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On presheaves satisfying property (F2) and a separation axiom

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Abstract

Suppose that $\mathcal{F}$ is a presheaf of sets, $\tilde{\mathcal{F}}$ is the associated sheaf and $\eta : \mathcal{F} \to \tilde{\mathcal{F}}$ is the canonical morphism of presheaves. We study the relationship between the surjectivity of η_D for any open set D and the "gluing" property (F2).

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1 Introduction

For a presheaf of sets $\mathcal{F}$ on a topological space X we denote by $\tilde{\mathcal{F}}$ the associated sheaf and by $\eta : \mathcal{F} \to \tilde{\mathcal{F}}$ the canonical morphism of presheaves. For basic notion and terminology regarding presheaves of sets on a topological space we refer to Godement's book [2]. For a presheaf of sets $\mathcal{F}$ on X and $V \subset U$ two open subsets of X we will denote the restrictions $\mathcal{F}(U) \to \mathcal{F}(V)$ either by ρ_V^U or by $s \to s_{|V}$.

We say that $\mathcal{F}$ has the property (F1) if for every open set U of X and every $s, t \in \mathcal{F}(U)$ if there exists an open covering $\{U_i\}_{i\in I}$ of U such that $s_{|U_i} = t_{|U_i}$ for every $i \in I$, then $s = t$.

We say that $\mathcal{F}$ has the property (F2) if for every family of open sets $\{U_i\}_{i\in I}$ of X and every family of sections $s_i \in \mathcal{F}(U_i)$ that satisfy $s_{i|U_i\cap U_j} = s_{j\,|U_i\cap U_j}$ for all $i, j \in I$, there exists $s \in \mathcal{F}(\cup_{i\in I} U_i)$ such that $s_{|U_i} = s_i$.

It is a standard fact and easy to prove that $\mathcal{F}$ satisfies (F1) if and only if $\eta_D : \mathcal{F}(D) \to \tilde{\mathcal{F}}(D)$ is injective for every open subset D of X and $\mathcal{F}$ satisfies both (F1) and (F2) if and only if η_D is bijective for every D. The natural question that one can ask is what is the relation between property (F2) and the surjectivity of η_D. The surjectivity of η_D can imply (F2) only for very simple topologies as Example 1 below shows. The other implication is much more subtle. In general this implication is not true either. We present here two examples. A very simple one, which in fact can serve as a motivation for the proof of Theorem 1, and a more interesting one. Namely we show that if we consider $\mathbb{C}^n$ endowed with the Zariski topology then we can find a presheaf of sets that satisfies (F2) and $\eta_{\mathbb{C}^n}$ is not surjective.

However, if X is a Hausdorff paracompact space and $\mathcal{F}$ satisfies (F2) then η_D is surjective for every open set D. For basic general topology notions we refer to [3] or [4]. This result is in fact an important technical ingredient. A proof of it can be found for example in [1], Chapter 1, Theorem 6.3.

Then the natural question that we can ask is if and how much the hypothesis that X is Hausdorff and paracompact can be relaxed and it is the purpose of our paper to explore this problem.

As it is well-known a Hausdorff paracompact space is in fact a normal (or a T4) space. While working on the above mentioned question we have been led to a weaker separation axiom. We call it (WS). It turns out that this condition is necessary as Theorem 1 shows.

2 Examples

For a presheaf of sets $\mathcal{F}$ on a topological space X and for a point $x \in X$ we denote by $\mathcal{F}_x = \tilde{\mathcal{F}}_x$ the inductive limit of $\mathcal{F}(U)$, U open subset of X with $x \in U$. That is $\mathcal{F}_x = (\bigsqcup\{\mathcal{F}(U) : x \in U,\ U \text{ open in } X\}) / \sim$ where the equivalence relation $\sim$ is defined as follows: if $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ then $s \sim t$ if and only if there exists W an open subset of X such that $x \in W$, $W \subset U \cap V$ and $s_{|W} = t_{|W}$. If U is an open subset of X, $x \in U$ and s in $\mathcal{F}(U)$ we denote by s_x the equivalence class of s in $\mathcal{F}_x$.

Example 1: Suppose that X is a topological space such that there exist U, V, U_1, U_2, V_1, V_2 proper and non-empty open subsets of X with the following properties: $U \cup V = X$ and $U \cap V \neq \emptyset$, $U_1 \cup U_2 = U$, $U_1 \neq U$, $U_2 \neq U$, $V_1 \cup V_2 = V$ and $V_1 \neq V$, $V_2 \neq V$.

