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Easy proofs of some well known facts via cleanness*

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Abstract

We give easy proofs for some well known facts by using some basic property of cleanness. We show that if $(R, \mathfrak{m})$ is a Noetherian local ring and M is a finitely generated almost clean R-module with the property that R/P is Cohen–Macaulay for all $P \in \operatorname{Ass}(M)$, then $\operatorname{depth}(M) = \min\{\dim(R/P) : p \in \operatorname{Ass}(M)\}$. Using this fact we show that if M is a finitely generated clean R-module such that R/P is Cohen–Macaulay and $\dim(M) = \dim(R/P)$ for all minimal prime ideals of M, then M is Cohen–Macaulay. This implies the well known fact that a pure shellable simplicial complex is Cohen–Macaulay.

Key Words: Prime filtration, Shellable simplicial complex, Monomial ideals, Clean and pretty clean modules.
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Introduction

Let R be a Noetherian ring and M an R-module. A chain $\mathcal{F}$: $(0) = M_0 \subset M_1 \subset \ldots \subset M_r = M$ of submodules of M is called a prime filtration of M if for all $i = 1, \ldots, r$ there exists a prime ideal $P_i \in \operatorname{Spec}(R)$ such that $M_i/M_{i-1} \cong R/P_i$. If M is finitely generated such a prime filtration of M always exists, see [5, Theorem 6.4]. The set of prime ideals $P_1, \ldots, P_r$ which define the cyclic quotients of $\mathcal{F}$ will be denoted by $\operatorname{Supp}(\mathcal{F})$. It follows from [5, Theorem 6.3] that if $\mathcal{F}$ is a prime filtration of M, then $\operatorname{Ass}(M) \subset \operatorname{Supp}(\mathcal{F}) \subset \operatorname{Supp}(M)$.

Dress [3] called the prime filtration $\mathcal{F}$ *clean* if $\operatorname{Supp}(\mathcal{F}) = \operatorname{Min}(M)$. The R-module M is called clean if it has a clean filtration.

Herzog and Popescu [4] generalized this concept and introduced pretty clean filtration. The prime filtration $\mathcal{F}$ is called *pretty clean* if for all $i < j$ with $P_i \subseteq P_j$ it follows that $P_i = P_j$. The R-module M is called pretty clean if it admits a pretty clean filtration. It follows from [4, Corollary 3.4] that if $\mathcal{F}$ is a pretty clean

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filtration of M, then $\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}(M)$. However there are examples which show that the converse of the above fact is not true, see [4, Example 3.6] and [6, Example 4.4]. We call an R-module M *almost clean* if it admits a prime filtration $\mathcal{F}$ with

$$\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}(M).$$

It is easy to see that

$$\text{clean} \Rightarrow \text{pretty clean} \Rightarrow \text{almost clean},$$

and if an R-module M has no embedded associated prime ideals, then

$$M \text{ is clean} \Leftrightarrow M \text{ is pretty clean} \Leftrightarrow M \text{ is almost clean}.$$

This paper is organized as follow. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. In Section 1 we give an upper bound for the depth of M in terms of the minimum of $\dim(R/P)$, where $P \in \mathrm{Supp}(\mathcal{F})$ and $\mathcal{F}$ is a prime filtration of M. By using this fact we show that if $\mathcal{F}$ is an almost clean filtration of M such that R/P is Cohen–Macaulay for all $P \in \mathrm{Supp}(\mathcal{F})$, then $\mathrm{depth}(M) = \min\{\dim(R/P) :\ P \in \mathrm{Ass}(M)\}$. This implies that if M is a clean R-module such that R/P is Cohen–Macaulay and $\dim(M) = \dim(R/P)$ for all $P \in \mathrm{Min}(M)$, then M is Cohen–Macaulay. In Section 2 we give an easy proof for a Theorem of Dress which says that a simplicial complex Δ on vertex set $[n] = \{1, 2, \ldots, n\}$ is shellable if and only if its Stanley–Reisner ring $K[\Delta] = K[x_1, x_2, \ldots, x_n]/I_\Delta$ is clean. Here K is a field and the Stanley–Reisner ideal I_Δ is a squarefree monomial ideal generated by all $x_{i_1}x_{i_2}\cdots x_{i_l}$, where $\{i_1, \ldots, i_l\} \notin \Delta$. Then as a corollary we get the well known fact that if Δ is a shellable simplicial complex, then Δ is sequentially Cohen–Macaulay. In particular a pure shellable simplicial complex is Cohen–Macaulay.

