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A mean value inequality for multifunctions

by
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Abstract

There are established mean value inequalities for multifunctions with values in metric spaces, and moreover, there are investigated Dini derivatives of multifunctions with values in normed spaces.

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1 Introduction

The aim of this paper is to extend some mean value inequalities in [2, p. 23], [5, p. 153], and [11, p. 16] from the function setting to the multifunction one.

Let (M, d) be a metric space, let $D \subseteq R$ be a set which contains at least two points, let $F : D \to M$ be a multifunction with nonempty values, and let $g : D \to R$ be a strictly increasing function.

It has been observed in [7, p. 239] and [8, pp. 145, 146] that for each couple of nonempty sets $P \subseteq M$ and $Q \subseteq M$ there hold the inequalities

$$\inf_{p\in P}\inf_{q\in Q} d(q,p) \le \sup_{p\in P}\inf_{q\in Q} d(q,p) \le \sup_{p\in P}\sup_{q\in Q} d(q,p)$$

and the intermediate item, which we denote by $\delta(Q, P)$, although asymmetric with regard to the couple (Q, P), does have interesting properties, including the "triangle inequality" $\delta(Q, O) \le \delta(Q, P) + \delta(P, O)$. That intermediate extended real number is suitable for our aim.

Note parenthetically that: $\delta(Q, P) = 0 \Leftrightarrow P \subseteq \overline{Q}$ (here, $\overline{Q}$ stands for the closure of Q); if $\epsilon > 0$, then $\delta(Q, P) < \epsilon \Rightarrow P \subseteq B(Q, \epsilon) \Rightarrow \delta(Q, P) \le \epsilon$ (here, $B(Q, \epsilon) = \cup_{q\in Q} B(q, \epsilon)$, whereas $B(q, \epsilon)$ stands for the open ball with center q and radius ϵ, i.e. $B(q, \epsilon) = \{m \in M; d(m, q) < \epsilon\}$).

Now, let $a \in D$ and $b \in D$ such that $a < b$. In view of a result in [11, p. 16], where F is a "function" and g is the identity on D, we expect for the *homogeneous mean value inequality*

$$\frac{\delta(F(b),F(a))}{g(b)-g(a)} \le \sup_{s\in[a,b)\cap D} \inf_{t\in(s,b]\cap D} \frac{\delta(F(t),F(s))}{g(t)-g(s)} \tag{1}$$

to hold. A less concise rephrasing of inequality (1),

$$\frac{\delta(F(b),F(a))}{g(b)-g(a)} \le \sup_{s\in[a,b)\cap D} \inf_{t\in(s,b]\cap D} \sup_{S\in F(s)} \frac{\delta(F(t),S)}{g(t)-g(s)},$$

suggests for us to consider the *heterogeneous mean value inequality*

$$\frac{\delta(F(b),F(a))}{g(b)-g(a)} \le \sup_{s\in[a,b)\cap D} \sup_{S\in F(s)} \inf_{t\in(s,b]\cap D} \frac{\delta(F(t),S)}{g(t)-g(s)}, \tag{2}$$

which is an improvement on the homogeneous one, for "sup inf $\le$ inf sup".

Obviously, the two inequalities coincide if F is a "function".

In the simplest case that $[a,b]\cap D = \{a,r,b\}$ for some $r \in (a,b)$, inequality (1) states

$$\frac{\delta(F(b),F(a))}{g(b)-g(a)} \le \max\left\{\min\left\{\frac{\delta(F(r),F(a))}{g(r)-g(a)}, \frac{\delta(F(b),F(a))}{g(b)-g(a)}\right\}, \frac{\delta(F(b),F(r))}{g(b)-g(r)}\right\},$$

whereas inequality (2) states

$$\frac{\delta(F(b),F(a)}{g(b)-g(a)} \le \max\left\{\sup_{S\in F(a)} \min\left\{\frac{\delta(F(r),S)}{g(r)-g(a)}, \frac{\delta(F(b),S)}{g(b)-g(a)}\right\}, \frac{\delta(F(b),F(r))}{g(b)-g(r)}\right\}.$$

