Bull. Math. Soc. Sci. Math. Roumanie
Tome 54(102) No. 2, 2011, 167–183

Step method for some integral equations from biomathematics

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Abstract

In this paper we study an integral equation from biomathematics using the step method and we prove that the sequence of successive approximation generated by the step method converges to the solution of this integral equation.

Key Words: Integral equation, step method, Picard operators, fibre contraction principle.
2010 Mathematics Subject Classification: Primary 47H10; Secondary 45G10, 45N05, 47N20.

1 Introduction

Let consider the following integral equation:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds, \quad t \in [a; b], \quad \tau > 0, \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [a - \tau; a]. \tag{2}$$

This equation appears in biomathematics and was studied by K.L. Cooke and J.L. Kaplan [3], D. Guo and V. Lakshmikantham [5], R. Precup [6], R. Precup and E. Kirr [7], I.A. Rus [8].

Relative to (1)+(2) we consider the following conditions:

(C_1) $(\mathbb{B}, \|\cdot\|)$ is a Banach space and $f \in C([a-\tau; b] \times \mathbb{B}, \mathbb{B})$, $\varphi \in C([a-\tau; a], \mathbb{B})$;

(C_2) there exists $L_f > 0$ such that:

$$\|f(t, u) - f(t, v)\| \leq L_f \cdot \|u - v\|, \quad \forall t \in [a - \tau; b], \ \forall u, v \in \mathbb{B}$$

(C_2') there exists $L_f > 0$ such that:

$$\|f(t,u) - f(t,v)\| \leq L_f \cdot \|u - v\|, \quad \forall t \in [a;b],\ \forall u,v \in \mathbb{B}$$

(C_3) $\varphi(a) = \int\limits_{a-\tau}^{a} f(s,\varphi(s))\,ds.$

The following result is well known (D. Guo and V. Lakshmikantham [5], R. Precup and E. Kirr [7], I.A. Rus [8], M. Dobriţoiu, I.A. Rus and M.A. Şerban [4]):

Theorem 1.1. *In the conditions* $(C_1)+(C_2)+(C_3)$ *the problem (1)+(2) has in* $C([a-\tau;b],\mathbb{B})$ *a unique solution* x^* *and the sequence of successive approximations,* $(x^n)_{n\in\mathbb{N}}$

$$x^{n+1}(t) = \begin{cases} \varphi(t), & t \in [a-\tau;a] \\ \int\limits_{t-\tau}^{t} f(s,x^n(s))\,ds, & t \in [a;b] \end{cases}$$

converges uniformly to x^* *for every* $x^0 \in C([a-\tau;b],\mathbb{B})$ *with* $x^0\big|_{[a-\tau;a]} = \varphi$.

On the other hand let $m \in \mathbb{N}^*$ be such that $a+(m-1)\tau \leq b$ and $a+m\tau > b$. In the conditions $(C_1)+(C_2')+(C_3)$ the step method for (1)+(2) consists in the following:

(e_0) $x_0(t) = \varphi(t)$, $t \in [a-\tau;a]$;

(e_1) $x_1(t) = \int\limits_{t-\tau}^{a} f(s,\varphi(s))\,ds + \int\limits_{a}^{t} f(s,x_1(s))\,ds$, $t \in [a;a+\tau]$;

(e_2) $x_2(t) = \int\limits_{t-\tau}^{a+\tau} f(s,x_1^*(s))\,ds + \int\limits_{a+\tau}^{t} f(s,x_2(s))\,ds$, $t \in [a+\tau,a+2\tau]$;

....

(e_m) $x_m(t) = \int\limits_{t-\tau}^{a+(m-1)\tau} f\left(s,x_{m-1}^*(s)\right)ds + \int\limits_{a+(m-1)\tau}^{t} f(s,x_m(s))\,ds$,
$t \in [a+(m-1)\tau,b]$;

where x_i^* is the unique solution of the step (e_i), $i = \overline{1,m}$.

So, we have the following result:

Theorem 1.2. *In the conditions* $(C_1)+(C_2')+(C_3)$ *we have:*

(a) the problem (1)+(2) has in $C([a-\tau;b],\mathbb{B})$ *a unique solution* x^*,

$$x^*(t) = \begin{cases} \varphi(t), & t \in [a-\tau;a] \\ x_1^*(t), & t \in [a;a+\tau] \\ \ldots\ldots & \\ x_m^*(t), & t \in [a+(m-1)\tau;b] \end{cases}$$

(b) for each $x_i^0 \in C([a+(i-1)\tau; a+i\tau], \mathbb{B})$, $i = \overline{1, m-1}$,
$x_m^0 \in C([a+(m-1)\tau; b], \mathbb{B})$, *the sequences defined by:*

$$
\begin{aligned}
x_1^{n+1}(t) &:= \int_{t-\tau}^{a} f(s, \varphi(s))\, ds + \int_{a}^{t} f(s, x_1^n(s))\, ds, \quad t \in [a; a+\tau],\\
x_2^{n+1}(t) &:= \int_{t-\tau}^{a+\tau} f(s, x_1^*(s))\, ds + \int_{a+\tau}^{t} f(s, x_2^n(s))\, ds, \quad t \in [a+\tau; a+2\tau]\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
x_m^{n+1}(t) &:= \int_{t-\tau}^{a+(m-1)\tau} f\left(s, x_{m-1}^*(s)\right) ds + \int_{a+(m-1)\tau}^{t} f(s, x_m^n(s))\, ds,\\
&\quad t \in [a+(m-1)\tau; b]
\end{aligned}
$$

converge and $\lim\limits_{n \to +\infty} x_k^n = x_k^*$, $k = \overline{1, m}$.