We define a presheaf of sets $\mathcal{F}$ on X as follows: $\mathcal{F}(W) = \{0,1\}$ (that means that we fix a set that has exactly two elements and for simplicity we choose these two elements to be the integers 0 and 1) if $W \supset U$ or $W \supset V$ and $\mathcal{F}(W) = \{0\}$ otherwise. If $W_1 \supset W_2$ are two open subsets of X then the restriction map $\rho^{W_1}_{W_2}$ is the identity if $\mathcal{F}(W_1) = \mathcal{F}(W_2)$ and the constant map $\{0,1\} \to \{0\}$ otherwise. Note that $\mathcal{F}(U_1) = \mathcal{F}(U_2) = \mathcal{F}(V_1) = \mathcal{F}(V_2) = \mathcal{F}(U \cap V) = \{0\}$ and $\mathcal{F}(U) = \mathcal{F}(V) = \mathcal{F}(X) = \{0,1\}$. It follows that $\mathcal{F}_x = \{0\}$ for every $x \in X$ and therefore for every open subset D of X we have that $\tilde{\mathcal{F}}(D)$ contains only one element, namely $s : D \to \bigsqcup_{x\in D} \mathcal{F}_x$, $s(x) = 0 \in \mathcal{F}_x$. As $\mathcal{F}(D) \neq \emptyset$ obviously η_D is surjective for every open subset D of X. On the other hand if we set $s_U \in \mathcal{F}(U)$, $s_U = 1$, and $s_V \in \mathcal{F}(V)$, $s_V = 0$ then $\rho^U_{U\cap V}(s_U) = 0 = \rho^U_{U\cap V}(s_V)$ and there is no $\sigma \in \mathcal{F}(X)$ such that $\rho^X_U(\sigma) = s_U$ and $\rho^X_V(\sigma) = s_V$, hence $\mathcal{F}$ does not satisfy (F2).

Example 2: Suppose that X is a set and U_1, U_2, V, V_1, V_2, W are subsets of

X such that $U_1 \cup U_2 = X$, $U_1 \cap U_2 = V$, $V_1 \cup V_2 = V$, $V_1 \cap V_2 = W$. Let $\mathcal{T} = \{X, U_1, U_2, V, V_1, V_2, W, \emptyset\}$. It is clear that $\mathcal{T}$ is a topology on X. We define a presheaf of sets on X (endowed with this topology) as follows: $\mathcal{F}(X) = \mathcal{F}(U_1) = \mathcal{F}(U_2) = \mathcal{F}(V) = \{0,1\}$ and $\mathcal{F}(V_1) = \mathcal{F}(V_2) = \mathcal{F}(W) = \mathcal{F}(\emptyset) = \{0\}$ and the restrictions $\rho_{D_2}^{D_1}$ are the identity if $\mathcal{F}(D_1) = \mathcal{F}(D_2)$ and the constant map otherwise. We note first that $\mathcal{F}$ satisfies (F2). Indeed, the only two open subsets of X that have non-trivial open coverings (non-trivial in the sense that no element of the covering is equal to the given open set) are $X = U_1 \cup U_2$ and $V = V_1 \cup V_2$. As $\mathcal{F}(V_1) = \mathcal{F}(V_2) = \{0\}$ the gluing property is automatically satisfied. Suppose that $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ are such that $s_{1|V} = s_{2|V}$. Then, because $\rho_V^{U_1} : \{0,1\} \to \{0,1\}$ and $\rho_V^{U_2} : \{0,1\} \to \{0,1\}$ are the identity functions it follows that $s_1 = s_2$ as elements of $\{0,1\}$ and then if we define $s \in \mathcal{F}(X)$, $s = s_1 = s_2$ (as elements of $\{0,1\}$) we get that $s_{|U_1} = s_1$ and $s_{|U_2} = s_2$. On the other hand $\mathcal{F}_x = \{0\}$ for every $x \in V$. Then if we set $s_1 \in \mathcal{F}(U_1)$, $s_1 = 0$, and $s_2 \in \mathcal{F}(U_2)$, $s_2 = 1$, s_1 and s_2 will determine a section $\tilde{s} \in \tilde{\mathcal{F}}(X)$. If $s \in \mathcal{F}(X)$ is such that $s_x = s_{1x}$ for some $x \in U_1 \setminus V$ then $s = s_1 = 0$ as elements of $\{0,1\}$. Similarly if $s \in \mathcal{F}(X)$ is such that $s_x = s_{2x}$ for some $x \in U_2 \setminus V$ then $s = s_2 = 1$ as elements of $\{0,1\}$. We deduce that there is no $s \in \mathcal{F}(X)$ with $\eta_X(s) = \tilde{s}$.