In case that $(s,t)\cap D \ne \emptyset$ for all $s \in [a,b)$ and for all $t \in (s,b]$, inequality (1) implies the fairly common *mean value inequality* (cf. [2, p. 23], [5, p. 153], where F is a "function")

$$\frac{\delta(F(b),F(a))}{g(b)-g(a)} \le \sup_{s\in[a,b)} \liminf_{t\downarrow s} \frac{\delta(F(t),F(s))}{g(t)-g(s)},$$

whereas inequality (2) implies the rather uncommon *mean value inequality*

$$\frac{\delta(F(b),F(a))}{g(b)-g(a)} \le \sup_{s\in[a,b)\cap D} \sup_{S\in F(s)} \liminf_{t\downarrow s} \frac{\delta(F(t),S)}{g(t)-g(s)}. \tag{3}$$

In Section 3, there are presented straightforward applications of inequality (3) to the theory of Dini derivatives of multifunctions with values in normed spaces. In Section 2, inequality (2) is proved in case the metric space M is complete and the multifunction F enjoys a convenient closure property.

2 Main result

To begin with, we introduce the convenient closure property which is useful for our purposes. Following [9, p. 208], we say the multifunction F is *closed* if its graph is closed in $R \times M$. This means $(t, T) \in graph(F)$ whenever there exists a sequence $(t_n, T_n) \in graph(F)$ which converges to (t, T).

Definition. The multifunction F is said to be *closed from the left* if $(t, T) \in graph(F)$ whenever there exists a sequence $(t_n, T_n) \in graph(F)$ which converges to (t, T), and moreover, the sequence $t_n \in R$ is strictly increasing.

The multifunction F is said to be *closed from the right* if $(t, T) \in graph(F)$ whenever there exists a sequence $(t_n, T_n) \in graph(F)$ which converges to (t, T), and moreover, the sequence $t_n \in R$ is strictly decreasing.

Obviously, F is closed if and only if F is closed from the left, F is closed from the right, and F has closed values.

Now we are in a position to state and prove our main result.

Theorem 1. *Let the metric space M be complete and let the restriction of F to $(a, b] \cap D$ be closed from the left. Then the* heterogeneous mean value inequality *(2) holds.*

Proof: Denote by μ the right hand side of inequality (2) and assume $\mu < +\infty$. We must show that $\delta(F(b), A) \leq (\mu + \epsilon)(g(b) - g(a))$ whenever $\epsilon > 0$ and $A \in F(a)$. Let $A \in F(a)$ and $\epsilon > 0$. We shall show that there exists $B \in F(b)$ such that $d(B, A) \leq (\mu + \epsilon)(g(b) - g(a))$.

To this purpose, we shall apply an ordering principle, the Corollary in the Appendix. The objects of this ordering principle are a nonempty set X, an ordering $\preceq$ on X, an extended real function $\phi : X \to R \cup \{+\infty\}$ which is increasing with regard to $\preceq$, and an extended real number $C \in R \cup \{+\infty\}$. Here, we shall apply the ordering principle to the set

$$X = graph(F) \cap ([a, b] \times M)$$

(note $(a, A) \in X$), to the relation $(s, S) \preceq (t, T)$ given on X through

$$s \leq t, d(T, S) \leq (\mu + \epsilon)(g(t) - g(s))$$

(note $\preceq$ is an ordering), to the real function ϕ given on X through $\phi(s, S) = s$ (note ϕ is increasing with regard to $\preceq$), and to the real number $b \in D$.

To rephrase our proof purpose, we shall show that there exists $(t, T) \in X$ such that $(a, A) \preceq (t, T)$ and $t = b$. We shall show a bit more, namely for every $(s, S) \in X$ such that $s < b$ there exists $(t, T) \in X$ such that $(s, S) \preceq (t, T)$ and $t = b$. But this is the latter condition in the conclusion of the ordering principle. To prove this latter condition, we shall check the hypothesis of the ordering principle as well as the former condition in its conclusion, namely for every $(s,S) \in X)$ such that $s < b$ there exists $(t,T) \in X$ such that $(s,S) \preceq (t,T)$ and $s < t \leq b$.