In this paper we shall study the following problem (see I.A. Rus [11] for an abstract setting)

Problem 1.1. *Can we put* x_{i-1}^n *instead of* x_{i-1}^*, $i = \overline{2, m}$ *in the conclusion* (*b*) *of the Theorem 1.2?*

In order to do this we need some notions and results from weakly Picard operator theory.

2 Fibre weakly Picard operator

Let (X, d) be a metric space and $A : X \to X$ an operator. In this paper we shall use the terminologies and notations from [9] and [10]. For the convenience of the reader we shall recall some of them.

We denote by $A^0 := 1_X$, $A^1 := A$, $A^{n+1} := A \circ A^n$, $n \in \mathbb{N}$ the iterate operators of the operator A. Also:

$$P(X) := \{Y \subseteq X \mid Y \neq \emptyset\}$$

$$F_A := \{x \in X \mid A(x) = x\}$$

$$I(A) := \{Y \in P(X) \mid A(Y) \subseteq Y\}$$

Definition 2.1. *$A : X \to X$ is called a Picard operator (briefly PO) if:*
(i) $F_A = \{x^*\}$;
(ii) $A^n(x) \to x^*$ *as* $n \to \infty$, *for all* $x \in X$.

The operator A is Picard if and only if the discrete dynamical system generated by A has an equilibrium state which is globally asymptotically stable.

Definition 2.2. *$A : X \to X$ is said to be a weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n \in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.*

If $A : X \to X$ is a WPO, then we may define the operator $A^\infty : X \to X$ by

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Obviously $A^\infty(X) = F_A$. Moreover, if A is a PO and we denote by x^* its unique fixed point, then $A^\infty(x) = x^*$, for each $x \in X$.

Theorem 2.1. *(Fibre contraction principle I.A. Rus [10]) Let (X, d) be a metric space and (Y, ρ) be a complete metric space. Let $B : X \to X$ and $C : X \times Y \to Y$ be two operators. We suppose that:*

(i) B is a WPO;

(ii) $C(x, \cdot) : Y \to Y$ is α-contraction, for all $x \in X$;

(iii) if $(x^*, y^*) \in F_A$, where $A : X \times Y \to X \times Y$, $A(x, y) = (B(x), C(x, y))$, then $C(\cdot, y^*)$ is continuous in x^*.

Then A is a WPO. Moreover, if B is PO then A is PO.

For other generalizations of *Fibre Contraction Principle* see S. András [1], C. Bacoţiu [2], M.A. Şerban [13], [14].

By induction, from the above result we have:

Theorem 2.2. *(I.A. Rus [9]) Let (X_i, d_i), $i = \overline{0, m}$, $m \geq 1$, be some metric spaces. Let*

$$A_i : X_0 \times ... \times X_i \to X_i, \quad i = \overline{0, m}$$

be some operator. We suppose that:

(i) (X_i, d_i), $i = \overline{1, m}$, are complete metric spaces;

(ii) the operator A_0 is WPO;

(iii) there exists $\alpha_i \in (0; 1)$ such that:

$$A_i(x_0, ..., x_{i-1}, \cdot) : X_i \to X_i, \quad i = \overline{1, m}$$

are α_i-contractions;

(iv) the operator A_i, $i = \overline{1, m}$, are continuous.

Then the operator $A : X_0 \times ... \times X_m \to X_0 \times ... \times X_m$,

$$A(x_0, ..., x_m) = (A_0(x_0), A_1(x_0, x_1), ..., A_m(x_0, ..., x_m))$$

is WPO. If A_0 is PO, then A is PO.

3 The main result

Theorem 3.1. *In the condition of Theorem 1.2, for each* $x_i^0 \in C([a+(i-1)\tau; a+i\tau], \mathbb{B})$*,* $i=\overline{1,m-1}$*,* $x_m^0 \in C([a+(m-1)\tau; b], \mathbb{B})$*, the sequences defined by:*

$$
\begin{aligned}
x_1^{n+1}(t) &:= \int_{t-\tau}^{a} f(s,\varphi(s))\,ds + \int_{a}^{t} f(s, x_1^n(s))\,ds, \quad t\in[a;a+\tau],\\
x_2^{n+1}(t) &:= \int_{t-\tau}^{a+\tau} f(s,x_1^n(s))\,ds + \int_{a+\tau}^{t} f(s, x_2^n(s))\,ds, \quad t\in[a+\tau;a+2\tau]\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_m^{n+1}(t) &:= \int_{t-\tau}^{a+(m-1)\tau} f\left(s,x_{m-1}^n(s)\right)ds + \int_{a+(m-1)\tau}^{t} f(s, x_m^n(s))\,ds,\\
&t\in[a+(m-1)\tau;b]
\end{aligned}
\tag{3}
$$

