1}{2} \end{cases} \tag{6.4}$$

and ,,cover" the domain $Y^- = Y_0^- \cup Y_1^-, \ Y_0^- := X^-(B^-)$ defined by:

$$\begin{aligned} & B^- := \begin{cases} (-\infty, 0) \times (-\infty, 0] \ if \ k \in (0, \frac{1}{2}] \\ (\frac{1}{2}\ln(1 - \frac{1}{2k}), 0) \times (-\infty, 0] \ if \ k > \frac{1}{2} \end{cases} \\ & Y_0^- = \begin{cases} \{x \in Y_0; \ x_1 < 1, \ x_2 \geq -\ln(1 - x_1)\} \ if \ k \in (0, \frac{1}{2}] \\ \{x \in Y_0; \ x_1 < 1, \ x_2 \in [-\ln(1 - x_1), -\ln(1 - \frac{1}{2k})) \ if \ k > \frac{1}{2} \end{cases} \\ & Y_1^- = (-\infty, 0) \times \{0\}. \end{aligned} \tag{6.5}$$

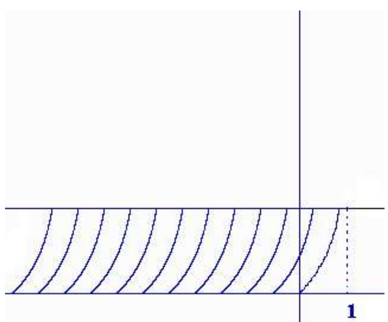

Fig. 3. $X^-(., \xi_1), k > \frac{1}{2}$

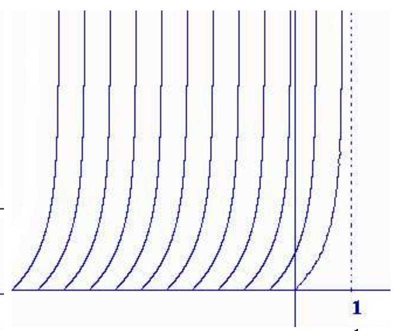

Fig. 4. $X^-(., \xi_1), k \in (0, \frac{1}{2}]$.

Continuation of the trajectories on the stratum Z_2^- *in the case* $k > \frac{1}{2}$.

Starting from the formulas which describe the ,,partial" Hamiltonian flow $X^*_{+,+}(.,.)$ in (5.11), together with the symmetry in (4.29), it follows the fact that the Hamiltonian flow $X^*_-(.,.)$ in (6.1) may be continuated for $t < \tau^-(k), \ k > \frac{1}{2}$ by the Hamiltonian flow $X^*_{-,-}(.,.)$ which is defined by:

$$\begin{cases} X_1^{-,-}(t, \xi_1) = -X_1^{+,+}(t, -\xi_1), \ t \in I^{-,-}(k) := (-\infty, \tau^-(k)) \\ X_2^{-,-}(t, \xi_1) = X_2^{+,+}(t, -\xi_1), \ \xi_1 \leq 0, \ k > \frac{1}{2} \\ P_1^{-,-}(t, \xi_1) = -P_1^{+,+}(t, -\xi_1) \\ P_2^{-,-}(t, \xi_1) = P_2^{+,+}(t, -\xi_1), \end{cases} \tag{6.6}$$

where $X^*_{+,+}(.,.) = (X^{+,+}(.,.), P^{+,+}(.,.))$ is the Hamiltonian flow defined in (5.11).

As in the other case, „geometrically”, the trajectories $X^{-,-}(.,\xi_1),\ \xi_1 \leq 0$ are the curves in Fig. 6 described by the equations:

$$x_1 = a^{-,-}(x_2,\xi_1) := -a^{+,+}(x_2,-\xi_1) = -2e^{-x_2} - x_2 - \ln(1-\tfrac{1}{2k}) + \\ +\xi_1 - \tfrac{1}{2k} + 2,\ x_2 > -\ln(1-\tfrac{1}{2k}) \tag{6.7}$$

and „cover” the domain $Y^{-,-} := Y_0^{-,-}$ defined by:

$$Y_0^{-,-} = X^{-,-}(B^{-,-}) = \{x \in Y_0; x_1 < 1, x_2 \in (-\ln(1-\tfrac{1}{2k}), \alpha^{-1}(-x_1)]\} \\ B^{-,-} = (-\infty, \tau^-(k)) \times (-\infty, 0]. \tag{6.8}$$

Finally, the trajectories $X^*_-(.,\xi_1),\ \xi_1 \leq 0$, in (6.1) together with $X^*_{-,-}(.,\xi_1)$, $\xi_1 \leq 0$ in (6.6) may be „concatenated” to obtain a new „extended” Hamiltonian flow, described by the formula:

$$X^*_{\ominus,\ominus}(t,\xi_1) := \begin{cases} X^*_-(t,\xi_1),\ t \in [\tau^-(k), 0),\ \xi_1 \leq 0,\ k > \frac{1}{2} \\ X^*_{-,-}(t,\xi_1),\ t < \tau^-(k). \end{cases} \tag{6.9}$$

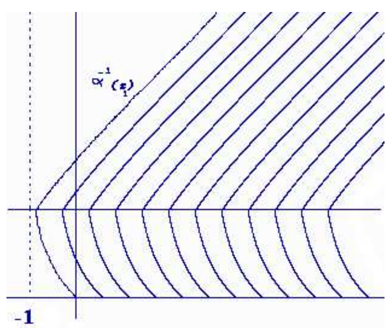

Fig. 5. $X^{\oplus,\oplus}(.,\xi_1)$

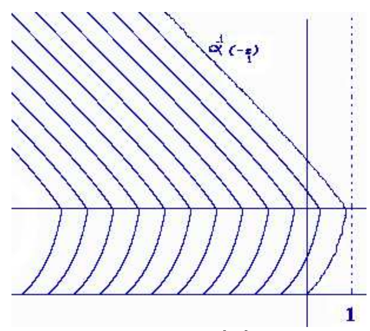

Fig. 6. $X^{\ominus,\ominus}(.,\xi_1)$.

7 The Hamiltonian flow ending on the stratum $Z_0^{0,+}$

In contrast with the previous cases, on the singular stratum $Z_0^{0,+}$, the differential inclusion in (4.2) coincides with the „proper differential inclusion” in (4.12) whose general solution is given in the form of the maximal flow $X^{0,*}_{0,+}(.;.,.)$ described by the formulas in (4.13).