In order to check the hypothesis of the ordering principle, let (t_n, T_n) be an increasing sequence in X such that the sequence t_n is strictly increasing. Clearly, t_n converges to some $t \leq b$ and $g(t_n)$ converges to some $\gamma \leq g(b)$. Further, $d(T_j, T_i) \leq (\mu + \epsilon)(g(t_j) - g(t_i))$ whenever $i \leq j$, so T_n is a Cauchy sequence in M. Denote by T its limit. Because $(t_n, T_n) \in X$ and F is closed from the left, it follows $(t,T) \in X$. Therefore $t \in D$ and $\gamma \leq g(t)$. Finally, $d(T, T_n) \leq (\mu + \epsilon)(g(t) - g(t_n))$, that is $(t_n, T_n) \preceq (t,T)$, and the hypothesis of the ordering principle is checked.

In the end of our proof, we check the former condition in the conclusion of the ordering principle. Let $(s,S) \in X$ such that $s < b$. According to the definition of μ,

$$\inf_{t \in (s,b]} \frac{\delta(F(t), S)}{g(t) - g(s)} < \mu + \epsilon,$$

so there exists $t \in (s,b]$ such that $\delta(F(t), F(s))/(g(t) - g(s)) < \mu + \epsilon$. Further, there exists $T \in F(t)$ such that $d(T,S)/(g(t) - g(s)) < \mu + \epsilon$. In other words, $(t,T) \in X$ as well as $(s,S) \preceq (t,T)$, the former condition in the conclusion of the ordering principle is checked, and the proof is accomplished. □

3 Dini derivatives of multifunctions

Throughout this section $(M, \|\cdot\|)$ is a normed space and $[a,b] \subseteq D$. Moreover, $s \in [a,b)$, $S \in F(s)$, and $t \in (s,b]$. The elementary inequality

$$\frac{\delta(F(t), S)}{g(t) - g(s)} \leq \frac{\delta(F(t), S + (g(t) - g(s))m)}{g(t) - g(s)} + \|m\|,$$

which holds for all $m \in M$, suggests for us to consider a set of Dini derivatives of F at (s,S):

$$\mathcal{D}_g^+ F(s,S) = \left\{ m \in M; \liminf_{t \downarrow s} \frac{\delta(F(t), S + (g(t) - g(s))m)}{g(t) - g(s)} = 0 \right\}.$$

This means $m \in \mathcal{D}_g^+ F(s,S)$ if and only if there exist a sequence $t_\nu > s$ which converges to s and a sequence $m_\nu \in M$ which converges to m such that $S + (g(t_\nu) - g(s))m_\nu \in F(t_\nu)$ for all ν. Such a multitude of limits has been considered for the first time by Dini [6, p. 178, ¶ 136] in case of real functions (there, g is the identity function):

> ... Nei punti o negli intervalli nei quali la derivata di una funzioni $f(x)$ non esiste, o almeno si è incerti alla esistenza di essa, non potendo considerare insieme, e talvolta neppure separatamente, i limiti del rapporto $\dfrac{f(x+h) - f(x)}{h}$ per h tendente a zero per valori positivi

> e per valori negativi, sarà naturale di prendere ad esaminare direttamente questo rapporto per ogni valore speciale di x fra a e b, o almeno i limiti fra i quali questo rapporto oscilla coll'impiccolire indefinitamente di h, e ciò considerando separatamente quello corrispondente ai valori positivi di h da quello corrispondente ai valori negativi; ...

In case of some particular functions g, the derivative concept $\mathcal{D}_g^+$ can be characterized through the tangency concept $\mathcal{T}$ which originates from Bouligand [1, p. 32] and Severi [10, p. 99]. Assume g is differentiable from the right at s, let $d^+g(s) = \lim_{t\downarrow s}(g(t)-g(s))/(t-s)$, and assume $d^+g(s) > 0$. Then $m \in \mathcal{D}_g^+(s,S)$ if and only if $(1, d^+g(s)\, m) \in \mathcal{T}_{\mathrm{graph}(F)}(s,S)$. Recall the latter relation means there exist a sequence $\rho_\nu > 0$ converging to 0 and a sequence $(\lambda_\nu, \Lambda_\nu) \in R \times M$ converging to $(1, d^+g(s)\, m)$ such that $(s,S) + \rho_\nu(\lambda_\nu, \Lambda_\nu) \in \mathrm{graph}(F)$ for all ν.