converge and $\lim\limits_{n\to+\infty} x_k^n = x_k^*$*,* $k=\overline{1,m}$*.*

Proof. We consider the following Banach spaces:

$$
X_0 = \left(C([a-\tau;a],\mathbb{B}), \|\cdot\|_0\right),
$$
$$
\|x\|_0 = \max_{t\in[a-\tau;a]}\left\{\|x(t)\|\cdot e^{-\lambda(t-a+\tau)}\right\},\ \lambda>0,
$$

$$
X_i = \left(C([a+(i-1)\tau;a+i\tau],\mathbb{B}), \|\cdot\|_i\right), \quad i=\overline{1,m-1}
$$
$$
\|x\|_i = \max_{t\in[a+(i-1)\tau;a+i\tau]}\left\{\|x(t)\|\cdot e^{-\lambda(t-a-(i-1)\tau)}\right\},\ \lambda>0,
$$

$$
X_m = \left(C([a+(m-1)\tau;b],\mathbb{B}), \|\cdot\|_m\right),
$$
$$
\|x\|_m = \max_{t\in[a+(m-1)\tau;b]}\left\{\|x(t)\|\cdot e^{-\lambda(t-a-(m-1)\tau)}\right\},\ \lambda>0,
$$

where λ will be specified later, and the operators:

$$
A_0 : X_0 \to X_0,
$$
$$
A_0(x_0)(t) = \varphi(t),
$$

for $t\in[a-\tau;a]$,

$$
A_i : X_{i-1}\times X_i \to X_i, \quad i=\overline{1,m-1},
$$
$$
A_i(x_{i-1},x_i)(t) = \int_{t-\tau}^{a+(i-1)\tau} f(s,x_{i-1}(s))\,ds + \int_{a+(i-1)\tau}^{t} f(s,x_i(s))\,ds,
$$

for $t\in[a+(i-1)\tau;a+i\tau]$,

$$
A_m : X_{m-1}\times X_m \to X_m
$$
$$
A_m(x_{m-1},x_m)(t) = \int_{t-\tau}^{a+(m-1)\tau} f(s,x_{m-1}(s))\,ds + \int_{a+(m-1)\tau}^{t} f(s,x_m(s))\,ds,
$$

for $t \in [a+(m-1)\tau; b]$, and

$$A : X_0 \times ... \times X_m \to X_0 \times ... \times X_m,$$
$$A(x_0,...,x_m) = (A_0(x_0), A_1(x_0,x_1), ..., A_m(x_{m-1},x_m)).$$

It is easy to see that for fixed $\left(x_0^0,...,x_m^0\right) \in X_0 \times ... \times X_m$ the sequence defined by (3) means

$$(x_0^n,...,x_m^n) = A^n\left(x_0^0,...,x_m^0\right).$$

To prove the conclusion we need to prove that the operator A is PO and for this we apply the Theorem 2.2.

Since $A_0 : X_0 \to X_0$ is a constant operator then A_0 is α_0 contraction with $\alpha_0 = 0$, so A_0 is PO and $F_{A_0} = \{x_0^*\}$, where $x_0^* = \varphi$. For $i = \overline{1,m}$ we have:

$$\|A_i(x_{i-1},x_i) - A_i(x_{i-1},y_i)\|_i \leq \frac{L_f}{\lambda} \cdot \|x_i - y_i\|_i$$

for all $x_{i-1} \in X_{i-1}$ and $x_i, y_i \in X_i$. Choosing $\lambda = L_f + 1$ we get that $A_i(x_{i-1},\cdot) : X_i \to X_i$ are α_i-contractions with $\alpha_i = \frac{L_f}{L_f+1}$, so we are in the conditions of the Theorem 2.2, therefore A is PO and $F_A = \{(x_0^*, x_1^* ..., x_m^*)\}$, thus

$$(x_0^n,...,x_m^n) = A^n\left(x_0^0,...,x_m^0\right) \to (x_0^*,...,x_m^*)$$

$x_0^n = \varphi$, for all $n \in \mathbb{N}$, and $x_1^n,...,x_m^n$ are defined by (3). From condition (C_3) and from the definitions of A_i, $i = \overline{1,m}$, we have

$$x_{i-1}^*(a+(i-1)\tau) = x_i^*(a+(i-1)\tau), \quad i = \overline{1,m},$$

therefore

$$x^*(t) = \begin{cases} \varphi(t), & t \in [a-\tau;a] \\ x_1^*(t), & t \in [a;a+\tau] \\ ..... & \\ x_m^*(t), & t \in [a+(m-1)\tau;b] \end{cases}$$

is the unique solution in $C([a-\tau;b],\mathbb{B})$. □

4 Generalization

In this section we consider a general case of integral equation

$$x(t) = \int_{t-\tau}^{t} K(t,s,x(s))\,ds + g(t,x(t-\tau)), \quad t \in [a;b], \quad \tau > 0, \tag{4}$$

$$x(t) = \varphi(t), \quad t \in [a-\tau;a]. \tag{5}$$

Let $x \in C([a-\tau;b],\mathbb{B})$. Relative to (4)+(5) we consider the following conditions:

(H_1) $(\mathbb{B}, \|\cdot\|)$ is a Banach space and $K \in C([a;b] \times [a-\tau;b] \times \mathbb{B}, \mathbb{B})$, $g \in C([a;b] \times \mathbb{B}, \mathbb{B})$, $\varphi \in C([a-\tau;a], \mathbb{B})$;

