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Topological degree for periodic solutions of non-autonomous differential delay equations*

by
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Abstract

By making use of topological degree theory for Poincáre map established in [1], we obtain the existence of periodic solutions for a class of non-autonomous differential delay equations, which can be transformed to planer Hamiltonian systems.

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1 Introduction

In this article we consider the existence of periodic solutions for the following non-autonomous differential delay equation

$$x'(t) = f(x(t-\tau) + g(t, x(t-\tau)), \tag{1.1}$$

where $\tau > 0$, f and g are both continuous. Some special autonomous differential delay equations similar to Eq.(1.1) have been studied by many authors through various methods. Lots of results of periodic solutions for those equations can be found in [1-7].

The previous works are mainly concerned with the autonomous cases. The purpose of the present paper is to use a new approach to study the existence of periodic solutions of the non-autonomous equation (1.1). More precisely, we shall use the topological degree theory for Poincáre map established in [1] to obtain the existence of periodic solutions of Eq.(1.1). We give our main result in the next section .

Throughout the present paper, we assume that the following conditions hold.

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(H1) Let $f(x) : \mathbb{R} \to \mathbb{R}$ be homogeneous, locally Lipschitz continuous and satisfying $f(x) > 0$, for all $x > 0$.

(H2) $g(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $T = 4\tau$-periodic in its first variable, locally Lipshitz continuous in its second variable.

(H3) There are two constants $\alpha > 0$ and $\beta \in [0, 1)$ and a function $h : \mathbb{R} \to \mathbb{R}$ such that

$$|g(t,x)| \le \alpha |x|^{\beta/2} \text{ for all } (t,x) \in \mathbb{R} \times \mathbb{R},$$

$$\lim_{\lambda \to +\infty} \frac{g(t, \lambda x)}{\lambda^\beta} = h(t), \forall \lambda > 0, \text{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}.$$

Remark 1.1. *Here we do not assume the condition that f and g are asymptotically linear both at origin and at infinity which plays a crucial role in the study of the existence and multiplicity of periodic solutions for those autonomous differential delay equations in previous papers.*

In Section 2 we give the main result of this paper and its proof. The proof will be carried out by applying the topological degree theory for Poincáre map constructed in [1]. In Section 3 an example will be given as an application of our result. In Section 4 a useful lemma will be proved by applying the ideas of [1].

2 Main result and its proof

In this section, we first reduce Eq.(1.1) to an associated planer Hamiltonian system. Consider the following system

$$J\frac{d}{dt}X(t) = \Phi(t, X(t)), \ where\ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.1}$$

$X(t) = (x_1(t), x_2(t))^\top$ and $\Phi(t, X) = (f(x_1) + g(t, x_1), f(x_2) + g(t, x_2))^\top$. It is not difficult to see that if $X(t)$ is a solution of (2.1) with the following symmetric structure

$$x_1(t) = -x_2(t-\tau), x_2(t) = x_1(t-\tau), \tag{2.2}$$

then $x(t) = x_1(t)$ gives a solution to Eq.(1.1) with the property $x(t) = -x(t-2\tau)$. In fact, by the system (2.1),

$$x_1'(t) = f(x_2(t)) + g(t, x_2(t)) = f(x_1(t-\tau)) + g(t, x_1(t-\tau)).$$

Therefore $x(t) = x_1(t)$ is a solution of Eq.(1.1).

Note that Eq.(2.1) can be written as the following planer system

$$Jy'(t) = \nabla H(y) + G(t, y), \tag{2.3}$$

where $H(y) = \int_0^{y_1} f(x)dx + \int_0^{y_2} f(x)dx$ and $G(t, y) = (g(t, y_1), g(t, y_2))^\top$ for each $y = (y_1, y_2)^\top \in \mathbb{R}^2$. Notice that the matrix J is a symplectic matrix. Thus the system (2.3) is a planer Hamiltonian system with the symplectic structure J. We call this planer Hamiltonian system the associated system to Eq.(1.1). Hence in the next, we only need to find periodic solutions of the system (2.3) satisfying (2.2).

For the two functions H and G, we have the following lemma.

Lemma 2.1. *The functions H and G have the following properties.*

(1) the function $H \in C^1(\mathbb{R}^2, \mathbb{R})$ with locally Lipschitz continuous gradient satisfies $H(\lambda y) = \lambda^2 H(y)$ and $min_{\|y\|=1} H(y) > 0$ for every $y \in \mathbb{R}^2$ and $\lambda > 0$.

(2) $G \in C^1(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ is T-periodic in its first variable and locally Lipschitz continuous in its second variable, and G satisfies $\|G(t, y)\| \leq \alpha(\|y\|^\beta + 1)$.

