Bull. Math. Soc. Sc. Math. Roumanie
Tome 49(97) No. 2, 2006, 189–191

A note on perturbations of authomorphisms of type II_1 factors

by
FLORIN POP

Abstract

We study perturbations of *-homorphisms of type II_1 factors with respect to the norm $\|\cdot\|_{2,\infty}$ introduced by Sinclair and Smith. We show that automorphisms close to the identity in this norm are implemented by unitary operators close to the identity in the Hilbert-Schmidt norm.

Key Words: type II_1 factor, subfactor, *-homomorphism.
2000 Mathematics Subject Classification: Primary: 46L40.

In [1] Kadison and Ringrose proved the remarkable fact that if an automorphism φ of a von Neumann algebra M satisfies $||\varphi - Id_M|| < 2$, then φ is inner, that is, $\varphi(x) = uxu^*$ for some unitary operator $u \in M$.

In this paper we consider a different metric on $Aut(M)$, the automorphism group of a type II_1 factor M. Our approach is inspired by the work of Popa, Sinclair and Smith on perturbations of subalgebras of type II_1 factors ([2], [3], [4]). This metric, denoted by $||.||_{2,\infty}$, is of Hilbert - Schmidt type and, from the perturbations viewpoint, has proved to be more flexible than the usual Hausdorff distance.

Throughout this paper M denotes a type II_1 factor with trace τ and unitary group $\mathcal{U}(M)$. $Aut(M)$ denotes the group of $*-$automorphisms of M, the identity automorphism being Id_M. The Hilbert-Schmidt norm on M is given by $||x||_2 = \tau(x^*x)^{1/2}$. If $A \subset M$ is a subalgebra and if f and g are bounded linear maps from A to M, define (following [3])

$$||f - g||_{2,\infty} = sup\{||f(x) - g(x)||_2,\ x \in A,\ ||x|| \leq 1\}$$

The main result is the following

Theorem. *Let M be a type II_1 factor and let $N \subset M$ be a subfactor with trivial relative commutant. If $\varphi : N \to M$ is a unital $*$-homomorphism such* *that* $||\varphi - Id_N||_{2,\infty} < \sqrt{2}$, *then* φ *is implemented by a unitary* v *in* M *satisfying* $||v - I||_2 \leq ||\varphi - Id_N||_{2,\infty}$.

Proof: Denote $||\varphi - Id_N||_{2,\infty} = t < \sqrt{2}$. Then $||\varphi(u)u^* - I||_2 \leq t$ $\forall\ u \in \mathcal{U}(N)$. We have

$$||\varphi(u)u^* - I||_2^2 = \tau((\varphi(u)u^* - I)(u\varphi(u^*) - I)) = 2 - 2Re\ \tau(\varphi(u)u^*) \leq t^2$$

It follows that $Re\ \tau(\varphi(u)u^*) \geq \frac{2-t^2}{2} > 0$. If a is the element of smallest 2-norm in the closed, convex hull of $\{\varphi(u)u^*\}$, then $\varphi(u)au^* = a$ and $||a - I||_2 \leq t$. Also, by the preceding remark, $Re\ \tau(a) > 0$, so $\tau(a) \neq 0$.

On the other hand, $\varphi(u^*)a = au^* \Rightarrow a^*\varphi(u) = ua^* \Rightarrow a^*\varphi(u)u^* = ua^*u^*$. There is a net of convex combinations of elements of the form $\varphi(u)u^*$ converging ultraweakly to a. By passing, if necessary, to a subnet, we may assume that the corresponding convex combinations of ua^*u^* converge ultraweakly to some d. By applying the trace, we get $a^*a = d \Rightarrow \tau(a^*a) = \tau(d) = \tau(a^*)$. This shows that $\tau(a) = \tau(a^*) = \tau(a^*a) = \lambda > 0$.

For all u in $\mathcal{U}(N)$ we have $\varphi(u)a = au$ and $a^*\varphi(u^*) = u^*a^* \Rightarrow a^*\varphi(u) = ua^*$. It follows that $a^*au = a^*\varphi(u)a = ua^*a$, therefore a^*a commutes with N and, since $N' \cap M$ is trivial, $a^*a = \tau(a^*a)I = \lambda I$. If we define $v = a/\sqrt{\lambda}$, then v is a unitary and satisfies $\varphi(x) = vxv^*$ for all x in N.