(H_2) there exists $L_K > 0$ such that:

$$\|K(t,s,u) - K(t,s,v)\| \leq L_K \cdot \|u-v\|, \ \forall t \in [a;b], \forall s \in [a-\tau;b], \ \forall u,v \in \mathbb{B}.$$

(H_2') there exists $L_K > 0$ such that:

$$\|K(t,s,u) - K(t,s,v)\| \leq L_K \cdot \|u-v\|, \quad \forall t,s \in [a;b], \ \forall u,v \in \mathbb{B}.$$

(H_3) there exists $L_g > 0$ such that:

$$\|g(t,u) - g(t,v)\| \leq L_g \cdot \|u-v\|, \quad \forall t \in [a;b], \ \forall u,v \in \mathbb{B}.$$

(H_4) $L_g < 1$;

(H_5) $\varphi(a) = \int\limits_{a-\tau}^{a} K(a,s,\varphi(s))\,ds + g(a,\varphi(a-\tau))$.

We consider the space $X = C([a-\tau;b], \mathbb{B})$ endowed with the norms $\|\cdot\|_C$ and $\|\cdot\|_B$ where

$$\|x\|_C = \max_{t\in[a-\tau;b]} \{\|x(t)\|\},$$

$$\|x\|_B = \max_{t\in[a-\tau;b]} \left\{\|x(t)\| \cdot e^{-\lambda(t-a+\tau)}\right\}, \ \lambda > 0,$$

where λ will be specified later.

We have:

$$\|x\|_C \leq \|x\|_B \cdot e^{\lambda(t-a+\tau)}, \quad \forall t \in [a-\tau;b]$$

Theorem 4.1. *In the conditions* $(H_1)-(H_5)$ *the problem (4)+(5) has in* $C([a-\tau;b], \mathbb{B})$ *a unique solution* x^* *and the sequence of successive approximations,* $(x^n)_{n\in\mathbb{N}}$

$$x^{n+1}(t) = \begin{cases} \varphi(t), & t \in [a-\tau;a] \\ \int\limits_{t-\tau}^{t} K(t,s,x^n(s))\,ds + g(t,x^n(t-\tau)), & t \in [a;b] \end{cases}$$

converges uniformly to x^* *for every* $x^0 \in C([a-\tau;b], \mathbb{B})$ *with* $x^0\big|_{[a-\tau;a]} = \varphi$.

Proof. $X_\varphi \subset X$

$$X_\varphi = \{x \in X \mid x(t) = \varphi(t), \ t \in [a-\tau;a]\}$$

and $A : X_\varphi \to X_\varphi$ defined by

$$A(x)(t) = \begin{cases} \varphi(t), & t \in [a-\tau;a], \\ \int\limits_{t-\tau}^{t} K(t,s,x(s))\,ds + g(t,x(t-\tau)), & t \in [a;b]. \end{cases}$$

X_φ is a closed subset of X, so $\left(X_\varphi, d_{\|\cdot\|_B}\right)$ is a complete metric space.

Let $x, y \in X_\varphi$, for $t \in [a-\tau; a]$ we have:

$$\|A(x)(t) - A(y)(t)\| = 0,$$

for $t \in [a; b]$, we have:

$$\|A(x)(t) - A(y)(t)\| \leq$$
$$\leq \int_{t-\tau}^{t} \|K(t,s,x(s)) - K(t,s,y(s))\| \, ds + \|g(t, x(t-\tau)) - g(t, y(t-\tau))\| \leq$$
$$\leq L_K \int_{t-\tau}^{t} \|x(s) - y(s)\| \cdot e^{-\lambda(s-a+\tau)} \cdot e^{\lambda(s-a+\tau)} ds + L_g \cdot \|x - y\|_C \leq$$
$$\leq \left(\tfrac{L_K}{\lambda} + L_g\right) \cdot \|x - y\|_B \cdot e^{\lambda(t-a+\tau)},$$

therefore

$$\|A(x) - A(y)\|_B \leq \left(\frac{L_K}{\lambda} + L_g\right) \cdot \|x - y\|_B$$

which proves that A is lipschitz with $L_A = \left(\frac{L_K}{\lambda} + L_g\right)$, from (H_4) we can choose λ sufficiently large such that $L_A = \left(\frac{L_K}{\lambda} + L_g\right) < 1$, so A is a contraction. Applying the Banach theorem we get the conclusion. □

Using the step method we can give a weaker result regarding the existence and uniqueness of the solution for the problem (4)+(5) by replacing the hypothesis (H_2) with (H_2') and droping the conditions (H_3) and (H_4). Let $m \in \mathbb{N}^*$ be such that $a + (m-1)\tau \leq b$ and $a + m\tau > b$. In the conditions $(H_1) + (H_2') + (H_5)$ the step method for (4)+(5) consists in the following:

(e$_0$) $x_0(t) = \varphi(t)$, $t \in [a-\tau; a]$;

(e$_1$) $x_1(t) = \int_{t-\tau}^{a} K(t,s,\varphi(s))\, ds + \int_{a}^{t} K(t,s,x_1(s))\, ds + g(t, \varphi(t-\tau))$, $t \in [a; a+\tau]$;

(e$_2$) $x_2(t) = \int_{t-\tau}^{a+\tau} K(t,s,x_1^*(s))\, ds + \int_{a+\tau}^{t} K(t,s,x_2(s))\, ds + g(t, x_1^*(t-\tau))$, $t \in [a+\tau, a+2\tau]$;

....