Example 3: Suppose that $X = \mathbb{C}^n$, $n \geq 2$, endowed with the Zariski topology. We pick two distinct points $a \neq b \in \mathbb{C}^n$. We define a presheaf of sets $\mathcal{F}$ as follows: if U is Zariski open set in $\mathbb{C}^n$ such that there exists an irreducible algebraic variety $Z \subset \mathbb{C}^n$ with $a, b \in Z$ and $U \subset \mathbb{C}^n \setminus Z$ then we set $\mathcal{F}(U) = \{0\}$, otherwise $\mathcal{F}(U) = \{0,1\}$. In particular if $U \cap \{a,b\} \neq \emptyset$ then $\mathcal{F}(U) = \{0,1\}$ and hence $\mathcal{F}(\mathbb{C}^n \setminus \{a\}) = \{0,1\}$, $\mathcal{F}(\mathbb{C}^n \setminus \{b\}) = \{0,1\}$. If U_1 and U_2 are two Zariski open sets such that $U_1 \subset U_2$ and $\mathcal{F}(U_1) = \{0,1\}$ (and therefore $\mathcal{F}(U_2) = \{0,1\}$) then the restriction map $\rho_{U_1}^{U_2}$ is the identity, otherwise $\rho_{U_1}^{U_2}$ is the constant function. Obviously this is a presheaf. We will show that $\mathcal{F}$ satisfies (F2) and η_X is not surjective. Suppose that $\{U_i\}_{i \in I}$ are Zariski open sets and $s_i \in \mathcal{F}(U_i)$ are sections such that $s_{i|U_i \cap U_j} = s_{j\,|U_i \cap U_j}$ for all $i, j \in I$. We would like to show that there exists $s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$. If $s_i = 0$ for every $i \in I$ there is nothing to prove; we simply choose $s = 0 \in \mathcal{F}(U)$. Suppose now that there exists $i_0 \in I$ such that $s_{i_0} = 1$. We need to prove that if $i \in I$ is such that $\mathcal{F}(U_i) = \{0,1\}$ then $s_i = 1$. If this is the case, we set $s = 1 \in \mathcal{F}(U)$. Choose such an $i \in I$. We have to consider two cases.

Case 1: $\mathcal{F}(U_i \cap U_{i_0}) = \{0,1\}$. Then $s_{i|U_{i_0} \cap U_i} = s_{i_0\,|U_{i_0} \cap U_i} = 1$, hence $s_i = 1$.

Case 2: $\mathcal{F}(U_i \cap U_{i_0}) = \{0\}$. By definition there exists an irreducible variety, Z, such that $\dim(Z) \geq 1$, $a, b \in Z$ and $U_{i_0} \cap U_i \subset \mathbb{C}^n \setminus Z$. Let $Y_0 = \mathbb{C}^n \setminus U_{i_0}$ and $Y_1 = \mathbb{C}^n \setminus U_i$. Then Y_0 and Y_1 are algebraic varieties in $\mathbb{C}^n$ and $Z \subset Y_0 \cup Y_1$. Since Z is irreducible it follows that $Z \subset Y_0$ or $Z \subset Y_1$. However this cannot happen because, according to our assumption, $\mathcal{F}(U_{i_0}) = \{0,1\}$ and $\mathcal{F}(U_i) = \{0,1\}$.

Next we will show that η_X is not onto. Notice that $\mathcal{F}_a = \{0,1\}$, $\mathcal{F}_b = \{0,1\}$ and, since for each point $x \in \mathbb{C}^n \setminus \{a,b\}$ there exists a positive dimensional irreducible algebraic variety Z such that $x \notin Z$ and $\{a,b\} \subset Z$, we have that

$\mathcal{F}_x = \{0\}$ for every point x other than a or b. If $s_1 \in \mathcal{F}(\mathbb{C}^n \setminus \{a\})$, $s_1 = 1$, and $s_2 \in \mathcal{F}(\mathbb{C}^n \setminus \{b\})$, $s_2 = 0$ then they determine a section in $\tilde{\mathcal{F}}(\mathbb{C}^n)$ and there is no $s \in \mathcal{F}(\mathbb{C}^n)$ such that $s_a = s_{1a} = 1$ and $s_b = s_{2b} = 0$.

3 Results

According to the definition of $\tilde{\mathcal{F}}$, the surjectivity of η_D, where D is an open subset of X, means the following: if $\{U_i\}_{i\in I}$ is an open covering of D and $s_i \in \mathcal{F}(U_i)$ are sections such that for every $i, j \in I$ and $s_{ix} = s_{jx}$ for every $x \in U_i \cap U_j$ then there exists $s \in \mathcal{F}(D)$ such that $s_x = s_{ix}$ for every $x \in U_i$. With Zorn's lemma we introduce the following:

Definition 1. *Let X be a topological space.*
a) If D is an open subset of X, we say that $\mathcal{F}$ satisfies property (O1)(D) if for every two open sets U_1 and U_2 such that $U_1 \cup U_2 = D$ and every two sections $s_1 \in \mathcal{F}(U_1)$, $s_2 \in \mathcal{F}(U_2)$ such that $s_{1x} = s_{2x}$ for every $x \in U_1 \cap U_2$ there exists $s \in \mathcal{F}(D)$ such that $s_x = s_{1x}$ $\forall x \in U_1$ and $s_x = s_{2x}$ $\forall x \in U_2$.
b) If D is an open subset of X, we say that $\mathcal{F}$ satisfies property (O2)(D) if for every totally ordered set $(I, <)$, every family of open subsets of D, $\{U_i\}_{i\in I}$, such that $U_i \subset U_j$ for every $i < j \in I$ and $\cup_{i\in I} U_i = D$, and every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_{ix} = s_{jx}$ for all $i < j \in I$ and $x \in U_i$, there exists $s \in \mathcal{F}(D)$ such that $s_x = s_{ix}$ for all $i \in I$ and $x \in U_i$.