No matter whether g is a particular function, the set $\mathcal{D}_g^+F(s,S)$ is closed. And, if it is nonempty, then $\liminf_{t\downarrow s} \delta(F(t),S)/(g(t)-g(s)) \le \|m\|$ for all $m \in \mathcal{D}_g^+F(s,S)$, therefore

$$\liminf_{t\downarrow s} \frac{\delta(F(t),S)}{g(t)-g(s)} \le \delta\left(\mathcal{D}_g^+F(s,S), 0\right).$$

Theorem 2. *Let M be a Banach space, let $[a,b] \subseteq D$, and let the restriction of F to $(a,b]$ be closed from the left. If $\mathcal{D}_g^+F(s,S)$ is nonempty for all $s \in [a,b)$ and for all $S \in F(s)$, then*

$$\frac{\delta(F(b),F(a))}{g(b)-g(a)} \le \sup_{s\in[a,b)} \sup_{S\in F(s)} \delta\left(\mathcal{D}_g^+F(s,S), 0\right).$$

To close this section, we note that if the set $\mathcal{D}_g^+F(s,S)$ is nonempty then

$$\liminf_{t\downarrow s} \frac{\delta(F(t),S)}{g(t)-g(s)} < +\infty,$$

but the converse may fail if the normed space M is not finite dimensional. For example, if $g : R \to R$ and $F : R \to l^2(R)$ are given by $g(r) = r$ and $F(r) = r\delta_r$ respectively (here $\delta_r(\rho)$ stands for the Kronecker symbol, i.e. $\delta_r(\rho) = 0$ if $r \ne \rho$, whereas $\delta_r(r) = 1$), then $\|F(t) - F(0)\|/(g(t) - g(0)) = 1$ for all $t > 0$, but $\mathcal{D}_g^+F(0,0)$ is empty. On the other hand, if the normed space M is finite dimensional, then the converse does hold, and moreover,

$$\liminf_{t\downarrow s} \frac{\delta(F(t),S)}{g(t)-g(s)} = \delta(\mathcal{D}_g^+(s,S), 0).$$

This statement follows from the next r-statement in case r stands for the left hand side of the preceding equality. Let the normed space X be finite dimensional and let $r > 0$ such that

$$\liminf_{t\downarrow s} \inf_{T\in F(t)} \left| \frac{d(T,S)}{g(t)-g(s)} - r \right| = o.$$

Then there exists $m \in \mathcal{D}_g^+ F(s,S)$ such that $\|m\| = r$. Indeed, if a sequence $t_\nu > s$ converges to s and a sequence $T_\nu \in F(t_\nu)$ renders the sequence $d(T_\nu, S)/(g(t_\nu) - g(s))$ convergent to r, then the sequence $m_\nu = (1/(g(t_\nu) - g(s))(T_\nu - S)$ is bounded. Since X is finite dimensional, we can suppose, taking a subsequence if necessary, that m_ν is convergent. Let $m \in M$ be its limit. Obviously, $\|m\| = r$. Moreover, $\delta(F(t_\nu), S + (g(t_\nu) - g(s))m) \leq d(T_\nu, S + (g(t_\nu) - g(s))m) = (g(t_\nu) - g(s))\|m_\nu - m\|$, therefore $m \in \mathcal{D}_g^+(s,S)$, and the r-statement is justified.

4 Appendix

Let X be a nonempty set, let $\preceq$ be an ordering on X, and let $\phi : X \to R \cup \{+\infty\}$ be an extended real function which is increasing with regard to $\preceq$. In the first part of this section we adjust the proof of Theorem 1 in [3, p. 355] to obtain a version of Corollary 1 in [3, p. 356].