(e$_m$) $x_m(t) = \int_{t-\tau}^{a+(m-1)\tau} K\left(t,s,x_{m-1}^*(s)\right) ds + \int_{a+(m-1)\tau}^{t} K(t,s,x_m(s))\, ds + g\left(t, x_{m-1}^*(t-\tau)\right)$, $t \in [a+(m-1)\tau, b]$;

where x_i^* is the unique solution of the step (e_i), $i=\overline{1,m}$.

So, we have the following result:

Theorem 4.2. *In the conditions* $(H_1)+(H_2')+(H_5)$ *we have:*

(a) the problem (4)+(5) has in $C([a-\tau;b],\mathbb{B})$ *a unique solution* x^*,

$$x^*(t)=\begin{cases}\varphi(t), & t\in[a-\tau;a]\\ x_1^*(t), & t\in[a;a+\tau]\\ \cdots\cdots & \\ x_m^*(t), & t\in[a+(m-1)\tau;b]\end{cases}$$

(b) for each $x_i^0\in C([a+(i-1)\tau;a+i\tau],\mathbb{B})$, $i=\overline{1,m-1}$, $x_m^0\in C([a+(m-1)\tau;b],\mathbb{B})$, *the sequences defined by:*

$$x_1^{n+1}(t):=\int_{t-\tau}^{a}K(t,s,\varphi(s))\,ds+\int_a^t K(t,s,x_1^n(s))\,ds+g(t,\varphi(t-\tau)),$$

for $t\in[a;a+\tau]$,

$$x_2^{n+1}(t):=\int_{t-\tau}^{a+\tau}K(t,s,x_1^*(s))\,ds+\int_{a+\tau}^{t}K(t,s,x_2^n(s))\,ds+g(t,x_1^*(t-\tau)),$$

for $t\in[a+\tau;a+2\tau]$,

....

$$\begin{aligned}x_m^{n+1}(t)\ :\ =\ &\int_{t-\tau}^{a+(m-1)\tau}K\big(t,s,x_{m-1}^*(s)\big)\,ds+\int_{a+(m-1)\tau}^{t}K(t,s,x_m^n(s))\,ds+\\ &+g\big(t,x_{m-1}^*(t-\tau)\big),\end{aligned}$$

for $t\in[a+(m-1)\tau;b]$, *converge and* $\lim\limits_{n\to+\infty}x_k^n=x_k^*$, $k=\overline{1,m}$.

Proof. For the first step we consider the Banach space $X_1=\left(C([a;a+\tau],\mathbb{B}),\|\cdot\|_1\right)$, where

$$\|x\|_1=\max_{t\in[a;a+\tau]}\left\{\|x(t)\|\cdot e^{-\lambda(t-a)}\right\},\ \lambda>0,$$

and $A_1:X_1\to X_1$ defined by

$$A_1(x)(t)=\int_{t-\tau}^{a}K(t,s,\varphi(s))\,ds+\int_a^t K(t,s,x(s))\,ds+g(t,\varphi(t-\tau)).$$

For $x, y \in X_1$ we have

$$\left\|A_1(x) - A_1(y)\right\|_1 \leq \frac{L_K}{\lambda} \cdot \|x - y\|_1$$

We can choose a $\lambda > 0$ such that $\frac{L_K}{\lambda} < 1$, (for example $\lambda = L_K + 1$), so A_1 is a contraction, therefore $F_{A_1} = \{x_1^*\}$.

For the next steps we consider the Banach spaces $X_i = (C([a + (i-1)\tau; a + i\tau], \mathbb{B}), \|\cdot\|_i)$, $i = \overline{2, m-1}$, where

$$\|x\|_i = \max_{t \in [a+(i-1)\tau; a+i\tau]} \left\{ \|x(t)\| \cdot e^{-\lambda(t-a-(i-1)\tau)} \right\}, \ \lambda > 0,$$

$X_m = (C([a + (m-1)\tau; b], \mathbb{B}), \|\cdot\|_m)$, where

$$\|x\|_m = \max_{t \in [a+(m-1)\tau; b]} \left\{ \|x(t)\| \cdot e^{-\lambda(t-a-(m-1)\tau)} \right\}, \ \lambda > 0,$$

with λ chosen such that $\frac{L_K}{\lambda} < 1$, (λ can be the same as for X_1) and the operators $A_i : X_i \to X_i$, $i = \overline{2, m}$ defined by

$$A_i(x)(t) = \int_{t-\tau}^{a+(i-1)\tau} K\left(t, s, x_{i-1}^*(s)\right) ds + \int_{a+(i-1)\tau}^{t} K(t, s, x(s)) ds + g\left(t, x_{i-1}^*(t-\tau)\right).$$

Also for $x, y \in X_i$ we have

$$\left\|A_i(x) - A_i(y)\right\|_i \leq \frac{L_K}{\lambda} \cdot \|x - y\|_i$$

so A_i is a contraction, therefore $F_{A_i} = \{x_i^*\}$, $i = \overline{2, m}$.