On this stratum, the admissible trajectories must satisfy the conditions:

$$X^{0,+}_{2,0}(t,z) > 0,\ P^{0,+}_{1,0}(t,z) = 0,\ P^{0,+}_{2,0}(t,z) > 0\ \forall\ t \in I_0^{0,+}(z), \tag{7.1}$$

on the maximal intervals $I_0^{0,+}(z) := (\tau_0^{0,+}(z), 0),\ z = (\xi, q) \in Z_1^*$, where the extremity $\tau_0^{0,+}(z) < 0$ is defined in the same way as in (5.4) such that (7.1) is satisfied.

From (4.13), (7.1) it follows that, the only possible case that satisfy the conditions in (7.1) is at the point $P_{1,0}^{0,+}(t) = q_1 = 0,\ P_{2,0}^{0,+}(t) = q_2 = \frac{1}{2}$ (taking into account the fact that, $z = ((0,0),(0,\frac{1}{2})) \in Z_{1,0}^*$) and therefore, the extremity $\tau_0^{0,+}(.)$ of the maximal interval $I_0^{0,+}$ is given by the formula:

$$\tau_0^{0,+} := -\infty \tag{7.2}$$

and the components of the ,,partial" Hamiltonian flow $X_{0,+}^{0,*}(.,.)$ on the stratum $Z_0^{0,+}$ are given by the formulas:

$$\begin{array}{l} X_0^{0,+}(t,\lambda) = (\lambda(e^{2t}-1), -2t),\ t \in (-\infty, 0),\ \lambda \in [-1,1] \\ P_0^{0,+}(t,\lambda) = (0, \frac{1}{2}); \end{array} \tag{7.4}$$

moreover, ,,geometrically", the trajectories $X_0^{0,+}(.,\lambda),\ \lambda \in [-1,1]$ are the curves in Fig. 7 described by the equations:

$$x_1 = \lambda(e^{-x_2} - 1),\ x_2 > 0,\ \lambda \in [-1,1] \tag{7.5}$$

and ,,cover" the domains:

$$\begin{array}{l} Y_{0,0}^{0,+} := X_0^{0,+}(B_0^{0,+}) = \{x \in Y_0;\ x_1 \in [e^{-x_2} - 1, 1 - e^{-x_2}],\ x_2 > 0\} \\ Y_{1,0}^{0,+} := \{(0,0)\},\ B_0^{0,+} := (-\infty, 0) \times [-1,1]. \end{array} \tag{7.6}$$

On the singular stratum $Z_0^{0,-}$ for which, $p_1 = 0,\ p_2 < 0$, it follows that, the only possible case that satisfy the similar conditions in (7.1) (except the third condition which must be replaced by: $P_{2,0}^{0,-}(t,z) < 0$) is at the point $P_{1,0}^{0,-} = q_1 = 0,\ P_{2,0}^{0,-} = q_2 = \frac{1}{2} > 0$ and therefore, there are no admissible trajectories though $d_S^{\#} H_0^{0,-}(x,p) \neq \emptyset$.

Thus, the Hamiltonian systems in (4.22), (4.24), (4.26),(4.28) generate the generalized characteristic flows $C_{\pm}^*(.,.) = (X_{\pm}^*(.,.), V(.,.)),\ C_{\pm,\pm}^*(.,.) = (X_{\pm,\pm}^*(.,.), V(.,.)),\ C_{0,+}^{0,*}(.,.) = (X_{0,+}^{0,*}(.,.), V(.,.))$ described explicitly in (4.6), (5.1), (5.11), (6.1), (6.6), (7.4) and which, according to the well known classical results (e.g. Mirică ([10])) satisfy the basic differential relation:

$$DV(t,s).(\bar{t},\bar{s}) =< P^{\pm}(t,s), DX^{\pm}(t,s).(\bar{t},\bar{s}) > \ \forall (\bar{t},\bar{s}) \in T_{(t,s)}B^{\pm}. \tag{7.7}$$

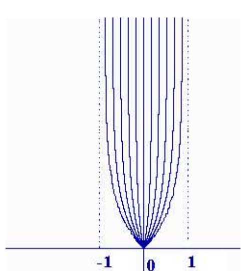

Fig. 7. $X_0^{0,+}(.,\lambda)$

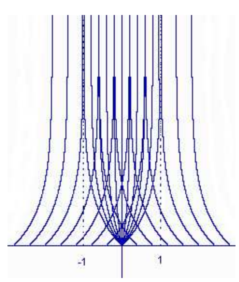

Fig. 8. Case $k \in (0, \frac{1}{2}]$

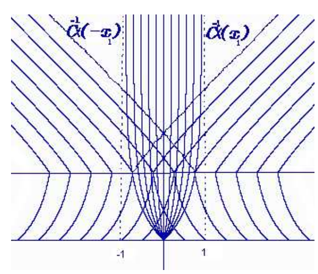

Fig. 9. Case $k > \frac{1}{2}$.

8 Partial value functions and feedback strategies

As indicated in the „theoretical algorithm" in Mirică ([12]), the „natural candidates" for value functions and optimal strategies in Problem 2.3 are the „extreme" ones, defined by:

$$\begin{aligned}
&W_0^m(y) := \inf_{X(t,a)=y} V(t,a),\ W_0^M(y) := \sup_{X(t,a)=y} V(t,a),\\
&\widehat{B}_m(y) := \{(t,a)\in B_0;\ X(t,a)=y,\ V(t,a)=W_0^m(y)\},\\
&\widehat{B}_M(y) := \{(t,a)\in B_0;\ X(t,a)=y,\ V(t,a)=W_0^M(y)\},\\
&\widetilde{U}_m(y) := \overline{U}(\widehat{B}_m(y)),\ \widetilde{V}_m(y) := \overline{V}(\widehat{B}_m(y))\\
&\widetilde{U}_M(y) := \overline{U}(\widehat{B}_M(y)),\ \widetilde{V}_M(y) := \overline{V}(\widehat{B}_M(y))\\
&\overline{U}(t,a) := \{u_a(t);\ u_a(.)\in\overline{\mathcal{U}}(a)\},\ \overline{V}(t,a) := \{v_a(t);\ v_a(.)\in\overline{\mathcal{V}}(a)\},
\end{aligned} \tag{8.1}$$

where $\overline{\mathcal{U}}(a)=\{u_a(.)\}$, $\overline{\mathcal{V}}(a)=\{v_a(.)\}$ denote the sets of control mappings that satisfy (4.3); one may note that:

$$\overline{U}(t,a)\subseteq\widehat{U}(X^*(t,a)),\ \overline{V}(t,a)\subseteq\widehat{V}(X^*(t,a)) \quad \forall\ (t,a)\in B_0,$$