1 Depth of clean modules

In this section we determine the depth of an almost clean R-module. For this we shall need the following result.

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. Assume that $\mathcal{F}$ is a prime filtration of M such that R/P is a Cohen–Macaulay ring for all $P \in \mathrm{Supp}(\mathcal{F})$. Then*

$$\mathrm{depth}(M) \geq \min\{\dim(R/P) :\ P \in \mathrm{Supp}\,\mathcal{F}\}.$$

Proof: Let $\mathcal{F}\colon\ (0) = M_0 \subset M_1 \subset \cdots \subset M_r = M$ be a prime filtration with $M_i/M_{i-1} \cong R/P_i$. We prove the assertion by induction on r, the length of the prime filtration $\mathcal{F}$. If $r = 1$, then $M \cong R/P_1$ and by our assumption M is Cohen–Macaulay. Hence $\mathrm{depth}(M) = \dim(M) = \dim(R/P_1)$. If $r \geq 2$, then

$$\mathcal{F}_1 :\ (0) = M_1/M_1 \subset M_2/M_1 \subset \cdots \subset M/M_1$$

is a prime filtration of M/M_1 which has length $r-1$. Since $\mathrm{Supp}(\mathcal{F}_1) \subset \mathrm{Supp}(\mathcal{F})$, by induction hypothesis we have $\mathrm{depth}(M/M_1) \geq \min\{\dim(R/P) : P \in \mathrm{Supp}\,\mathcal{F}_1\}$. Therefore the assertion follows if we apply the depth Lemma [1, Proposition 1.2.9] to the following short exact sequence

$$(0) \to M_1 \to M \to M/M_1 \to (0).$$

□

Now we recall the following well known fact.

Proposition 1.2. *([1, Proposition 1.2.13]) Let* $(R, \mathfrak{m})$ *be a Noertherian local ring and* M *a finitely generated* R*-module. Then* $\mathrm{depth}(M) \leq \min\{\dim(R/P) : P \in \mathrm{Ass}(M)\}$.

If we combine Proposition 1.2 with Theorem 1.1 we get

Corollary 1.3. *Let* $(R, \mathfrak{m})$ *be a Noertherian local ring and* M *a finitely generated* R*-module. If* M *is an almost clean* R*-module with the property that* R/P *is Cohen–Macaulay for all* $P \in \mathrm{Ass}(M)$*, then*

$$\mathrm{depth}(M) = \min\{\dim(R/P) : P \in \mathrm{Ass}(M)\}.$$

As an immediate consequence of Corollary 1.3 we get the following.

Corollary 1.4. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* M *a finitely generated* R*-module with a clean filtration* $\mathcal{F}$ *such that* R/P *is Cohen–Macaulay and* $\dim(R/P) = \dim(M)$ *for all* $P \in \mathrm{Supp}(\mathcal{F})$*. Then* M *is Cohen–Macaulay.*

Let R be a Noetherian local rind and M a finitely generated R-module. A finite filtration

$$(0) \subset M_1 \subset M_2 \subset \cdots \subset M_s = M$$

of submodules of M is called a CM filtration, if each quotient M_i/M_{i-1} is Cohen–Macaulay and

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_s/M_{s-1}).$$

The R-module M is called sequentially Cohen–Macaulay if M admit a CM filtration.

In [4, Theorem 4.1] Herzog and Popescu proved that a pretty clean R-module M which satisfies some extra conditions is sequentially Cohen–Maculay. In the following we give an easy proof for the same fact.

Corollary 1.5. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* M *a finitely generated* R*-module with a pretty clean filtration*

$$\mathcal{F}\colon\ (0) = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that R/P_i *is Cohen–Macaulay and* $\dim(R/P_i) \leq \dim(R/P_{i+1})$ *for all* i*. Then* M *is sequentially Cohen–Macaulay.*