Proposition 1. *Suppose that $\mathcal{F}$ is a presheaf of sets on the topological space X such that $\mathcal{F}$ satisfies (O1)(D) and (O2)(D) for every open subset D of X. Then $\eta_D : \mathcal{F}(D) \to \tilde{\mathcal{F}}(D)$ is surjective for every open subset D of X.*

Proof: Let Ω be an open subset of X and let $\{U_i\}_{i\in I}$ be an open covering of Ω and let $s_i \in \mathcal{F}(U_i)$ be such that $s_{ix} = s_{jx}$ for every $i, j \in I$ and every $x \in U_i \cap U_j$. We define the following subset of the power set of I:

$$\mathcal{A} = \{J : J \subset I \text{ and there exist } \sigma \in \mathcal{F}(\cup_{j\in J} U_j),\ \sigma_x = s_{jx} \forall j \in J\}$$

On $\mathcal{A}$ we consider the order given by the inclusion. Since $\mathcal{F}$ satisfies (O2)(D) for every open set D, $\mathcal{A}$ is directed and hence by Zorn's lemma it has a maximal element, call it J_0. If we show that J_0 is actually equal to I the Proposition is proved. Suppose that $J_0 \neq I$ and let $k \in I \setminus J_0$ and $\sigma \in \mathcal{F}(\cup_{j\in J_0} U_j)$, $\sigma_x = s_{jx} \forall j \in J_0$. Obviously $\sigma_x = s_{kx}$ for all $x \in U_k \cap (\cup_{j\in J_0} U_j)$. As $\mathcal{F}$ satisfies (O1)($\cup_{i\in J_0\cup\{k\}} U_i$), there exists $\sigma' \in \mathcal{F}(\cup_{i\in J_0\cup\{k\}} U_i)$ such that $\sigma'_x = \sigma_x$ for $x \in \cup_{j\in J_0} U_j$ and $\sigma'_x = s_{kx}$ for $x \in U_k$. It follows that $\sigma'_x = s_{ix}$ for all $i \in J_0 \cup \{k\}$ and $x \in U_i$. This implies that $J_0 \cup \{k\} \in \mathcal{A}$, which contradicts the maximality of J_0. □

Definition 2. *a) Let X be a topological space and F_1 and F_2 two disjoint closed subsets of X. We say that F_1 and F_2 have the property (WS) if for every $\{\Omega_i\}_{i\in I}$, an open covering of $X \setminus (F_1 \cup F_2)$, there exist $\mathcal{A}_1$ and $\mathcal{A}_2$ open coverings for F_1 and F_2 respectively such that for every $U_1 \in \mathcal{A}_1$ and $U_2 \in \mathcal{A}_2$ there exists $i \in I$ such that $U_1 \cap U_2 \subset \Omega_i$.*
b) We say that X is a (WN)-space if every two closed disjoint subsets of X have the property (WS).

Theorem 1. *If X is a topological space such that every presheaf of sets $\mathcal{F}$ on X that has the property (F2) has also the property (O1)(X) then every two disjoint closed subsets of X, F_1 and F_2, have the property (WS).*

Proof: Suppose that there exists two disjoint closed subsets of X, F_1 and F_2, that do not have the property (WS). We will produce a presheaf of sets $\mathcal{F}$ on X that has the property (F2) and does not have the property (O1)(X).

Let $\{\Omega_i\}_{i\in I}$ be an open covering of $X \setminus (F_1 \cup F_2)$ such that for every $\mathcal{A}_1$ and $\mathcal{A}_2$ open coverings for F_1 and F_2, respectively, there exist $U_1 \in \mathcal{A}_1$ and $U_2 \in \mathcal{A}_2$ such that $U_1 \cap U_2 \not\subset \Omega_i$ for every $i \in I$.