From condition (H_5) we get $\varphi(a) = x_1^*(a)$ and from the definitions of A_i, $i = \overline{1, m}$, we have

$$x_{i-1}^*(a + (i-1)\tau) = x_i^*(a + (i-1)\tau), \quad i = \overline{1, m},$$

therefore

$$x^*(t) = \begin{cases} \varphi(t), & t \in [a-\tau; a] \\ x_1^*(t), & t \in [a; a+\tau] \\ \ldots\ldots & \\ x_m^*(t), & t \in [a + (m-1)\tau; b] \end{cases}$$

is the unique solution in $C([a-\tau; b], \mathbb{B})$. ∎

Further, we study the Problem 1.1 in the case of Theorem 4.2, which means if the conclusion (b) from Theorem 4.2 remains true if we put x_i^n instead of x_i^*, $i = \overline{2, m}$. Thus, we have:

Theorem 4.3. *In the condition of Theorem 4.2, for each* $x_i^0 \in C([a+(i-1)\tau; a+i\tau], \mathbb{B})$, $i = \overline{1, m-1}$, $x_m^0 \in C([a+(m-1)\tau; b], \mathbb{B})$, *the sequences defined by:*

$$
\begin{array}{rl}
x_1^{n+1}(t) & = \int\limits_{t-\tau}^{a} K(t,s,\varphi(s))\,ds + \int\limits_{a}^{t} K(t,s,x_1^n(s))\,ds + g(t,\varphi(t-\tau)), \\
& for\ t \in [a; a+\tau], \\
x_2^{n+1}(t) & := \int\limits_{t-\tau}^{a+\tau} K(t,s,x_1^n(s))\,ds + \int\limits_{a+\tau}^{t} K(t,s,x_2^n(s))\,ds + g(t,x_1^n(t-\tau)), \\
& for\ t \in [a+\tau; a+2\tau], \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
x_m^{n+1}(t) & = \int\limits_{t-\tau}^{a+(m-1)\tau} K\left(t,s,x_{m-1}^n(s)\right)ds + \int\limits_{a+(m-1)\tau}^{t} K(t,s,x_m^n(s))\,ds + \\
& +g\left(t,x_{m-1}^n(t-\tau)\right), \quad for\ t \in [a+(m-1)\tau; b]
\end{array}
\tag{6}
$$

converge and $\lim\limits_{n\to+\infty} x_k^n = x_k^*$, $k = \overline{1, m}$.

Proof. We consider the following Banach spaces $X_0 = (C([a-\tau; a], \mathbb{B}), \|\cdot\|_0)$, where

$$
\|x\|_0 = \max_{t\in[a-\tau;a]} \left\{ \|x(t)\| \cdot e^{-\lambda(t-a+\tau)} \right\}
$$

and $X_i = (C([a+(i-1)\tau; a+i\tau], \mathbb{B}), \|\cdot\|_i)$, $i = \overline{2, m-1}$, $X_m = (C([a+(m-1)\tau; b], \mathbb{B}), \|\cdot\|_m)$ as in the proof of Theorem 4.2 and the operators:

$$
\begin{gathered}
A_0 : X_0 \to X_0, \\
A_0(x_0)(t) = \varphi(t), \quad t \in [a-\tau; a],
\end{gathered}
$$

$$
A_i : X_{i-1} \times X_i \to X_i, \quad i = \overline{1, m-1},
$$

$$
\begin{aligned}
A_i(x_{i-1}, x_i)(t) \quad = & \int\limits_{t-\tau}^{a+(i-1)\tau} K(t,s,x_{i-1}(s))\,ds + \\
& + \int\limits_{a+(i-1)\tau}^{t} K(t,s,x(s))\,ds + g(t,x_{i-1}(t-\tau)),
\end{aligned}
$$

for $t \in [a+(i-1)\tau; a+i\tau]$,

$$
A_m : X_{m-1} \times X_m \to X_m
$$

$$\begin{aligned} A_m(x_{m-1}, x_m)(t) &= \int_{t-\tau}^{a+(m-1)\tau} K(t, s, x_{m-1}(s))\, ds + \\ &+ \int_{a+(m-1)\tau}^{t} K(t, s, x_m(s))\, ds + g(t, x_{m-1}(t-\tau)), \end{aligned}$$

for $t \in [a + (m-1)\tau; b]$ and

$$\begin{gathered} A : X_0 \times ... \times X_m \to X_0 \times ... \times X_m, \\ A(x_0, ..., x_m) = (A_0(x_0), A_1(x_0, x_1), ..., A_m(x_{m-1}, x_m)). \end{gathered}$$