and also that if $X(.,.)$ is invertible at $(t,a)\in B_0$ with inverse

$$\widehat{B}_0(y) := (X(.,.))^{-1}(y),$$

then one has:

$$W_0^m(y)=W_0^M(y)=V(\widehat{B}_0(y)),\ \widehat{B}_m(y)=\widehat{B}_M(y)=\widehat{B}_0(y); \tag{8.2}$$

moreover, it follows from (7.8) that if, in addition, the function $\widetilde{W}_0(.) := W_0^m(.) = W_0^M(.)$ is differentiable at the point $y\in Int(\widetilde{Y}_0)$, then its derivative is given by:

$$D\widetilde{W}_0(y)=\widetilde{P}(y) := P(\widehat{B}_0(y)) \tag{8.3}$$

and verifies the relations:

$$\begin{aligned}
&D\widetilde{W}_0(y).f(y,\overline{u},\overline{v})+f_0(y,\overline{u},\overline{v})=0 \quad \forall\overline{u}\in\widetilde{U}(y),\ \overline{v}\in\widetilde{V}(y),\\
&\widetilde{U}(y) := \widehat{U}(y,\widetilde{P}(y)),\quad \widetilde{V}(y) := \widehat{V}(y,\widetilde{P}(y))
\end{aligned} \tag{8.4}$$

and $\widetilde{U}(.)$, $\widetilde{V}(.)$ are the corresponding „candidates" for optimal feedback strategies; moreover, from (3.2) and (4.3) it follows that in this case $\widetilde{W}(.)$ verifies *Isaacs' basic equation*:

$$\begin{aligned}
&\min_{u\in U(y)}\max_{v\in V(y)}[D\widetilde{W}_0(y).f(y,u,v)+f_0(y,u,v)]=\\
&=\max_{v\in V(y)}\min_{u\in U(y)}[D\widetilde{W}_0(y).f(y,u,v)+f_0(y,u,v)]=0.
\end{aligned} \tag{8.5}$$

Due to these relations, the computations and arguments to follow may be significantly simplified if the characteristic flow may be „split" into a finite collection of the smooth „invertible" characteristic flows $C_j^*(.,.)$, $1\le j\le k$ so that the ,,marginal characteristic value functions" in (8.1) may be represented as:

$$W_0^m(y) = \min_{1\le j\le k} W_0^j(y), \quad W_0^M(y) = \max_{1\le j\le k} W_0^j(y) \quad \forall\, y \in \widetilde{Y}_0, \tag{8.6}$$

where $W_0^j(.) : Y_0^j \subseteq Y_0 \to R$ are differentiable functions of the form in (8.2) satisfying relations of the forms in (8.3)-(8.5) and which characterize ,,partial solutions" $(\widetilde{U}_j(.), \widetilde{V}_j(.))$ of the original problem DG_A; however, they may be considered ,,complete solutions" to the restrictions $DG_A|_{Y_j}$ of *Problem* 2.3 to the subsets $Y_0^j \subseteq Y_0$ (see *Remark* 2.3).

In what follows we shall prove that in the particular case of Problem 2.3 the ,,extreme solutions" in (8.1) may be expressed as in (8.6); the main ,,ingredient" is the following ,,quasi-elementary" result:

Lemma 8.1. (1) *The mapping* $X^+(.,.) : B^+ \to Y^+$ *defined in* (5.1) is a C^1- *stratified diffeomorphism whose inverse* $\widehat{B}^+(.)$ *is described by*:

$$\begin{array}{l} \widehat{B}^+(x) := (\widehat{t}^+(x), \widehat{\xi}_1^+(x)),\ x = (x_1, x_2) \in Y^+ \\ \widehat{t}^+(x) := -\frac{1}{2}x_2,\ \widehat{\xi}_1^+(x) := x_1 - e^{-x_2} + 1. \end{array} \tag{8.7}$$

(2) *Symmetrically, the mapping* $X^-(.,.) : B^- \to Y^-$ *defined in* (6.1) *is a* C^1- *stratified diffeomorphism whose inverse* $\widehat{B}^-(.)$ *is described by*:

$$\begin{array}{l} \widehat{B}^-(x) := (\widehat{t}^-(x), \widehat{\xi}_1^-(x)),\ x = (x_1, x_2) \in Y^- \\ \widehat{t}^-(x) := -\frac{1}{2}x_2,\ \widehat{\xi}_1^-(x) := x_1 + e^{-x_2} - 1. \end{array} \tag{8.8}$$

(3) *If* $k > \frac{1}{2}$ *then the mapping* $X^{+,+}(.,.) : B^{+,+} \to Y_0^{+,+}$ *defined in* (5.11) *is a* C^1- *diffeomorphism whose inverse* $\widehat{B}^{+,+}(.)$ *is described by*:

$$\begin{array}{l} \widehat{B}^{+,+}(x) := (\widehat{t}^{+,+}(x), \widehat{\xi}_1^{+,+}(x)),\ x = (x_1, x_2) \in Y_0^{+,+} \\ \widehat{t}^{+,+}(x) := -[x_2 + \frac{1}{2}\ln(1 - \frac{1}{2k})],\ k > \frac{1}{2} \\ \widehat{\xi}_1^{+,+}(x) := x_1 - 2e^{-x_2} - x_2 - \ln(1 - \frac{1}{2k}) - \frac{1}{2k} + 2. \end{array} \tag{8.9}$$

(4) *Symmetrically, if* $k > \frac{1}{2}$ *then the mapping* $X^{-,-}(.,.) : B^{-,-} \to Y_0^{-,-}$ *defined in* (6.6) *is a* C^1- *diffeomorphism whose inverse* $\widehat{B}^{-,-}(.)$ *is described by*:

$$\begin{array}{l} \widehat{B}^{-,-}(x) := (\widehat{t}^{-,-}(x), \widehat{\xi}_1^{-,-}(x)),\ x = (x_1, x_2) \in Y_0^{-,-} \\ \widehat{t}^{-,-}(x) := -[x_2 + \frac{1}{2}\ln(1 - \frac{1}{2k})],\ k > \frac{1}{2} \\ \widehat{\xi}_1^{-,-}(x) := x_1 + 2e^{-x_2} + x_2 + \ln(1 - \frac{1}{2k}) + \frac{1}{2k} - 2. \end{array} \tag{8.10}$$