Suppose that D is an open subset of X and $\mathcal{U}$ is an open covering of D. A function $\alpha : \mathcal{U} \to \{0,1\}$ is called *consistent* (with $\{\Omega_s\}$) if the following two conditions are satisfied:
- for all $U \in \mathcal{U}$, if there exists $i \in I$ such that $U \subset \Omega_i$ then $\alpha(U) = 0$
- for all $U, V \in \mathcal{U}$, if there is no $i \in I$ such that $U \cap V \subset \Omega_i$ then $\alpha(U) = \alpha(V)$

Suppose that $D_1 \subset D$ are open subsets of X, $\mathcal{U}$ is an open covering of D and $\alpha : \mathcal{U} \to \{0,1\}$ is consistent. Then we set $\mathcal{U}_{|D_1} := \mathcal{U} \cap D_1 = \{U \cap D_1 : U \in \mathcal{U}\}$ and $\alpha_{|D_1} : \mathcal{U}_{|D_1} \to \{0,1\}$, where, for every $U \in \mathcal{U}$, $\alpha_{|D_1}(U \cap D_1) = \alpha(U)$ if there is no $i \in I$ such that $U \cap D_1 \subset \Omega_i$ and $\alpha_{|D_1}(U \cap D_1) = 0$ otherwise. It is easy to see that $\alpha_{|D_1}$ is also consistent. Note that if $U \in \mathcal{U}$ and $U \subset D_1$ then $\alpha_{|D_1}(U) = \alpha(U)$

An open covering $\mathcal{U}$ of D is called *complete* if whenever U and V are open subsets of X such that $U \in \mathcal{U}$ and $V \subset U$ then $V \in \mathcal{U}$. Note that if $\mathcal{U}$ is a complete covering of D and D_1 is an open subset of D then $\mathcal{U}_{|D_1}$ is a complete covering of D.

Remark: If $\mathcal{U}$ is an open covering of D we set $co(\mathcal{U}) := \{V : V$ is open in X and $\exists U \in \mathcal{U}, V \subset U\}$. Hence $\mathcal{U}$ is complete if and only if $\mathcal{U} = co(\mathcal{U})$. If $\mathcal{U}$ is an open covering of D and $\alpha : \mathcal{U} \to \{0,1\}$ is consistent then we can extend it to a consistent function $co(\alpha) : co(\mathcal{U}) \to \{0,1\}$ as follows: if $V \subset U$ are open in X and $U \in \mathcal{U}$ then we set $co(\alpha)(V) = \alpha(U)$ if there is no $i \in I$ such that $V \subset \Omega_i$ and $co(\alpha)(V) = 0$ otherwise.

We define now a presheaf of sets on X as follows: if D is an open subset of X then $\mathcal{F}(D) = \{(\mathcal{U}, \alpha) : \mathcal{U}$ is a complete covering of $D, \alpha : \mathcal{U} \to \{0,1\}$ is consistent$\}$. If $D_1 \subset D_2$ are open sets, $(\mathcal{U}, \alpha) \in \mathcal{F}(D_2)$ then $\rho^{D_2}_{D_1}(\mathcal{U}, \alpha) = (\mathcal{U}_{|D_1}, \alpha_{|D_1})$.

It is clear that for $D_0 \subset D_1 \subset D_2$, we have $\rho^{D_2}_{D_0} = \rho^{D_1}_{D_0} \circ \rho^{D_2}_{D_1}$. Hence $\mathcal{F}$ is a presheaf. We will check next that $\mathcal{F}$ has the property (F2). Let $\{D_l\}_{l\in L}$, be open sets in X and $(\mathcal{U}_l, \alpha_l) \in \mathcal{F}(D_l)$ be such that, for every $l, k \in L$, $\rho^{D_l}_{D_l \cap D_k}(\mathcal{U}_l, \alpha_l) =$

$\rho^{D_k}_{D_l\cap D_k}(\mathcal{U}_k,\alpha_k)$. Let $\mathcal{U}=\cup_{l\in L}\mathcal{U}_l$. Clearly $\mathcal{U}$ is an open covering for $D=\cup_{l\in L}D_l$. It is also a complete covering: if $U\in\mathcal{U}$ and V is an open subset of U then there exists $l\in L$ such that $U\in\mathcal{U}_l$. Since $\mathcal{U}_l$ is complete then $V\in\mathcal{U}_l$ and hence $V\in\mathcal{U}$.

Note now that $\alpha_{k|\mathcal{U}_l\cap\mathcal{U}_k}=\alpha_{l|\mathcal{U}_l\cap\mathcal{U}_k}$. Indeed, if $U\in\mathcal{U}_l\cap\mathcal{U}_k$ then $U\subset D_l\cap D_k$ and hence $U\in\mathcal{U}_{l|D_l\cap D_k}$. By definition $\alpha_l(U)=\alpha_{l|D_l\cap D_k}(U)$ and similarly $\alpha_k(U)=\alpha_{k|D_l\cap D_k}(U)$. The compatibility condition $\rho^{D_l}_{D_l\cap D_k}(\mathcal{U}_l,\alpha_l)=\rho^{D_k}_{D_l\cap D_k}(\mathcal{U}_k,\alpha_k)$ implies that $\alpha_{l|D_l\cap D_k}(U)=\alpha_{k|D_l\cap D_k}(U)$.