For fixed $\left(x_0^0, ..., x_m^0\right) \in X_0 \times ... \times X_m$ we have that the sequence defined by (6) is

$$(x_0^n, ..., x_m^n) = A^n\left(x_0^0, ..., x_m^0\right).$$

Since $A_0 : X_0 \to X_0$ is a constant operator then A_0 is an α_0 contraction with $\alpha_0 = 0$, so A_0 is PO and $x_0^* = \varphi$. For $i = \overline{1, m}$ we have:

$$\|A_i(x_{i-1}, x_i) - A_i(x_{i-1}, y_i)\|_i \leq \frac{L_K}{\lambda} \cdot \|x_i - y_i\|_i$$

for all $x_{i-1} \in X_{i-1}$ and $x_i, y_i \in X_i$. Choosing $\lambda = L_K + 1$ we get that $A_i(x_{i-1}, \cdot) : X_i \to X_i$ are α_i-contractions with $\alpha_i = \frac{L_K}{L_K+1}$, so we are in the conditions of the Theorem 2.2, therefore A is PO and $F_A = \{(x_0^*, ..., x_m^*)\}$, thus

$$(x_0^n, ..., x_m^n) = A^n\left(x_0^0, ..., x_m^0\right) \to (x_0^*, ..., x_m^*),$$

$x_0^n = \varphi$, for all $n \in \mathbb{N}$, and $x_1^n, ..., x_m^n$ are defined by (6). From condition (H_5) and from the definitions of A_i, $i = \overline{1, m}$, we have

$$x_{i-1}^*(a + (i-1)\tau) = x_i^*(a + (i-1)\tau), \quad i = \overline{1, m},$$

therefore

$$x^*(t) = \begin{cases} \varphi(t), & t \in [a-\tau; a] \\ x_1^*(t), & t \in [a; a+\tau] \\ ..... & \\ x_m^*(t), & t \in [a + (m-1)\tau; b] \end{cases}$$

is the unique solution in $C([a-\tau; b], \mathbb{B})$. □

5 Numerical example

In this section we give a numerical example to illustrate the convergence of the sequence (3) to the solution. We consider the following integral equation:

$$\begin{aligned} &x(t) = \int_{t-\tau}^{t} \cos(x(s))\, ds, \quad t \in [1; 10],\ \tau = 1, \\ &x(t) = \lambda t, \quad t \in [0; 1], \end{aligned}$$

where λ is chosen such that condition (C_3) is satisfied

$$\lambda = \int_0^1 \cos(\lambda s)\, ds = \frac{\sin\lambda}{\lambda},$$

therefore $\lambda = 0.876\,726\,215\,395\,06$. Obviously conditions (C_1) and (C_2') are satisfied for $\mathbb{B} = \mathbb{R}$, so all conditions of Theorem 3.1 are fulfilled.

The example analysis. If $\lim\limits_{t\to+\infty} x(t) = c$ then c verifies the equation

$$c = \cos c,$$

so $c = 0.739\,085\,133\,215\,16$. This value will be used for the estimation of numerical solution accuracy $x(t)$ in $t = 10$.

Differentiating the integral equation we get

$$\begin{aligned} &x'(t) = \cos(x(t)) - \cos(x(t-1)), \quad t \in [1; 10], \\ &x(t) = \lambda t, \quad t \in [0; 1]. \end{aligned}$$

This problem can be solved numerically with the Matlab command `dde23` and the obtained solution will be used for efficiency estimation of the algorithm proposed by Theorem 3.1.

Numerical method. (For more details see N.L. Trefethen [15], D. Trif [16]) We divide the working interval by the points $P_k = k$, $k = 0, 1, ..., M$, (concretely $M = 10$ and represents the number of subintervals). On each subinterval $I_k = [P_{k-1}; P_k]$, $k = 1, ..., M$, we find the numerical solution by the form

$$x_k(t) = c_{0,k}\frac{T_0}{2} + c_{1,k}T_1(\xi) + c_{2,k}T_2(\xi) + ... + c_{n-1,k}T_{n-1}(\xi),$$

where $T_i(\xi) = \cos(i \arccos(\xi))$ are Chebishev polynomials of i degree, $i = 0, ..., n-1$, (concretely $n = 8$), and $t = \alpha\xi + \beta$ with $\alpha = (P_k - P_{k-1})/2$, respectively $\beta = (P_k + P_{k-1})/2$.

Choosing a mesh ξ_j, $j = 1, ..., n$, on interval $[-1; 1]$ consisting by the knots of Gauss quadrature formula generated by Matlab subprogram `[csi,w]=pd(n)`, the transformation $t = \alpha\xi + \beta$ corresponding to each interval $I_k = [P_{k-1}; P_k]$ construct a local mesh on that subinterval. The coefficients $c_{i,k}$ of x_k expansion after the Chebishev polynomials T_i are obtained from x_k values on the local mesh using Fast Fourier Transforms (if n is large) or using a matrix T generated by the subprogram `T=x2t(n,csi)` (for n small)

$$\begin{pmatrix} c_{0,k} \\ c_{1,k} \\ \vdots \\ c_{n-2,k} \\ c_{n-1,k} \end{pmatrix} = (T')^{-1} \cdot \begin{pmatrix} x_k(t_1) \\ x_k(t_2) \\ \vdots \\ x_k(t_{n-1}) \\ x_k(t_n) \end{pmatrix}.$$

The same formula allows the quick pass from the local coefficients to the values on local mesh.