(5) *The mapping* $X_0^{0,+}(.,.) : B_0^{0,+} \to Y_{0,0}^{0,+} \cup Y_{1,0}^{0,+}$ *defined in* (7.4) *is a* C^1- *diffeomorphism whose inverse* $\widehat{B}_0^{0,+}(.)$ *is described by*:

$$\begin{array}{l} \widehat{B}_0^{0,+}(x) := (\widehat{t}_0^{0,+}(x), \widehat{\lambda}_0^{0,+}(x)),\ x = (x_1, x_2) \in Y_{0,0}^{0,+} \cup Y_{1,0}^{0,+} \\ \widehat{t}_0^{0,+}(x) := -\frac{1}{2}x_2, \quad \widehat{\lambda}_0^{0,+}(x) = \frac{x_1}{e^{-x_2} - 1}. \end{array} \tag{8.11}$$

Proof. (1) If $x = (x_1, x_2) \in Y^+$, then it follows from (5.1) that a point $(t, \xi_1) \in B^+$ for which $X^+(t, \xi_1) = x$ is characterized by the equations:

$$e^{2t} + \xi_1 - 1 = x_1, \quad -2t = x_2,$$

this implies the fact that $t := \widehat{t}^+(x) = -\frac{1}{2}x_2$ and $\xi_1 := \widehat{\xi}_1^+(x) = x_1 - e^{-x_2} + 1$ therefore, the components $\widehat{t}^+(.)$, $\widehat{\xi}_1^+(.)$ are stratified functions since $Y^+ = Y_0^+ \cup Y_1^+$ is a stratified set.

(2) The proof of this symmetric statement is entirely similar so we may omit it; one may note here that it follows from the relations above that the ,,symmetric" functions in (8.8) may equivalently be described by:

$$\widehat{t}^-(x) = \widehat{t}^+(-x_1, x_2), \quad \widehat{\xi}_1^-(x) = -\widehat{\xi}_1^+(-x_1, x_2), \ x \in Y_0^-. \tag{8.12}$$

In order to prove the other statements, we use the same type of computations and arguments as above taking into account the following relations:

$$\widehat{t}^{-,-}(x) = \widehat{t}^{+,+}(-x_1, x_2), \quad \widehat{\xi}_1^{-,-}(x) = -\widehat{\xi}_1^{+,+}(-x_1, x_2), \ x \in Y_0^{-,-}. \tag{8.13}$$

The results in Lemma 8.1 show that the characteristic flows $C^*_\pm(.,.)$, $C^*_{\pm,\pm}(.,.)$, $C^{0,*}_{0,+}(.,.)$ described in the *Sections* 5, 6, 7 are ,,*invertible*" in the sense of (8.2) and define the ,,*stratified partial proper value functions*", since from (4.6) and (5.1) it follows that:

$$\begin{aligned}
&W_0^\pm(x) = k|\widehat{\xi}_1^\pm(x)| - \widehat{t}^\pm(x) = \pm k\widehat{\xi}_1^\pm(x) - \widehat{t}^\pm(x) = k(\pm x_1 - e^{-x_2} + 1) + \\
&\qquad + \tfrac{1}{2}x_2, \ x \in Y_0^\pm \\
&W_0^{\pm,\pm}(x) = \pm k\widehat{\xi}_1^{\pm,\pm}(x) - \widehat{t}^{\pm,\pm}(x) = k(\pm x_1 - x_2 + 1) - 2ke^{-x_2} + x_2 + \\
&\qquad + (\tfrac{1}{2} - k)[\ln(1 - \tfrac{1}{2k}) - 1], \ x \in Y_0^{\pm,\pm} \\
&W_{0,0}^{0,+}(x) = -\widehat{t}_0^{0,+}(x) = \tfrac{1}{2}x_2, \ x \in Y_{0,0}^{0,+},
\end{aligned} \tag{8.14}$$

which may be naturally extended by $W^\pm(\xi) := 0 \ \forall \ \xi \in Y_1^\pm$, $W^{0,+}(\xi) := 0 \ \forall \ \xi \in Y_{1,0}^{0,+}$ to the corresponding ,,terminal sets" defined in (5.8), (6.5), (7.6).

Moreover, from (3.3) it follows that the corresponding ,,feedback strategies" in (8.4) are given by:

$$\begin{aligned}
&\widetilde{u}^\pm(x) := 0, \quad \widetilde{v}^\pm(x) := \pm 1 \quad x \in Y_0^\pm, \\
&\widetilde{u}^{\pm,\pm}(x) := \mp 1, \quad \widetilde{v}^{\pm,\pm}(x) := \pm 1 \quad x \in Y_0^{\pm,\pm}, \\
&\widetilde{u}_0^{0,+}(x) := 0, \quad v_0^{0,+}(x) := \widehat{\lambda}_0^{0,+}(x) = \tfrac{x_1}{e^{-x_2} - 1}, \ x \in Y_{0,0}^{0,+}.
\end{aligned} \tag{8.15}$$

Corollary 8.2. (1) *The functions* $W_0^\pm(.)$, $W_0^{\pm,\pm}(.)$, $W_{0,0}^{0,+}(.)$ *defined in* (8.14) *are stratified solutions of Isaacs' equation in* (8.5) *on the corresponding domains* $Y_0^\pm$, $Y_0^{\pm,\pm}$, $Y_{0,0}^{0,+}$; *moreover, each of them is the value function in the sense of* (2.11) *of the corresponding feedback strategies in* (8.15).

(2) *The feedback strategies* $(\widetilde{u}^\pm(.), \widetilde{v}^\pm(.))$, $(\widetilde{u}^{\pm,\pm}(.), \widetilde{v}^{\pm,\pm}(.))$, $(\widetilde{u}_0^{0,+}(.), \widetilde{v}_0^{0,+}(.))$ in (8.15) *are optimal in the sense of* (2.12), (2.15) *for the restrictions* $DG_A|_{Y_0^\pm}$, $DG_A|_{Y_0^{\pm,\pm}}$, $DG_A|_{Y_{0,0}^{0,+}}$, *respectively of the differential game in Problem* 2.3.

Proof. (1) The fact that $W_0^{\pm}(.)$ (respectively; $W_0^{\pm,\pm}(.)$) in (8.14) are stratified solutions of (8.5) (on their domains, $Y_0^{\pm}$ (respectively; $Y_0^{\pm,\pm}$, $Y_{0,0}^{0,+}$)) follows from Lemma 8.1 and the classical theory of smooth Hamiltonian-Jacobi equations (e.g. Mirică [10]) using the basic differential relations in (7.7); on the other hand, the fact that the functions $W_{0,0}^{0,+}(.)$ on the set $Y_{0,0}^{0,+}$ has the same property follows by ,,direct inspection", since from (2.7), (8.5), (8.14), (8.15) one has

$$\min_{u\in[-1,1]}\max_{v\in[-1,1]}[DW_{0,0}^{0,+}(x).f(x,u,v)+f_0(x,u,v)]=\frac{1}{2}\min_{u\in[-1,1]}|u|=0.$$

(2) The optimality in the sense of Problem 2.3-(B) follows from the verification Theorem 5.2 in [10] for the stratified value functions, taking into account the ,,technical" result in Lemma 8.1.