The following function is well defined then: $\alpha:\mathcal{U}\to\{0,1\}$, $\alpha_{|\mathcal{U}_l}=\alpha_l$, for all $l\in L$. We claim that it is consistent: let $U,V\in\mathcal{U}$ and let $l,k\in L$ be such that $U\in\mathcal{U}_l$ and $V\in\mathcal{U}_k$. If $U\subset\Omega_i$ for some $i\in I$ then, as α_l is consistent, $\alpha_l(U)=0$ and therefore $\alpha(U)=0$. At the same time $U\cap V\in\mathcal{U}_l$ and $U\cap V\in\mathcal{U}_k$ (by the completeness of $\mathcal{U}_l$ and $\mathcal{U}_k$). If there is no $i\in I$ such that $U\cap V\subset\Omega_I$ then, using the consistency of α_l and α_k we get:
$\alpha(U)=\alpha_l(U)=\alpha_l(U\cap V)=\alpha_k(U\cap V)=\alpha_k(V)=\alpha(V)$.

What is left to notice is that $\mathcal{U}_{|D_l}=\mathcal{U}_l$. This is in fact the main point where the completeness assumption comes into place. It is clear that $\mathcal{U}_{|D_l}\supset\mathcal{U}_l$. Let $U\in\mathcal{U}$. Say that $U\in\mathcal{U}_k$. Since $\mathcal{U}_{l|D_l\cap D_k}=\mathcal{U}_{l|D_l\cap D_k}$ it follows that there exists $V\in\mathcal{U}_l$ such that $V\cap D_l\cap D_k=U\cap D_l\cap D_k$. Since $\mathcal{U}_l$ is complete we have that $V\cap D_l\cap D_k\in\mathcal{U}_l$ and therefore $U\cap D_l\cap D_k=(U\cap D_k)\cap D_l=U\cap D_l\in\mathcal{U}_l$.

We will prove that $\mathcal{F}$ does not have the property (O1)(X). Let $D_1=X\setminus F_2$ and $D_2=X\setminus F_1$. Let $\mathcal{U}_1=\{U:U \text{ is open in } D_1\}$, $\mathcal{U}_2=\{U:U \text{ is open in } D_2\}$, $\alpha_1:\mathcal{U}_1\to\{0,1\}$, $\alpha_1(U)=1$ if there is no $i\in I$ such that $U\subset\Omega_i$ and $\alpha_1(U)=0$ otherwise, $\alpha_2:\mathcal{U}_2\to\{0,1\}$, $\alpha_2(U)=0$ for every $U\in\mathcal{U}_2$. We have that $(\mathcal{U}_1,\alpha_1)\in\mathcal{F}(D_1)$ and $(\mathcal{U}_2,\alpha_2)\in\mathcal{F}(D_2)$. At the same time, for every $x\in D_1\cap D_2=X\setminus(F_1\cup F_2)$ there exists $i\in I$ such that $x\in\Omega_i$. As $\rho^{D_1}_{\Omega_i}(\mathcal{U}_1,\alpha_1)=\rho^{D_2}_{\Omega_i}(\mathcal{U}_2,\alpha_2)$ it follows that $(\mathcal{U}_1,\alpha_1)_x=(\mathcal{U}_2,\alpha_2)_x$ for every $x\in D_1\cap D_2$.

Suppose that there exists $(\mathcal{U},\alpha)\in\mathcal{F}(X)$ such that $(\mathcal{U},\alpha)_x=(\mathcal{U}_1,\alpha_1)_x$ for every $x\in D_1$ and $(\mathcal{U},\alpha)_x=(\mathcal{U}_2,\alpha_2)_x$ for every $x\in D_2$. We set $\mathcal{A}_1=\{U\in\mathcal{U}_1:U\cap F_1\neq\emptyset\}$, $\mathcal{A}_2=\{U\in\mathcal{U}_2:U\cap F_2\neq\emptyset\}$ which are obviously open coverings for F_1 and F_2 respectively. Let $U\in\mathcal{A}_1$ and let $x\in U\cap F_1$. Because $(\mathcal{U},\alpha)_x=(\mathcal{U}_1,\alpha_1)_x$, there exists an open set V in X such that $x\in V\subset(U\cap D_1)$ and $\rho^{D_1}_V(\mathcal{U}_1,\alpha_1)=\rho^U_V(\mathcal{U},\alpha)$. By definition $\Omega_i\subset X\setminus(F_1\cup F_2)$ for every $i\in I$. In particular there is no $i\in I$ such that $V\subset\Omega_i$. This implies that $\alpha_{1|V}(V)=1$ and hence $\alpha_{|V}(V)=1$. By the consistency of α we deduce that $\alpha(U)=\alpha(V)=1$. The same argument shows that for every $U\in\mathcal{A}_2$, $\alpha(U)=0$.