Proof. (1) By (H1) and the definition of H, $\nabla H(y) = (f(y_1), f(y_2))$. Then $H \in C^1(\mathbb{R}^2, \mathbb{R})$ and $\nabla H(y)$ is locally Lipschitz continuous.

For $\lambda > 0$, note that $H(\lambda y) = \int_0^{\lambda y_1} f(x)dx + \int_0^{\lambda y_2} f(x)dx$. Let $x = \lambda v$. (H2) yields

$$\begin{aligned} H(\lambda y) = &\int_0^{y_1} \lambda f(\lambda v)dv + \int_0^{y_2} \lambda f(\lambda v)dv \\ =&\lambda^2 (\int_0^{y_1} f(v)dv + \int_0^{y_2} f(v)dv) \\ =&\lambda^2 H(y). \end{aligned}$$

By virtue of mean value theorem for integrals and (H1),

$$\begin{aligned} H(y) &= f(\theta y_1)y_1 + f(\theta y_2)y_2 \\ &= \theta(f(y_1)y_1 + f(y_2)y_2) > 0 \text{ for some } \theta \in (0, 1). \end{aligned}$$

That means $min_{\|y\|=1} H(y) > 0$.

(2) By (H2), we only need to show $\|G(t, y)\| \leq \alpha(\|y\|^\beta + 1)$ or $\|G(t, y)\|^2 \leq \alpha^2(\|y\|^\beta + 1)^2$. For $\alpha > 0$ and $\beta \in [0, 1)$, notice that $|g(t, x)| \leq \alpha |x|^{\beta/2}$.

$$\|G(t, y)\|^2 = g^2(t, y_1) + g^2(t, y_2) \leq \alpha^2(|y_1|^\beta + |y_2|^\beta).$$

Set $y_1 = rcos\theta, y_2 = rsin\theta$. Thus

$$\begin{aligned} \|G(t, y)\|^2 \leq &\alpha^2(r^\beta |cos^\beta \theta| + r^\beta |sin^\beta \theta|) \\ \leq &2\alpha^2 r^\beta < \alpha^2(r^{2\beta} + 2r^\beta + 1) \\ = &\alpha^2(\|y\|^\beta + 1)^2. \end{aligned}$$

This completes the proof of Lemma 2.1. □

Let $E = \{\varphi(t) : \varphi(t) \in C^1([0, T] \to \mathbb{R}^2), \varphi(0) = \varphi(T)\}$. Define an action σ on E by

$$\sigma\varphi(t) = J\varphi(t - \tau).$$

Then $G = \{\sigma, \sigma^2, \sigma^3, \sigma^4\}$ is a compact group action over E. Moreover if $\sigma\varphi(t) = \varphi(t)$, then $\varphi(t)$ has the symmetric structure (2.2). Denote $E_0 = \{\varphi(t) \in E : \sigma\varphi(t) = \varphi(t)\}$. A direct verification shows that E_0 is a subspace of E. Since solutions of (2.3) in E_0 have the symmetric structure (2.2), they will give solutions of Eq.(1.1). So that in the next, we only find solutions of (2.3) in E_0.

As in [1], we now fix a solution of the following autonomous system

$$Jy' = \nabla H(y).$$

Let $\varphi = (\varphi_1(t), \varphi_2(t)) \in E_0$ be such that

$$J\varphi'(t) = \nabla H(\varphi) \text{ and } H(\varphi(t)) = \frac{1}{2},$$

for each $t \in \mathbb{R}$. Then $\langle \varphi'(t), J\varphi(t)\rangle < 0$ for each $t \in \mathbb{R}$. Define

$$\Omega = \{\rho\varphi(\theta) : \theta \in [0, T), \rho \in [0, 1)\},$$

which is strictly star-shaped with respect to the origin, i.e. every ray emanating from the origin crosses the orbit of φ at precisely one point.

Let $P : \bar{\Omega} \to \mathbb{R}^2$ be a continuous function such that $\forall t \in \mathbb{R}, P(\varphi(t)) \neq \varphi(t)$ and $P(\varphi(t)) \neq (0, 0)$. That means $(P - Id)|_{\partial\Omega} \neq 0$. For $\delta > 0$, define

$$\Omega_\delta = \{\frac{1}{\delta}\rho\varphi(\theta) : \theta \in [0, T), \rho \in [0, 1)\}.$$

Note that $\Omega_1 = \Omega$. Define two functions Φ and Ψ by

$$\Phi(\theta) = \int_0^T \langle F(t, \varphi(t + \theta)), \varphi(t + \theta)\rangle dt,$$
$$\Psi(\theta) = \int_0^T \langle F(t, \varphi(t + \theta)), \varphi'(t + \theta)\rangle dt,$$

where $F(t, y) = \lim_{\lambda\to+\infty} \frac{G(t,\lambda y)}{\lambda^\beta} = (h(t), h(t))$. Since $g(t, x)$ is continuous, $h(t)$ is also continuous. Therefore Φ and Ψ are well defined.