To see how close v is to I, note that

$$||v - I||_2^2 = \tau((v - I)(v^* - I)) = \tau(2I - v - v^*) = 2 - \frac{1}{\sqrt{\lambda}}(\tau(a) + \tau(a^*))$$

$$= 2 - \frac{2\tau(a)}{\sqrt{\lambda}} = 2 - 2\sqrt{\lambda} = \frac{2(1-\lambda)}{1+\sqrt{\lambda}} \leq 2(1-\lambda)$$

On the other hand,

$$\lambda = \tau(a) = Re\ \tau(a) \geq \frac{2-t^2}{2} \Rightarrow 2(1-\lambda) \leq 2(1 - \frac{2-t^2}{2}) = t^2$$

which implies $||v - I||_2^2 \leq t^2 \Rightarrow ||v - I||_2 \leq ||\varphi - Id_N||_{2,\infty}$. □

Corollary. (i) *Let* $\varphi : M \to M$ *be a unital* $*$*-homomorphism. If*

$$||\varphi - Id_M||_{2,\infty} < \sqrt{2},$$

then φ *is implemented by a unitary* v *in* $\mathcal{U}(M)$ *satisfying*

$$||v - I||_2 \leq ||\varphi - Id_M||_{2,\infty}.$$

In particular, φ *is an automorphism.*

(ii) If φ and ψ are automorphisms of M such that $||\varphi-\psi||_{2,\infty}<\sqrt{2}$, then φ and ψ are conjugate via a unitary v in $\mathcal{U}(M)$ satisfying $||v-I||_2 \leq ||\varphi-\psi||_{2,\infty}$.

Remark. In the theorem we cannot drop the condition $N'\cap M=\mathbf{C}I$. Take $M=N\otimes M_n$ and let φ be an outer automorphism of N. Let $\overline{N}$ consist of diagonal operators $x\oplus x\oplus ...\oplus x$ and define $\theta:\overline{N}\to N\otimes M_n$ by $\theta(x\oplus x\oplus ...\oplus x)=x\oplus x\oplus ...\oplus x\oplus\varphi(x)$. For all $x\in N$ with $||x||\leq 1$ we have

$$||\theta(x\oplus ...\oplus x)-(x\oplus ...\oplus x)||_2^2=\frac{1}{n}||\varphi(x)-x||_2^2\leq\frac{4}{n}\Rightarrow ||\theta-Id_{\overline{N}}||_{2,\infty}\leq\frac{2}{\sqrt{n}}$$

We will show that θ is not implemented by any unitary in M. To get a contradiction, suppose there exists a unitary $U=(a_{ij})$ in M satisfying

$$U\theta(x\oplus ...\oplus x)=(x\oplus ...\oplus x)U$$

Then, for all $1\leq i,j\leq n-1$, $a_{ij}x=xa_{ij}$, hence a_{ij} are scalar multiples of the identity of N. Denote $a_{in}=b_i$. Since $UU^*=I_M$, it is easily seen that $b_ib_i^*=t_i^2I$ for all $1\leq i\leq n-1$ and for some $t_i\geq 0$. If $t_i\neq 0$, then $w_i=b_i/t_i$ is unitary and $w_i\varphi(x)=xw_i$, which is impossible, since φ is outer. This shows that $b_i=0$ for all $1\leq i\leq n-1$, which forces $w=a_{nn}$ to be a unitary in N such that $w\varphi(x)=xw$, contradiction. □

References

[1] R. V. Kadison and J. R. Ringrose, Derivations and automorphisms of operator algebras, Comm. Math. Phys., 4(1967), 32-63.

[2] S. Popa, A. Sinclair and R. R. Smith, Perturbations of subalgebras of type II_1 factors, J. Funct. Anal., 213(2004), 346-379.

[3] A. Sinclair and R. R. Smith, Strongly singular masas in type II_1 factors, Geom. Funct. Anal. (GAFA), 12(2002), 199-216.

[4] A. Sinclair and R. R. Smith, The laplacian masa in a free group factor, Trans. Amer. Math. Soc., 355(2003), 465-475.

Received: 05.01.2005

Department of Mathematics and Computer Science,
Wagner College, Staten Island,
New York 10301, U. S. A.
E-mail: fpop@wagner.edu