For $U_1\in\mathcal{A}_1$ and $U_2\in\mathcal{A}_2$, since $\alpha(U_1)\neq\alpha(U_2)$ and α is consistent, it follows that there exists $i\in I$ such that $U_1\cap U_2\subset\Omega_i$ which is a contradiction with the choice of $\{\Omega_i\}_{i\in I}$. □

Proposition 2. *Let X be a topological space such that every two disjoint closed subsets of X satisfy the property (WS), then every presheaf of sets on X, $\mathcal{F}$, that satisfies property (F2) satisfies also the property (O1)(X).*

Proof: Suppose that D_1 and D_2 are open subsets of X such that $D_1 \cup D_2 = X$ and $s_1 \in \mathcal{F}(D_1)$, $s_2 \in \mathcal{F}(D_2)$ are sections such that $s_{1x} = s_{2x}$ for every $x \in D_1 \cap D_2$. Let $F_2 = X \setminus D_1$ and $F_1 = X \setminus D_2$. Therefore F_1 and F_2 are two disjoint closed subsets of X and hence they have the property (WS).

Since $s_{1x} = s_{2x}$ for every $x \in D_1 \cap D_2 = X \setminus (F_1 \cup F_2)$, there exists $\{\Omega_i\}_{i\in I}$ an open covering for $X \setminus (F_1 \cup F_2)$ such that $s_{1|\Omega_i} = s_{2|\Omega_i}$ for every $i \in I$. On the other hand, since F_1 and F_2 have (WS) there exist two open coverings $\mathcal{A}_1$ and $\mathcal{A}_2$ of F_1 and F_2, respectively, such that for every $U_1 \in \mathcal{A}_1$ and $U_2 \in \mathcal{A}_2$ there exists $i \in I$ such that $U_1 \cap U_2 \subset \Omega_i$

We consider the following open covering for X: $\mathcal{U} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{\Omega_i : i \in I\}$. We consider also the following collection of sections corresponding to $\mathcal{U}$: $s_{1|U_1} \in \mathcal{F}(U_1)$ for all $U_1 \in \mathcal{A}_1$, $s_{2|U_2} \in \mathcal{F}(U_2)$ for all $U_2 \in \mathcal{A}_2$ and $s_{1|\Omega_i} = s_{2|\Omega_i} \in \mathcal{F}(\Omega_i)$ for all $i \in I$.

Now if $U, V \in \mathcal{U}$ and $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ are the corresponding sections then $s_{|U\cap V} = t_{|U\cap V}$. Indeed: if $U \in \mathcal{A}_1$ and $V \in \mathcal{A}_2$ then $s = s_{1|U}$ and $t = s_{2|V}$. On the other hand, by the choice of $\mathcal{A}_1$ and $\mathcal{A}_2$, there exists $i \in I$ such that $U \cap V \subset \Omega_i$. It follows that

$$s_{|U\cap V} = (s_{1|\Omega_i})_{|U\cap V} = (s_{2|\Omega_i})_{|U\cap V} = t_{|U\cap V}.$$

The other cases are trivial.

Since $\mathcal{F}$ has the property $(F2)$ there exists $\sigma \in \mathcal{F}(X)$ such that $\sigma_{|U} = s$ for every $U \in \mathcal{U}$ and s the corresponding section. In particular $\sigma_x = s_{1x}$ for every $x \in D_1$ and $\sigma_x = s_{2x}$ for every $x \in D_2$. ◻

Theorem 2. *Let X be a topological space. Suppose that every open subset of X is a (WN)-space and that for every totally ordered set $(I, <)$ and every family of open subsets of X, $\{U_i\}_{i\in I}$ with $U_i \subset U_j$ for every $i < j \in I$ there exists an increasing function $\iota : \mathbb{N} \to I$ and $\{V_n\}_{n\in\mathbb{N}}$, a sequence of open subsets of X, such that $\cup_{n\in\mathbb{N}} V_n = \cup_{i\in I} U_i$, $V_n \subset U_{\iota(n)}$, and $\overline{V}_n \subset V_{n+1}$. Then for every presheaf of sets $\mathcal{F}$ on X that satisfies property $(F2)$ and every open subset D of X the canonical mapping $\eta_D : \mathcal{F}(D) \to \tilde{\mathcal{F}}(D)$ is surjective.*

Proof: Let $\mathcal{F}$ be an arbitrary presheaf of sets on X that satisfies property $(F2)$ and D an open subset of X. Since every open subset of X is a (WN)-space, by Proposition 2, $\mathcal{F}$ satisfies property (O1)(D). According to Proposition 1 we have to check that for every open set $D \subset X$ the presheaf $\mathcal{F}$ satisfies (O2)(D). Let $(I, <)$ be a totally ordered set and $\{U_i\}_{i\in I}$ be a family of open subsets of D such that $\cup_{i\in I} U_i = D$ and $U_i \subset U_j$ for every $i < j \in I$. Let $\{V_n\}_{n\in\mathbb{N}}$ and $\iota : \mathbb{N} \to I$ be as in the hypothesis.We will construct inductively a sequence of sections $\sigma_n \in \mathcal{F}(V_{2n})$ such that

a) $\sigma_{nx} = s_{\iota(2n)_x}$ for every $n \in \mathbb{N}$ and every $x \in V_{2n}$

b) $\sigma_{n|V_{2n-4}} = \sigma_{n-1|V_{2n-4}}$ for every $n \geq 2$.