Then our main result reads as follows.

Theorem 2.1. *Assume that (H1)-(H3) hold. Then*

$$\begin{aligned}\Phi(\theta) =& \int_0^T h(t)(\varphi_1(t + \theta) + \varphi_1(t + \theta - \tau))dt,\\ \Psi(\theta) =& \int_0^T h(t)(\varphi_1'(t + \theta) + \varphi_1'(t + \theta - \tau))dt\\ =&\Phi'(\theta).\end{aligned}$$

Suppose

$$\forall\theta \in \mathbb{R}, |\Phi(\theta)| + |\Psi(\theta)| \neq 0.$$

If Ψ changes sign more than twice on the zeros of Φ in $[0, T)$, then Eq.(1.1) has a T-periodic solution.

Denote by $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ the Poincáre map for the period T associated to (2.3). In order to prove Theorem 2.1, it is enough for us to verify that topological degree for the map $\mathcal{P} - Id$ on Ω_δ never vanishes, i.e. $\deg(\mathcal{P} - Id, \Omega_\delta) \neq 0$. We need the following lemma.

Lemma 2.2. *Assume that (H1)-(H3) hold and*

$$\forall \theta \in \mathbb{R}, |\Phi(\theta)| + |\Psi(\theta)| \neq 0.$$

Then for every sufficiently small δ*,*

$$\begin{aligned}\deg(\mathcal{P} - Id, \Omega_\delta) =& 1 - \deg(\Phi, (a, a+T) \cap \{\Psi > 0\}),\\ =& 1 + \deg(\Phi, (a, a+T) \cap \{\Psi < 0\}),\end{aligned}$$

where a *is chosen such that* $\Phi(a) \neq 0$.

Proof. Since the proof is similar to Theorem 2 of [1], for the readers' convenience, we give the proof of Lemma 2.2 in Appendix. □

Proof of Theorem 2.1. Notice that $\varphi(t) \in E_0$. $\varphi = (\varphi_1, \varphi_2)$ has the symmetric structure (2.2). From (H3) $F(t, y) = \lim_{\lambda\to\infty} \frac{G(t,\lambda y)}{\lambda^\beta} = (h(t), h(t))$. Hence

$$\begin{aligned}\Phi(\theta) &= \int_0^T \langle F(t, \varphi(t+\theta)), \varphi(t+\theta)\rangle dt\\ &= \int_0^T h(t)(\varphi_1(t+\theta) + \varphi_2(t+\theta))dt\\ &= \int_0^T h(t)(\varphi_1(t+\theta) + \varphi_1(t+\theta-\tau))dt.\end{aligned}$$

A direct computation yields $\Psi(\theta) = \Phi'(\theta)$.

If $\Psi(\theta) > 0$, then $\Phi'(\theta) > 0$. This implies that the zeros of Φ in $\{\Psi > 0\}$ are all simple with positive derivative. By assumption, there are at least two of them in $[a, a+T) \cap \{\Psi > 0\}$, where a is chosen so that $\Phi(a) \neq 0$. Then it follows from Lemma 2.2 that

$$deg(\mathcal{P} - Id, \Omega_\delta) = 1 - deg(\Phi, (a, a+T) \cap \{\Psi > 0\}) \leq -1.$$

If $\Psi(\theta) < 0$, then $\Phi'(\theta) < 0$. By using similar arguments, the conclusion of Theorem 2.1 holds. Therefore the proof is complete. □

3 Applications

In order to illustrate some applications for our result, we consider the following differential delay equation

$$x' = x(t-\tau) + \lambda^\beta h(t) + \mu e(t) sin(x(t-\tau)^{\nu(x(t-\tau))}), \tag{3.1}$$

where $h(t)$ and $e(t)$ are both T-periodic, $\lambda > 0$, $\nu(x)$ is a continuous function satisfying

$$\nu(x) = \begin{cases} \beta_1, & \text{as } |x| \geq 1, \\ \beta_2, & \text{as } |x| < 1, \end{cases}$$

where $\beta_1 \leq \frac{\beta}{2}$ and $\beta_2 > \frac{\beta}{2}$.

Take $f(x) = x$ and $g(t,x) = \lambda^\beta h(t) + \mu e(t) sin(x^{\nu(x)})$. Then f and g satisfy the conditions (H1)-(H3).