We note that the solutions in Corollary 8.2 are ,,unsatisfactory" since from some initial points there exist other trajectories than the ones generated by the feedback strategies in (8.15).

9 A Complete Solution in the case $k \in (0, \frac{1}{2}]$

In the case $k \in (0, \frac{1}{2}]$ the situation is simpler since only the sets $Y_0^{\pm}$ and $Y_{0,0}^{0,+}$ are present. Due to results in *Section* 8 above, the ,,extreme value functions" in (8.6) are given by:

$$\begin{aligned}&\widetilde{W}_0^m(x)=\min\{W_0^{\pm}(x),W_{0,0}^{0,+}(x)\},\quad x\in Y_0^+\cap Y_0^-\cap Y_{0,0}^{0,+}\\&\widetilde{W}_0^M(x)=\max\{W_0^{\pm}(x),W_{0,0}^{0,+}(x)\},\end{aligned}\tag{9.1}$$

on the intersection of the domains $Y_0^{\pm}$, $Y_{0,0}^{0,+}$; to characterize these functions we use first the following obvious result:

Proposition 9.1. *The functions* $W_0^{\pm}(.)$, $W_0^{\pm,\pm}(.)$, $W_{0,0}^{0,+}(.)$ *defined in* (8.14) *satisfy the relations*:

$$\begin{aligned}&W_0^+(x)-W_0^-(x)=2kx_1,\ x\in Y_0^+\cap Y_0^-\\&W_0^+(x)>W_0^-(x)\ if\ x\in Y_0^+\cap Y_0^-,\ x_1\in(0,1)\\&W_0^+(x)<W_0^-(x)\ if\ x\in Y_0^+\cap Y_0^-,\ x_1\in(-1,0)\\&W_0^+(x)=W_0^-(x)\ if\ x_1=0,\end{aligned}\tag{9.2}$$

$$\begin{aligned}&W_0^+(x)-W_{0,0}^{0,+}(x)=k[x_1-(e^{-x_2}-1)],\ x\in Y_{0,0}^{0,+}=Y_0^+\cap Y_{0,0}^{0,+}\\&W_0^+(x)>W_{0,0}^{0,+}(x)\ if\ x_1>e^{-x_2}-1\\&W_0^+(x)=W_{0,0}^{0,+}(x)\ if\ x_1=e^{-x_2}-1,\end{aligned}\tag{9.3}$$

$$\begin{aligned}&W_0^-(x)-W_{0,0}^{0,+}(x)=-k[x_1-(1-e^{-x_2})],\ x\in Y_{0,0}^{0,+}=Y_0^-\cap Y_{0,0}^{0,+}\\&W_0^-(x)>W_{0,0}^{0,+}(x)\ if\ x_1<1-e^{-x_2}\\&W_0^-(x)=W_{0,0}^{0,+}(x)\ if\ x_1=1-e^{-x_2},\end{aligned}\tag{9.4}$$

and from these inequalities it follows that:

$$\begin{array}{l} W_0^+(x) > W_0^-(x) > W_{0,0}^{0,+}(x) \ if \ x_1 \in (0, 1-e^{-x_2}), \ x \in Y_{0,0}^{0,+} \\ W_0^-(x) > W_0^+(x) > W_{0,0}^{0,+}(x) \ if \ x_1 \in (e^{-x_2}-1, 0). \end{array} \tag{9.5}$$

In the case $k \in (0, \frac{1}{2}]$ *one has:*

$$\begin{array}{l} W_0^{+,+}(x) - W_0^{-,-}(x) = 2kx_1, \ x \in Y_0^{+,+} \cap Y_0^{-,-} \\ W_0^{+,+}(x) > W_0^{-,-}(x) \ if \ x_1 \in (0,1) \\ W_0^{+,+}(x) < W_0^{-,-}(x) \ if \ x_1 \in (-1,0) \\ W_0^{+,+}(x) = W_0^{-,-}(x) \ if \ x_1 = 0, \end{array} \tag{9.6}$$

and therefore the value function in (9.1) *are given by:*

$$\begin{array}{l} \widetilde{W}_0^m(x) = \begin{cases} W_0^+(x), \ x \in Y_0^+ \setminus Y_{0,0}^{0,+} \\ W_0^-(x), \ x \in Y_0^- \setminus Y_{0,0}^{0,+} \\ W_{0,0}^{0,+}(x), \ x \in Y_{0,0}^{0,+}, \end{cases} \\ \widetilde{W}_0^M(x) = \begin{cases} W_0^+(x), \ x \in \widetilde{Y}_0^+ := R_+^* \times R_+ \\ W_0^-(x), \ x \in \widetilde{Y}_0^- := R_-^* \times R_+ \\ W_0^+(x) = W_0^-(x), \ x \in \widetilde{Y}_0^0 := \{0\} \times R_+. \end{cases} \end{array} \tag{9.7}$$

Moreover, the functions $\widetilde{W}_0^m(.)$, $\widetilde{W}_0^M(.)$ *as well as their natural extensions to* $Y := Y_0 \cup Y_1$ *defined by:*

$$\widetilde{W}^m(x) := \begin{cases} \widetilde{W}_0^m(x) \ if \ x \in Y_0 \\ 0 \qquad\quad if \ x \in Y_1 \end{cases}, \quad \widetilde{W}^M(x) := \begin{cases} \widetilde{W}_0^M(x) \ if \ x \in Y_0 \\ 0 \qquad\quad if \ x \in Y_1, \end{cases} \tag{9.8}$$

are C^1*-stratified and have the following additional regularity properties:*

(i) *the ,,maximal" value function* $\widetilde{W}^M(.)$ *is locally-Lipschitz;*

(ii) *the ,,minimal" value function* $\widetilde{W}^m(.)$ *it is only lower semicontinuous (l.s.c) with* **discontinuity points** *in the subsets:*

$$\begin{array}{l} Y_0^{0,+} := \{x = (x_1, -\ln(1+x_1)); \ x_1 \in (0,1)\} \\ Y_0^{0,-} := \{x = (x_1, -\ln(1-x_1)); \ x_1 \in (-1,0)\}. \end{array} \tag{9.9}$$

Proof. (i) The function $\widetilde{W}^M(.)$ in (9.8) is locally-Lipschitz since the derivatives $D\widetilde{W}_0^{\pm}(x)$ remain bounded as $x \to \xi \in Y_1^{\pm}$ and also as $x \to y = (0, y_2) \in \widetilde{Y}_0^0$.