We set $\sigma_0 = s_{\iota(0)}$ and $\sigma_1 = s_{\iota(2)}$. We assume that we have constructed $\sigma_0, \sigma_1, \ldots, \sigma_n$, $n \geq 1$, and we will construct σ_{n+1}.

Because $\sigma_{nx} = s_{\iota(2n+2)_x}$ for every $x \in V_{2n} \setminus \overline{V}_{2n-1}$ it follows, just by definition, that each $x \in V_{2n} \setminus \overline{V}_{2n-1}$ has a neighborhood say Ω_x such that $\sigma_{n|\Omega_x} = s_{\iota(2n+2)|\Omega_x}$. Replacing Ω_x with $\Omega_x \cap (V_{2n} \setminus \overline{V}_{2n-1})$ we get that $\{\Omega_x\}_{x \in V_{2n} \setminus \overline{V}_{2n-1}}$ is an open covering for $V_{2n} \setminus \overline{V}_{2n-1}$. Changing the notation for the index set we conclude that there exists $\{\Omega_k\}_{k \in K}$ an open covering for $V_{2n} \setminus \overline{V}_{2n-1}$, $\Omega_k \subset V_{2n} \setminus \overline{V}_{2n-1}$ $\forall k \in K$, such that $\sigma_{n|\Omega_k} = s_{\iota(2n+2)|\Omega_k}$ for every $k \in K$.

Let $F_1 = \overline{V}_{2n-1} \setminus \overline{V}_{2n-2}$ and $F_2 = V_{2n+1} \setminus V_{2n}$. They are closed disjoint subsets of $V_{2n+1} \setminus \overline{V}_{2n-2}$ which is a (WN)-space and therefore F_1 and F_2 satisfy (WS). Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be open coverings of F_1 and F_2, respectively, such that for every $U_1 \in \mathcal{A}_1$ and $U_2 \in \mathcal{A}_2$ there exists $k \in K$ such that $U_1 \cap U_2 \subset \Omega_k$. Note that each $U \in \mathcal{A}_1 \cup \mathcal{A}_2$ is an open subset of $V_{2n+1} \setminus \overline{V}_{2n-2}$ hence it is open in X as well.

We consider the following open covering for V_{2n+2}: $\mathcal{U} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{\Omega_k : k \in K\} \cup \{V_{2n+2} \setminus \overline{V}_{2n}, V_{2n-1}\}$. We consider also the following collection of sections corresponding to $\mathcal{U}$:

- $\sigma_{n|U_1} \in \mathcal{F}(U_1)$ for all $U_1 \in \mathcal{A}_1$, $s_{\iota(2n+2)|U_2} \in \mathcal{F}(U_2)$ for all $U_2 \in \mathcal{A}_2$
- $\sigma_{n|\Omega_k} = s_{\iota(2n+2)|\Omega_k} \in \mathcal{F}(\Omega_k)$ for all $k \in K$.
- $s_{\iota(2n+2)|V_{2n+2} \setminus \overline{V}_{2n}} \in \mathcal{F}(V_{2n+2} \setminus \overline{V}_{2n})$, $\sigma_{n|V_{2n-1}} \in \mathcal{F}(V_{2n-1})$.

As in the proof of Proposition 2 there exists $\sigma_{n+1} \in \mathcal{F}(V_{2n+2})$ such that $\sigma_{|U} = s$ for every $U \in \mathcal{U}$ and s the corresponding section. In particular $\sigma_{n|V_{2n-2}} = \sigma_{n+1|V_{2n-2}}$ and, at the same time, $\sigma_{n+1x} = s_{\iota(2n+2)_x}$ for every $x \in V_{2n+2}$. The construction of $\{\sigma_n\}_n$ is completed.

If we define now $\tau_n \in \mathcal{F}(V_{2n})$, $\tau_n = \sigma_{n+1|V_{2n}}$ then $\tau_{n+1|V_n} = \tau_n$ and as $\mathcal{F}$ has the property (F2) there exists $\tau \in \mathcal{F}(\cup_{n \in \mathbb{N}} V_{2n})$ such that $\tau_{|V_{2n}} = \tau_n$. However $\cup_{n \in \mathbb{N}} V_{2n} = D$ and for every $x \in V_{2n}$ we have that $\tau_x = \tau_{nx} = \sigma_{n+1x} = s_{\iota(2n+2)_x} = s_{ix}$ whenever $x \in U_i$. □

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