It is easy to see $H(y) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2$. By a simple computation, a nontrivial periodic solution $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ for the autonomous system $Jy' = \nabla H(y)$ is $\varphi_0(t) = (\frac{1}{2}sint, \frac{1}{2}cost)$, which satisfies $H(\varphi_0(t)) = \frac{1}{2}$. Moreover take $\tau = \frac{3}{2}\pi$, we can verify easily that $J\varphi_0(t-\tau) = \varphi_0(t)$, i.e. $\sigma\varphi_0(t) = \varphi_0(t)$. Thus $\varphi_0(t) \in E_0$ and has the symmetric structure (2.2).

A direct computation shows that

$$\Phi(\theta) = \frac{1}{2}\int_0^T h(t)(sin(t+\theta) + cos(t+\theta))dt,$$
$$\Psi(\theta) = \frac{1}{2}\int_0^T h(t)(cos(t+\theta) - sin(t+\theta))dt.$$

Assume that

$$\forall\theta \in \mathbb{R}, |\Phi(\theta)| + |\Psi(\theta)| \neq 0$$

and Ψ changes sign more than twice on the zeros of Φ in $[0,T)$, then Eq.(3.1) has a T-periodic solution.

4 Appendix

In this section, we give a skeleton of the proof of Lemma 2.2. The main idea comes from [1].

For the continuous map P, we can define two continuous functions $R, \Theta : \mathbb{R} \to \mathbb{R}$ such that

$$P(\varphi(t)) = R(t)\varphi(t + \Theta(t))$$

with $R(t) > 0$ for all $t \in \mathbb{R}$.

Let $\tau_{opp}(t)$ denote the function defined by

$$\tau_{opp}(t) \in (0,\tau) \text{ and } \frac{\varphi(t)}{\|\varphi(t)\|} = -\frac{\varphi(t+\tau_{opp}(t))}{\|\varphi(t+\tau_{opp}(t))\|} \tag{4.1}$$

We have the following theorem.

Theorem 4.1. *(Theorem 1 of [1]) Let $P : \bar{\Omega} \to \mathbb{R}^2$ be continuous and such that*

$$P(\varphi(t)) = R(t)\varphi(t + \Theta(t)),$$

where $R, \Theta : \mathbb{R} \to \mathbb{R}$ *are continuous and, for every* $t \in \mathbb{R}$

$$R(t) > 0 \tag{4.2}$$

$$\tau_{opp}(t) - \tau < \Theta(t) < \tau_{opp}(t) \tag{4.3}$$

and

$$|R(t) - 1| + |\Theta(t)| \neq 0. \tag{4.4}$$

Then,

$$\begin{aligned}\deg(P - Id, \Omega) &= 1 + \deg(\Theta, (a, a+\tau) \cap \{R > 1\}) \\ &= 1 - \deg(\Theta, (a, a+\tau) \cap \{R < 1\}),\end{aligned}$$

where a *is chosen so that* $\Theta \neq 0$*.*

Now for some $\delta > 0$, by change of variables $z = \delta y$, (2.3) becomes

$$Jz' = \nabla H(z) + \delta G(t, \frac{z}{\delta}). \tag{4.5}$$

Denote by $\tilde{\mathcal{P}_\delta} : \mathbb{R}^2 \to \mathbb{R}^2$ the Poincáre map associated to (4.5). Then one has $\tilde{\mathcal{P}_\delta}(z) = \delta\mathcal{P}(\frac{z}{\delta})$. Moreover

$$\deg(\mathcal{P} - Id, \Omega_\delta) = \deg(\tilde{\mathcal{P}_\delta} - Id, \Omega).$$

For each $\theta_0 \in [0, \tau)$, write

$$\tilde{\mathcal{P}_\delta}(\varphi(\theta_0)) = r_1\varphi(\theta_1).$$

According to [1], we can evaluate θ_1 and r_1 as

$$\begin{aligned}\theta_1 &= \theta_0 + \delta\int_0^T \frac{1}{r(t)}\langle G(t, \frac{r(t)}{\delta}\varphi(t+\theta(t))), \varphi(t+\theta(t))\rangle \\ r_1 &= 1 - \delta\int_0^T \langle G(t, \frac{r(t)}{\delta}\varphi(t+\theta(t))), \varphi'(t+\theta(t))\rangle,\end{aligned}$$

where $r(t) > 0$ and $r(0) = 1$ and $z(t) = r(t)\varphi(t + \theta(t))$, where $z(t)$ is a solution of (4.5).

We have the following lemma.