Proposition. *Assume that, for each increasing sequence x_n in X such that the sequence $\phi(x_n)$ is strictly increasing, there exists $x \in X$ such that $x_n \preceq x$ for all n. Then, for each $x \in X$, there exists $y \in X$ such that $x \preceq y$ as well as $\phi(y) = \phi(z)$ whenever both $z \in X$ and $y \preceq z$.*

Proof: We can suppose, replacing the function $x \to \phi(x)$ with the function $x \to \arctan(\phi(x))$ if necessary, that ϕ is bounded above. For each $x \in X$, let $S(x) = \{y \in X; x \preceq y\}$ and observe that $\phi(x) \in \phi(S(x)) \subseteq [\phi(x), +\infty)$. Further, let $\rho(x) = \sup \phi(S(x))$ and observe that $\phi(x) \leq \rho(x)$. Moreover, if $y \in S(x)$, then $\phi(x) \leq \phi(y) \leq \rho(y) \leq \rho(x)$. We have to show that, for each $x \in X$, there exists $y \in S(x)$ such that $\phi(y) = \rho(y)$. Let $x \in X$ and define, by induction, a sequence y_n in X such that: $y_1 = x$; $y_{n+1} \in S(y_n)$ and $(\phi(y_n) + \rho(y_n))/2 \leq \phi(y_{n+1})$. Note $\phi(y_n) \leq \phi(y_{n+1}) \leq \rho(y_{n+1}) \leq \rho(y_n)$ and $\rho(y_{n+1}) - \phi(y_{n+1}) \leq (\rho(y_n) - \phi(y_n))/2$. Therefore the sequence $\phi(y_n)$ is increasing, the sequence $\rho(y_n)$ is decreasing, and both sequences converge to the same limit. If $\phi(y_n) = \phi(y_{n+1})$ for some n, then $\phi(y_n) = \rho(y_n)$, so we can take $y = y_n$. If $\phi(y_n) < \phi(y_{n+1})$ for all n, then, by hypothesis, there exists $y \in X$ such that $y_n \preceq y$ for all n. Further, $\phi(y_n) < \phi(y) \leq \rho(y) \leq \rho(y_n)$, so $\phi(y) = \rho(y)$, and the proof is accomplished. □

In the second part of the section we state and prove a variant of Corollary 2 in [3, p. 356]. Let $C \in R \cup \{+\infty\}$ be an extended real number.

Corollary. *Assume that, for each increasing sequence x_n in X such that the sequence $\phi(x_n)$ is strictly increasing and $\phi(y_n) \leq C$ for all n, there exists $x \in X$ such that $x_n \preceq x$ for all n and $\phi(x) \leq C$. Then the following two conditions are equivalent:*

> *for each $x \in X$ such that $\phi(x) < C$, there exists $y \in X$ such that $x \leq y$ and $\phi(x) < \phi(y) \leq C$;*

for each $x \in X$ such that $\phi(x) < C$, there exists $y \in X$ such that $x \leq y$ and $\phi(y) = C$.

Proof: Clearly, the latter condition in the conclusion implies the former one. Now, let the former condition in the conclusion be satisfied and consider the set $X_0 = \{x \in X; \phi(x) \leq C\}$. If $x \in X$ and $\phi(x) < C$, then $x \in X_0$. It follows from Proposition above, applied in X_0, that there exists $y \in X_0$ such that both $x \preceq y$ as well as $\phi(y) = \phi(z)$ whenever $z \in X_0$ and $y \preceq z$. We must have $\phi(y) = C$. Otherwise, $\phi(y) < C$ would imply, through the former condition, the existence of a $z \in X_0$ such that $y \preceq z$ and $\phi(y) < \phi(z) \leq C$, a contradiction. □

In contrast with the ordering hypotheses in [3], which concerns increasing sequences x_n in X, the ordering hypotheses above concern only increasing sequences x_n in X such that the corresponding sequences $\phi(x_n)$ are strictly increasing. Therefore the hypotheses above do hold if, for example, $\phi(X)$ is a finite subset of $R \cup \{+\infty\}$.

The fact that ϕ may have extended real values is useful while discussing existence of saturated solutions to differential systems (cf. [4, p. 382]).

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