The formulae

$$\int_{\xi-\tau}^{\xi} T_i(s)\,ds = \frac{T_{i+1}(\xi)}{2(i+1)} - \frac{T_{i-1}(\xi)}{2(i-1)}$$

allow to obtain the coefficients C_i of a primitive F for a function f given by its coefficients c_i, from multiplication of them with a sparse matrix J generated by the subprogram `J=tchej(n)`

$$\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_{n-2} \\ C_{n-1} \end{pmatrix} = J \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{pmatrix}.$$

Of course, if the primitive is calculated for other interval $[P_{k-1}; P_k]$ instead of $[-1; 1]$, the matrix J is replaced by αJ, where $\alpha = (P_k - P_{k-1})/2$.

The algorithm from Theorem 3.1 is implemented in the following way in program `[X,sol]=step_meth`, which can be obtained from the authors:

Step 0. We generate a global mesh X on $[0; 10]$ by the union of all local meshes on which we also add the points P_k of subintervals. We calculate the values of $x^{(0)}$ on the global mesh from the values of the function φ on the local mesh of the first interval and from the constant value $\varphi(1)$ on the other knots.

Step K. Taking the values of $x^{(K)}$ on the global mesh, we obtain the values of $\cos\left(x^{(K)}\right)$ on the local mesh, we calculate the coefficients of $\cos\left(x^{(K)}\right)$ on that subinterval, then we get the coefficients of a primitive for $\cos\left(x^{(K)}\right)$ on that subinterval and finally we obtain the values of that primitive on the local mesh. The implementation of the formulae from Theorem 3.1 is now immediately, getting the values of the new iteration $x^{(K+1)}$ on the global mesh. We notice that the delay τ transforms the local mesh from an interval into the local mesh of the previous interval.

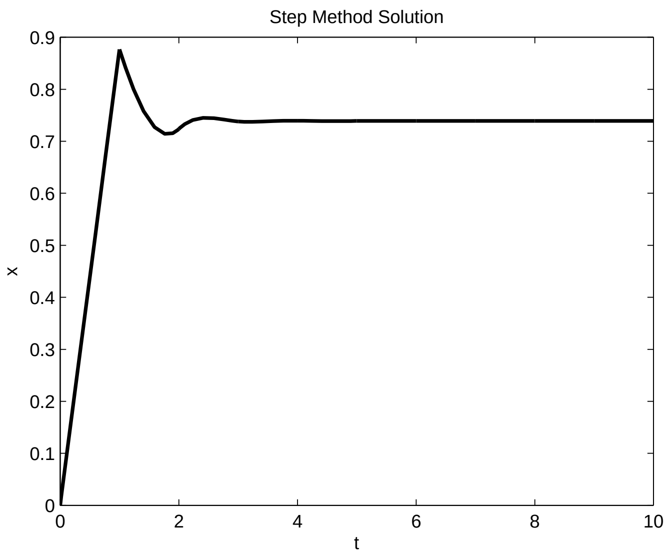
Stoping test. We evaluate the difference in norm between the values of $x^{(K)}$ and $x^{(K+1)}$ and iterations stop when this is below than a chosen value (concretely 10^{-9}). We display the last value of solution (in $t = 10$) and we represent the graph of solution and the norm of difference for different K.

For the efficiency estimation of this algorithm, the integral equation is written in the form of delay differential equation and we use the Matlab command `dde23` to solve it. We impose the relative error to 10^{-9} and the absolute error to 10^{-12} to obtain a accuracy comparable with the step method. We display the last value calculated in $t = 10$ and we represent the graph of solution in the same window with numerical solution of the step method.

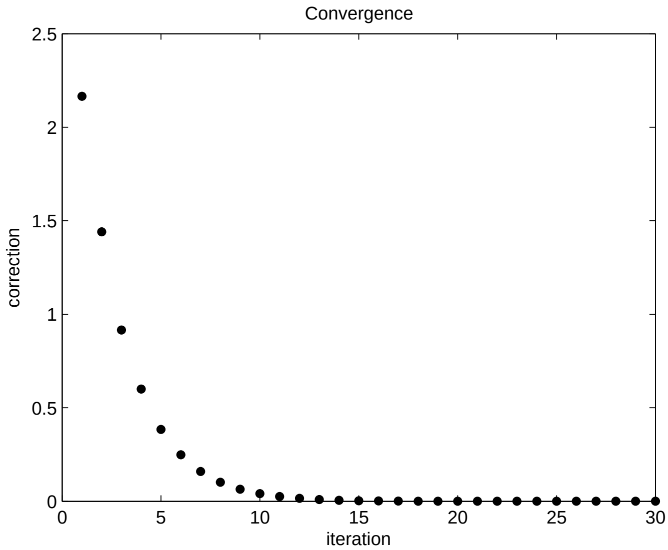
Results. Running the program we get the following results:

>>[X,sol]=step_meth;
Step method solution
Elapsed time is 0.010798 seconds.
ans =
0.73908513302906
Matlab solution
Elapsed time is 0.331578 seconds.
ans =
0.73908513279153
exact stationary solution
ans =
0.73908513321516

The graph of solutions and the evolution of the differences between two successive iterations are given below:



Conclusions. For the chosen example, the step method obtains the stationary solution in 30 iterations with an error of 2×10^{-10} in 0.01 seconds CPU. The Matlab program `dde23` needs 0.33 seconds CPU (30 times bigger) for a precision of 5×10^{-10}. The above comparisons validate the step method from the accuracy and efficiency point of view.



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Received: 23.05.2009,
Revised: 16.11.2010,
Accepted: 16.12.2010.

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