Lemma 4.1. *(Lemma 2 of [1]) We have*

$$\begin{aligned}\theta_1 &= \theta_0 + \delta^{1-\beta}[\Phi(\theta_0) + R_1(\theta_0, \delta)] \\ r_1 &= 1 - \delta^{1-\beta}[\Psi(\theta_0) + R_2(\theta_0, \delta)],\end{aligned}$$

where Φ *and* Ψ *are defined in Section 2,* R_1 *and* R_2 *are such that*

$$\lim_{\delta\to 0+} R_1(\theta_0, \delta) = \lim_{\delta\to 0+} R_2(\theta_0, \delta) = 0$$

uniformly for $\theta_0 \in [0, \tau)$*.*

The proof of Lemma 4.1 bases on Lemma 2.1. Here we omit the details. Now we give a brief proof of Lemma 2.2.

Proof of Lemma 2.2. Let $rot(\tilde{\mathcal{P}}_\delta - Id, \partial\Omega)$ denote the rotation number of $\tilde{\mathcal{P}}_\delta - Id$ on the curve $\partial\Omega$. In fact,

$$\deg(\tilde{\mathcal{P}}_\delta - Id, \Omega) = \deg(\tilde{\mathcal{P}}_\delta - Id, \partial\Omega).$$

Since $|\Phi(\theta)| + |\Psi(\theta)| \neq 0$, we can chose a constant $c > 0$ such that

$$|\Phi(\theta)| + |\Psi(\theta)| \geq 2c.$$

For $\lambda \in [0,1]$, consider the functions $P_{\delta,\lambda} : \partial\Omega \to \mathbb{R}^2$ defined by

$$P_{\delta,\lambda}(\varphi(\theta_0)) = r_1^\lambda \varphi(\theta_1^\lambda).$$

Now we prove $P_{\delta,\lambda} - Id$ never vanishes on $\partial\Omega$ for each $\lambda \in [0,1]$ and for δ sufficiently small, i.e. $P_{\delta,\lambda}$ has no fixed points on $\partial\Omega$. By Lemma 4.1, we can chose $\bar{\delta} > 0$ such that

$$0 < \delta < \bar{\delta} \Rightarrow |R_1(\theta_0, \delta)| < c, |R_2(\theta_0, \delta)| < c.$$

Note that

$$\begin{aligned}
|\Phi(\theta_0)| &= \left| \frac{\theta_1^\lambda - \theta_0}{\delta^{1-\beta}} - \lambda R_1(\theta_0, \delta) \right| < \frac{|\theta_1^\lambda - \theta_0|}{\delta^{1-\beta}} + c \\
|\Psi(\theta_0)| &= \left| \frac{r_1^\lambda - \theta_0}{\delta^{1-\beta}} - \lambda R_2(\theta_0, \delta) \right| < \frac{|r_1^\lambda - \theta_0|}{\delta^{1-\beta}} + c.
\end{aligned}$$

One has $(\theta_1^\lambda, r_1^\lambda) \neq (\theta_0, 1)$, or $|\Phi(\theta)| + |\Psi(\theta)| < 2c$, a contradiction. So we have the above conclusion. Thus

$$rot(P_{\delta,1} - Id, \partial\Omega) = rot(P_{\delta,0} - Id, \partial\Omega).$$

Let us focus on $P_{\delta,0} : \partial\Omega \to \mathbb{R}^2$. It is easy to see that $P_{\delta,0} = r_1^0 \varphi(\theta_1^0)$, where $r_1^0 = 1 - \delta^{1-\beta}\Psi(\theta_0)$ and $\theta_1^0 = \theta_0 + \delta^{1-\beta}\Phi(\theta_0)$. We extend $P_{\delta,0}$ to a continuous function $P : \Omega \to \mathbb{R}^2$. So that

$$P(\varphi(\theta_0)) = R(\theta_0)\varphi(\theta_0 + \Theta(\theta_0))$$

with $R(\theta_0) = 1 - \delta^{1-\beta}\Psi(\theta_0)$ and $\Theta(\theta_0) = \delta^{1-\beta}\Phi(\theta_0)$. For sufficiently small δ, we can check (4.2), (4.3) and (4.4) are satisfied. Therefore by Theorem 4.1, we have

$$\begin{aligned}
\deg(P - Id, \Omega) &= 1 - \deg(\Theta, (a, a+\tau) \cap \{R < 1\}) \\
&= 1 - \deg(\Phi, (a, a+\tau) \cap \{\Psi > 0\}),
\end{aligned}$$

where a is chosen such that $\Phi(a) \neq 0$.

It follows from the excision property of topological degree that the second formula holds. The proof of Lemma 2.2 is complete. □

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