On the other hand, the ,,minimal" function $\widetilde{W}_0^m(.)$ in (9.7) is *l.s.c* at each point but discontinuous at the points in (9.9) since if, for instance, $x \in Y_0^{0,+} \subset Y_0^+ \cap Y_{0,0}^{0,+}$, then it follows from (9.3) that:

$$\widetilde{W}_0^m(x) = \liminf_{y \to x} \widetilde{W}_0^m(y) < W_0^+(x) = \limsup_{y \to x} \widetilde{W}_0^m(y). \tag{9.10}$$

It is also very easy to prove the fact that the ,,maximal-type" value function $\widetilde{W}_0^M(.)$ in (9.7) defines as in (8.1) the pair $(\widetilde{U}_M(.), \widetilde{V}_M(.))$ of ,,admissible feedback strategies" while the ,,minimal" one, $\widetilde{W}_0^m(.)$ in (9.7) does not have this property.

Proposition 9.2. *The feedback strategies in* (8.1) *defined by the value functions in* (9.7) *are given by:*

$$(\widetilde{U}_m(x), \widetilde{V}_m(x)) = \begin{cases} \{(\widetilde{u}^+(x), \widetilde{v}^+(x))\} \ if\ x \in Y_0^+ \setminus Y_{0,0}^{0,+} \\ \{(\widetilde{u}^-(x), \widetilde{v}^-(x))\} \ if\ x \in Y_0^- \setminus Y_{0,0}^{0,+} \\ \{(\widetilde{u}_0^{0,+}(x), \widetilde{v}_0^{0,+}(x))\} \ if\ x \in Y_{0,0}^{0,+}, \end{cases} \tag{9.11}$$

$$\begin{aligned} \widetilde{U}_M(x) &= \begin{cases} \{\widetilde{u}^+(x)\} & if\ x \in \widetilde{Y}_0^+ \\ \{\widetilde{u}^-(x)\} & if\ x \in \widetilde{Y}_0^- \\ \{\widetilde{u}^+(x), \widetilde{u}^-(x)\} & if\ x \in \widetilde{Y}_0^0, \end{cases} \\ \widetilde{V}_M(x) &= \begin{cases} \{\widetilde{v}^+(x)\} & if\ x \in \widetilde{Y}_0^+ \\ \{\widetilde{v}^-(x)\} & if\ x \in \widetilde{Y}_0^- \\ \{\widetilde{v}^+(x), \widetilde{v}^-(x)\} & if\ x \in \widetilde{Y}_0^0, \end{cases} \end{aligned} \tag{9.12}$$

where $(\widetilde{u}^{\pm}(.), \widetilde{v}^{\pm}(.))$, $(\widetilde{u}_0^{0,+}(.), \widetilde{v}_0^{0,+}(.))$ *are the mappings in* (8.15); *moreover, the pair of ,,maximal-type" feedback strategies* $(\widetilde{U}_M(.), \widetilde{V}_M(.))$ *in* (9.12) *is admissible in the sense of statement (A) of Problem 2.3 while the ,,minimal type"* $(\widetilde{U}_m(.), \widetilde{V}_m(.))$ *in* (9.11) *does not have this property.*

Proof. As one may see by ,,direct inspection", for each point $y \in Y_0^{\pm} \setminus Y_{0,0}^{0,+}$, the components $X^{\pm}(.,.)$ of the Hamiltonian flows define as in (4.4) the unique admissible controls and, respectively, trajectories:

$$\begin{aligned} &(u_y^{\pm}(t), v_y^{\pm}(t)) = (\widetilde{u}^{\pm}(x_y^{\pm}(t)), \widetilde{v}^{\pm}(x_y^{\pm}(t)))\ \forall\ t \in [0, -\widehat{t}^{\pm}(y)] \\ &x_y^{\pm}(t) := X^{\pm}(t + \widehat{t}^{\pm}(y), \widehat{\xi}_1^{\pm}(y)) \in Y_0^{\pm} \setminus Y_{0,0}^{0,+}, \end{aligned} \tag{9.13}$$

along which the value of the cost functional in (2.2) is given by:

$$\mathcal{C}(y; u_y^{\pm}(.), v_y^{\pm}(.)) = W_0^{\pm}(y) = \widetilde{W}_0^M(y). \tag{9.14}$$

Similarly, for any point $y \in Y_{0,0}^{0,+}$, the component $X_0^{0,+}(.,.)$ in (7.4) defines the unique admissible control and, respectively, trajectories:

$$\begin{aligned} &(u_y^0(t), v_y^0(t)) = (\widetilde{u}_0^{0,+}(x_y^0(t)), \widetilde{v}_0^{0,+}(x_y^0(t))), \\ &x_y^0(t) = X_0^{0,+}(t + \widehat{t}_0^{0,+}(y), \widehat{\lambda}_0^{0,+}(y)) \end{aligned} \tag{9.15}$$

for $t \in [0, -\widehat{t}_0^{0,+}(y)]$, along which the value of the cost functional in (2.2) is given by:

$$\mathcal{C}(y; u_y^0(.), v_y^0(.)) = W_{0,0}^{0,+}(y) = \widetilde{W}_0^m(y) = \frac{1}{2}x_2. \tag{9.16}$$

Next, it follows from the definitions in (9.12) that for each initial point $y \in \widetilde{Y}_0^+ \cup \widetilde{Y}_0^-$ the feedback differential inclusion:

$$x' \in f(x, \widetilde{U}_M(x), \widetilde{V}_M(x)), \quad x(0) = y, \tag{9.17}$$

has the unique admissible trajectory $\widetilde{x}_y(.) := x_y^{\pm}(.)$ in (9.13) while for each initial point $y \in \widetilde{Y}_0^0 = \{0\} \times R_+$, the feedback differential inclusion in (9.17) has exactly two admissible trajectories, $x_y^+(.)$, $x_y^-(.)$, for which the relation in (9.14) holds, hence the pair $(\widetilde{U}_M(.), \widetilde{V}_M(.))$ in (9.12) is admissible.

On the other hand, it follows from the definitions in (9.11) that for each initial point $y \in Y_0^{0,+} \cup Y_0^{0,-}$ in (9.9) the feedback differential inclusion:

$$x' \in f(x, \widetilde{U}_m(x), \widetilde{V}_m(x)), \quad x(0) = y, \tag{9.18}$$

has exactly two admissible trajectories, $x_y^{\pm}(.)$, $x_y^0(.)$, respectively, for which, according to the relations in (9.3)-(9.4) one has:

$$\mathcal{C}(y; u_y^{\pm}(.), v_y^{\pm}(.)) = W_0^{\pm}(y) > W_{0,0}^{0,+}(y) = \mathcal{C}(y; u_y^0(.), v_y^0(.)), \tag{9.19}$$

hence $(\widetilde{U}_m(.), \widetilde{V}_m(.))$ in (9.11) is not admissible.