Proof: Let t_1 be the largest integer such that $\dim(M_{t_1}/M_{t_1-1}) = \dim(R/P_{t_1-1}) = \dim(R/P_1)$. Then M_{t_1} is clean and hence by Corollary 1.4 is Cohen–Macaulay of dimension $\dim(R/P_1)$. We know that $M_{t_1+1}/M_{t_1} \cong R/P_{t_1}$, and $\dim(R/P_{t_1}) > \dim(R/P_1)$. Again let t_2 be the largest integer such that $\dim(M_{t_2}/M_{t_2-1}) = \dim(R/P_{t_1})$. Then M_{t_2}/M_{t_1} is clean and therefore Cohen–Macaulay of dimension $\dim(R/P_{t_1})$. If we continue in this way after a finite number of steps we obtain the chain $(0) \subset M_{t_1} \subset M_{t_2} \subset \cdots \subset M_{t_s} = M$ of submodule of M which is indeed a CM filtration. □

2 The relation between cleanness and shellablity

Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables. For a monomial $u \in S$ we denote $\text{supp}(u) = \{i\colon\ x_i \mid u\}$. Let I be a monomial ideal in S. We say that I is (pretty) clean if S/I is (pretty) clean. Cleanness is the algebraic counterpart of shellability for simplicial complexes.

A *simplicial complex* Δ over a set of vertices $[n] = \{1, \ldots, n\}$ is a collection of subsets of $[n]$ with the property that $i \in \Delta$ for all $i \in [n]$, and if $F \in \Delta$, then all the subsets of F are also in Δ. An element of Δ is called a *face* of Δ, and the maximal faces of Δ under inclusion are called *facets*. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. The *dimension* of a face F is defined as $\dim F = |F| - 1$, where $|F|$ is the number of vertices of F. The dimension of the simplicial complex Δ is the maximal dimension of its facets. A simplicial complex Δ is called *pure* if all facets of Δ have the same dimension. We denote the simplicial complex Δ with facets $F_1, \ldots, F_t$ by $\Delta = \langle F_1, \ldots, F_t\rangle$. If Δ is a simplicial complex on vertex set $[n]$, then the Stanley–Reisner ideal, I_Δ, is the squarefree monomial ideal generated by all monomials $x_{i_1}x_{i_2}\cdots x_{i_t}$ such that $\{i_1, i_2, \ldots, i_t\} \notin \Delta$. If $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$, then $I_\Delta = \bigcap_{i=1}^m P_{F_i}$, where $P_{F_i} = (x_j : j \notin F_i)$, see [1, Theorem 5.4.1]. We say the simplicial complex Δ is Cohen–Macaulay if S/I_Δ is Cohen–Macaulay.

According to Björner and Wachs [2] an order $F_1, \ldots, F_t$ of the facets of Δ is called a (non-pure) shelling of Δ if the simplicial complex $\langle F_1, \ldots, F_{i-1}\rangle \cap \langle F_i\rangle$ is pure and $(\dim F_i - 1)$-dimensional for all $i = 2, \ldots, t$. Given a shelling $F_1, \ldots, F_t$ of Δ, we denote by Δ_i the simplicial complex with facets $F_1, \ldots, F_i$. We follow the notation in [2] and define the restriction of facet F_k by

$$R(F_k) = \{i \in F_k \colon F_k \setminus \{i\} \in \Delta_{k-1}\}.$$

Then $R(F_k) \subset F_k$ is the unique minimal face which is not in Δ_{k-1}, see [2, lemma 2.4]. In other words

$$\langle F_k\rangle \setminus \Delta_{k-1} = [R(F_k), F_k] = \{B \colon R(F_k) \subset B \subset F_k\}.$$

Therefore the simplicial complex Δ splits up into disjoint union of Boolean intervals

$$\Delta = \bigcup_{i=1}^{t} [R(F_i), F_i].$$

It is easy to see from the definition of Stanley–Reisner ideal that

$$I_{\Delta_{k-1}} = (I_{\Delta_k}, u)_{\text{supp}(u)\in(F_k\setminus\Delta_{k-1})} = (I_{\Delta_k}, X_{R(F_k)}), \quad \text{where} \quad X_{R(F_k)} = \prod_{j\in R(F_k)} x_j$$

The following fact was proved by Dress [3]. Here we give an easy proof of it by using induction.

Theorem 2.1. *Let Δ be a simplicial complex with facets $F_1, \dots, F_t$. The following are equivalent:*

(a) An order $F_1, \dots, F_t$ of facets of Δ is a shelling of Δ ;

(b) $\mathcal{F}\colon I = I_\Delta \subset I_{\Delta_{t-1}} \subset \cdots \subset I_{\Delta_1} \subset I_{\Delta_0} = S$ is a clean filtration of S/I_Δ with $I_{\Delta_{i-1}}/I_{\Delta i} \cong S/P_{F_i}$ for $i = 1, \dots, t$.