The main result in this section is the following:

Theorem 9.3. *The admissible pair of feedback strategies* $(\widetilde{U}_M(.), \widetilde{V}_M(.))$ *in* (9.12) *with the value function* $\widetilde{W}^M(.)$ *in* (9.8) *is optimal in the sense of Problem* 2.3.

Proof. Taking into account the fact that, the value functions $\widetilde{W}^M(.)$ in (9.8) is continuous on Y_1 and locally-Lipschitz on the its domain $\widetilde{Y}_0^M$, to prove the optimality of the admissible pair of feedback strategies $(\widetilde{U}_M(.), \widetilde{V}_M(.))$ in (9.12) we must use the verification Theorem 5.4 in [10] for locally-Lipschitz value functions which reduces to the verification some differential inequalities.

According to the Corollary 8.2, the functions $W_0^{\pm}(.)$ are solutions of class C^1 of the *Isaacs' equation* in (8.5) on the their domains $Y_0^{\pm} \supset \widetilde{Y}_0^{\pm}$ therefore, the differential inequalities (5.22)-(5.23) in [10] are automatically satisfied on the domains $\widetilde{Y}_0^{\pm}$. It remains only to check these inequalities at the „junctions points" $y \in \widetilde{Y}_0^0$; to this end, we note first that from the definitions of the extreme contingent derivatives (e.g. [9], [10],etc.) that at such points it follows:

$$\begin{aligned} &\overline{D}_K^{\pm} \widetilde{W}_0^M(x; w) = \max\{DW_0^+(x).w, DW_0^-(x).w\}, \ w \in T_x \widetilde{Y}_0^0 \\ &\underline{D}_K^{\pm} \widetilde{W}_0^M(x; w) = \min\{DW_0^+(x).w, DW_0^-(x).w\}, \end{aligned} \tag{9.20}$$

hence if $u \in U$, $\overline{v} := \widetilde{v}^+(x) = 1$ then one has:

$$\begin{aligned} &\max\{\overline{D}_K^+, \overline{D}_K^-\} \widetilde{W}_0^M(x; f(x, u, \overline{v})) + f_0(x, u, \overline{v}) \geq \\ &\geq DW_0^+(x).f(x, u, \overline{v}) + f_0(x, u, \overline{v}) \geq 0, \end{aligned}$$

while if $u \in U$, $\overline{v} := \widetilde{v}^-(x) = -1$, then one has:

$$\begin{aligned} &\max\{\overline{D}_K^+, \overline{D}_K^-\} \widetilde{W}_0^M(x; f(x, u, \overline{v})) + f_0(x, u, \overline{v}) \geq \\ &\geq DW_0^-(x).f(x, u, \overline{v}) + f_0(x, u, \overline{v}) \geq 0, \end{aligned}$$

which proves the inequality (5.22) in [10]; the symmetric inequality (5.23) in [10] follows in the same way using Corollary 8.2 according to which the functions $W_0^{\pm}(.)$ satisfy *Isaacs' equation* on their corresponding domains $Y_0^{\pm} \supset \widetilde{Y}_0^{\pm}$; see Fig.11.

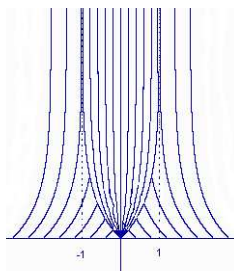

Fig. 10.Case $k \in (0, \frac{1}{2}]$

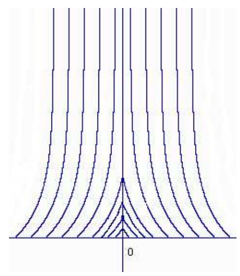

Fig. 11. Case $k \in (0, \frac{1}{2}]$

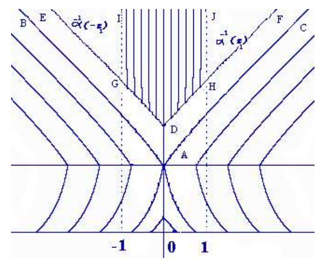

Fig. 12. Case $k > \frac{1}{2}$.

10 A partial solution in the case $k > \frac{1}{2}$

In the case $k > \frac{1}{2}$, as one may see in Fig. 9, the domains $Y_0^{\pm}$, $Y_0^{\pm,\pm}$, $Y_{0,0}^{0,+}$ do not cover any more the set Y_0 of initial states and, moreover, the extremal value functions have the more complicated forms:

$$\begin{array}{l} \widetilde{W}_0^m(x) = \min\{W_0^{\pm}(x), W_{0,0}^{0,+}(x), W_0^{\pm,\pm}(x)\}, x \in Y_0^{\pm} \cap Y_0^{\pm,\pm} \cap Y_{0,0}^{0,+} \\ \widetilde{W}_0^M(x) = \max\{W_0^{\pm}(x), W_{0,0}^{0,+}(x), W_0^{\pm,\pm}(x)\}, \end{array} \tag{10.1}$$

on the intersection of the domains.

As in Proposition 9.2, one may prove that the ,,minimal-type" value function, $\widetilde{W}_0^m(.)$ is not admissible so we shall concentrate on the ,,maximal" one, $\widetilde{W}_0^M(.)$.

From Proposition 9.1 and some additional computations it follows that the function $\widetilde{W}_0^M(.)$ in (10.1) is given, more explicitely by the formulas:

$$\widetilde{W}_0^M(y) := \left\{ \begin{array}{l} W^{\pm}(y),\ y \in \widetilde{Y}_0^{\pm} \\ W^{+}(y) = W^{-}(y),\ y \in \widetilde{Y}_0^{0} \\ W^{\pm,\pm}(y), y \in \widetilde{Y}_0^{\pm,\pm} \\ W^{+,+}(y) = W^{-,-}(y),\ y \in \widetilde{Y}_{0,0}^{0,0} \\ W_{0,0}^{0,+}(x),\ x \in \widetilde{Y}_{0,0}^{0,+} \end{array} \right. \tag{10.2}$$

where the sets $\widetilde{Y}_0^{\pm}$, $\widetilde{Y}_0^{\pm,\pm}$, $\widetilde{Y}_0^{0}$, $\widetilde{Y}_{0,0}^{0,0}$, $\widetilde{Y}_{0,0}^{0,+}$ are defined in the formulas of the corresponding feedback strategies:

$$\widetilde{U}_M(y) := \begin{cases} \{\widetilde{u}^{\pm}(y)\},\ y \in \widetilde{Y}_0^{\pm} := R_{\pm}^* \times (0, -\ln(1-\frac{1}{2k})] \\ \{\widetilde{u}^+(y), \widetilde{u}^-(y)\},\ y \in \widetilde{Y}_0^0 := \{0\} \times (0, -\ln(1-\frac{1}{2k})] \\ \{\widetilde{u}^{\pm,\pm}(y)\}, y \in \widetilde{Y}_0^{\pm,\pm} = \{x; x_1 \in R_{\pm}^*, x_2 \in (-\ln(1-\frac{1}{2k}), \\ \qquad \alpha^{-1}(\pm x_1)\} \\ \{\widetilde{u}^{+,+}(y), \widetilde{u}^{-,-}(y)\}, y \in \widetilde{Y}_{0,0}^{0,0} = \{0\} \times (-\ln(1-\frac{1}{2k}), +\infty) \\ \{\widetilde{u}_0^{0,+}(x)\}, x \in \widetilde{Y}_{0,0}^{0,+} := \{x; x_1 \in (0,1), x_2 > \alpha^{-1}(x_1)\} \cup \\ \qquad \{x; x_1 \in (-1,0), x_2 > \alpha^{-1}(-x_1)\} \end{cases} \tag{10.3}$$

$$\widetilde{V}_M(y) := \begin{cases} \{\widetilde{v}^{\pm}(y)\},\ y \in \widetilde{Y}_0^{\pm} := R_{\pm}^* \times (0, -\ln(1-\frac{1}{2k})] \\ \{\widetilde{v}^+(y), \widetilde{v}^-(y)\}, y \in \widetilde{Y}_0^0 := \{0\} \times (0, -\ln(1-\frac{1}{2k})] \\ \{\widetilde{v}^{\pm,\pm}(y)\}, y \in \widetilde{Y}_0^{\pm,\pm} = \{x; x_1 \in R_{\pm}^*, x_2 \in (-\ln(1-\frac{1}{2k}), \\ \qquad \alpha^{-1}(\pm x_1)\} \\ \{\widetilde{v}^{+,+}(y), \widetilde{v}^{-,-}(y)\}, y \in \widetilde{Y}_{0,0}^{0,0} = \{0\} \times (-\ln(1-\frac{1}{2k}), +\infty) \\ \{\widetilde{v}_0^{0,+}(x)\}, x \in \widetilde{Y}_{0,0}^{0,+} := \{x; x_1 \in (0,1), x_2 > \alpha^{-1}(x_1)\} \cup \\ \qquad \{x; x_1 \in (-1,0), x_2 > \alpha^{-1}(-x_1)\} \end{cases}$$

The admissible trajectories generated by the feedback strategies $(\widetilde{U}(.), \widetilde{V}(.))$ are shown in Fig. 12, from which one may see that only on the subset $\widetilde{Y}_0^1$ situated below the curve BAC in Fig. 12 the function $\widetilde{W}_0^M$ in (10.2) is the value function associated to these trajectories. Therefore, as in Theorem 9.3, it follows that $(\widetilde{U}(.), \widetilde{V}(.))|_{\widetilde{Y}_0^1}$ is a solution of the restriction $DG_A|_{\widetilde{Y}_0^1}$ which is not a „**satisfactory**" solution for the initial problem since the subset $\widetilde{Y}_0^1$ is obviosely not invariant with respect to the differential system (2.3), (2.7).

On the other hand, at each point $y \in \widetilde{Y}_0^2 := \widetilde{Y}_0^+ \setminus \widetilde{Y}_0^1$, the feedback strategies $(\widetilde{U}(.), \widetilde{V}(.))$ generate a unique trajetory, $\widetilde{x}_y(.)$ reaches the point $y = A(0, \overline{x}_2)$, $\overline{x}_2 = -\ln(1-\frac{1}{2k})$ from which one may be continuated by one of the trajectories of the previous solution.

Therefore, for each point $y \in \widetilde{Y}_0^2$ (above the curve BAC), there exists a moment, $\widetilde{t}(y) > 0$ such that the trajectory $\widetilde{x}_y(.)$ (generated by $(\widetilde{U}(.), \widetilde{V}(.))$) reaches the point $A(0, \overline{x}_2)$, at $\widetilde{t}(y)$, ie. $\widetilde{x}_y(\widetilde{t}(y)) = \overline{y} = (0, \overline{x}_2)$; therefore, according to (2.2), (2.7), the cost functional along the trajectory $\widetilde{x}_y(.)$ is given by:

$$\widetilde{W}_0(y) = \mathcal{C}(y; \widetilde{u}_y(.), \widetilde{v}_y(.)) = \widetilde{t}(y) + \widetilde{W}_0^M(\overline{y}) \tag{10.4}$$

which actually defines the value function $\widetilde{W}_0(.)$, associated to the feedback strategies $(\widetilde{U}(.), \widetilde{V}(.))$.

Direct, though tedious computations show that the function $\widetilde{t}(.)$ described above is *locally-Lipschitz* (and also stratified) which as in Theorem 9.3, reads to the conclusion that the function $\widetilde{W}_0(.)$ defined by:

$$\widetilde{W}_0(y) := \begin{cases} \widetilde{W}_0^M(y)\ if\ y \in \widetilde{Y}_0^1 \\ \widetilde{t}(y) + \widetilde{W}_0^M(y)\ if\ y \in \widetilde{Y}_0^2 := \widetilde{Y}_0^M \setminus \widetilde{y}_0^1 \end{cases} \tag{10.5}$$

is the value function associated to the feedback strategies $(\widetilde{U}(.), \widetilde{V}(.))$ in (10.3) and is optimal for the restriction $DG_A|_{\widetilde{Y}_0^M}$.

However, to be a complete solution, one needs to prove that the domain $\widetilde{Y}_0^M$ described in (10.3) (see Fig. 12) is invariant with respect to the differential system in (2.3), (2.7); this argument requires some more computations and arguments.

We note that in [4] only the obvious non-complete solution, $\widetilde{W}_0^M(.)|_{\widetilde{Y}_0^1}$ has been pointed aut and not quite rigorously justified.

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University of Bucharest,
Faculty of Mathematics and Informatics,
Academiei 14, 010014 Bucharest,
Romania.
E-mail: mirica@fmi.unibuc.ro
E-mail: bouremani@yahoo.com