Proof: (a) $\Rightarrow$ (b): We use induction on t, the number of facets of Δ. If $t = 1$, then Δ is a simplex and we are done. Let $t > 1$. Then the order $F_1, \dots, F_{t-1}$ is a shelling for Δ_{t-1}. Therefore by induction hypothesis

$$\mathcal{F}_1\colon I_{\Delta_{t-1}} \subset \cdots \subset I_{\Delta_1} \subset I_{\Delta_0} = S$$

is a clean filtration of $S/I_{\Delta_{t-1}}$ with $I_{\Delta_{i-1}}/I_{\Delta i} \cong S/P_{F_i}$ for $i = 1, \dots, t-1$. From the fact that $I_{\Delta_{t-1}} = (I_\Delta, X_{R(F_t)})$, one has $I_{\Delta_{t-1}}/I_\Delta \cong S/(I_\Delta : X_{R(F_t)}) \cong S/P_{F_t}$, and therefore $\mathcal{F}$ is a clean filtration of S/I_Δ.

(b) $\Rightarrow$ (a): Again we prove by induction on t, the length of clean filtration $\mathcal{F}$. If $t = 1$, then I_Δ ia a monomial prime ideal. Therefore Δ is a simplex and shellable. Let $t > 1$. Then

$$\mathcal{F}_1\colon I_{\Delta_{t-1}} \subset \cdots \subset I_{\Delta_1} \subset I_{\Delta_0} = S$$

is a clean filtration of $S/I_{\Delta_{t-1}}$. Hence by induction hypothesis the order $F_1, \dots, F_{t-1}$ is a shelling for Δ_{t-1}. On the other hand since $I_\Delta/I_{\Delta_{t-1}} \cong S/P_{F_t}$, one has $I_{\Delta_{t-1}} = (I_\Delta, u)$, where u is the unique minimal monomial in $I_{\Delta_{t-1}} \setminus I_\Delta$. Hence $\text{supp}(u) = \{i\colon\ x_i \mid u\}$ is the unique minimal face of $\Delta \setminus \Delta_{t-1}$. Since F_t is the unique facet of $\Delta \setminus \Delta_{t-1}$, we see that $\Delta \setminus \Delta_{t-1} = [\text{supp}(u), F_t]$. Let F be a facet of $\Delta_{t-1} \cap F_t$. Then $|F| < |F_t|$. On the other hand since $F_t - \{j\}$ is a facet of $\Delta_{t-1} \cap F_t$ for each $j \in \text{supp}(u)$, we have $\Delta_{t-1} \cap F_t$ is $\dim F_t - 1$ dimensional simplicial complex. Therefore the order $F_1, \dots, F_t$ of facets of Δ is a shelling of Δ. $\square$

It is well known that a pure simplicial complex which is shellable is Cohen–Macaulay, see [1, Theorem 5.1.13]. In the following as a corollary of our result we give a simple proof of it.

Corollary 2.2. *Let Δ be a shellable simplicial complex. Then Δ is sequentially Cohen–Macaulay. Moreover if Δ is pure, then it is Cohen–Macaulay.*

Proof: If Δ is shellable, then by [2, Lemma 2] there exists a shelling $F_1, \dots, F_t$ of Δ such that $|F_i| \geq |F_{i+1}|$. Hence by Theorem 2.1 S/I_Δ is clean and by Corollary 1.5 sequentially Cohen–Macaulay. Now if Δ is pure, then $\dim(S/I_\Delta) = \dim(S/P)$ for all P which appear in the clean filtration. Therefore by Corollary 1.4 S/I_Δ is Cohen–Macaulay. □

Let $I \subset S$ be a monomial ideal. If S/I is pretty clean, then it is shown in [6] that there exists a pretty clean filtration $\mathcal{F} : I \subset I_1 \subset \cdots \subset I_r = S$ of S/I with $I_i/I_{i-1} \cong S/P_i$ such that $\dim(S/P_i) < \dim(S/P_{i+1})$ for all i. Hence from Corollary 1.5 we get the following result which was proved by Herzog and Popescu in [4].

Corollary 2.3. *Let $I \subset S$ be a monomial ideal. If S/I is pretty clean, then S/I is sequentially Cohen–Macaulay.*

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