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On the intervals of a third between Farey fractions

by
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Dedicated to the memory of Laurenţiu Panaitopol (1940-2008)
on the occasion of his 70th anniversary

Abstract

The spacing distribution between Farey points has drawn attention in recent years. It was found that the gaps $\gamma_{j+1} - \gamma_j$ between consecutive elements of the Farey sequence produce, as $Q \to \infty$, a limiting measure. Numerical computations suggest that for any $d \geq 2$, the gaps $\gamma_{j+d} - \gamma_j$ also produce a limiting measure whose support is distinguished by remarkable topological features. Here we prove the existence of the spacing distribution for $d = 2$ and characterize completely the corresponding support of the measure.

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1 Introduction

Let $\mathfrak{F}_Q = \{\gamma_1, \ldots, \gamma_N\}$ be the Farey sequence of order Q, which is defined to be the set of all subunitary irreducible fractions with denominators $\leq Q$, arranged in ascending order. For any interval $\mathcal{I} \subset [0,1]$, we write $\mathfrak{F}_Q(\mathcal{I}) = \mathfrak{F}_Q \cap \mathcal{I}$. The cardinality of $\mathfrak{F}_Q(\mathcal{I})$ is well known to be $N_{\mathcal{I}}(Q) = 3|\mathcal{I}|Q^2/\pi^2 + O(Q\log Q)$. When $\mathcal{I} = [0,1]$ we write shortly $N(Q)$ instead of $N_{[0,1]}(Q)$. Since $\mathfrak{F}_Q$ contains a large number of fractions obtained by a combined process of division, sieving and sorting of integers from $[1, Q]$, one would apriori expect little or even no special structure in the set of all differences between consecutive fractions (which we also call *intervals of a second*). Though, this expectation is not fulfilled. This is sustained from many points of view by a series of authors, such as Franel [4],

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Kanemitsu, Sita Rama Chandra Rao and Siva Rama Sarma [9], Hall and Tenenbaum [7], [8], Hall [5], Augustin, Boca and the authors [1], who have studied the set of gaps between consecutive Farey fractions. A regularity is expected also in the set of larger gaps $\gamma^{(d+1)} - \gamma'$, where γ' runs over $\{\gamma_1, \ldots, \gamma_{N-d}\}$ and $d \geq 2$. (We use up-scripts, such as $\gamma', \gamma'', \gamma''', \ldots$ to write consecutive elements of $\mathfrak{F}_Q$.) It is our object to treat here the case $d = 2$, that is, the case of *intervals of a third.*

Geometrical representations of the set of pairs of neighbor intervals of fractions from $\mathfrak{F}_Q$ created for different values of Q reveals sets of points whose density concentrates on different parts of the plane. The aesthetical qualities of the pictures catches attention immediately. For any $d \geq 1$ they look like a swallow and the main topological distinctions are in the number of folds of the tail. Thus, when $d = 1$ (neighbor pairs of intervals of a second) the swallow has a one-fold tail (see [1]). When $d = 2$, the case treated in the present paper, the swallow has a two-fold tail (see Figure 1) and in Section 3 we have calculated explicitely the equations of the frontier. In the cases $d \geq 3$ the tail appears always to have a three-folded tail, but this is more complex and its characterization will appear in a separate paper.

Given N real numbers $x_1 \leq x_2 \leq \cdots \leq x_N$ with mean spacing 1, we consider *the h-th level of intervals of a third probability* $\mu_{2,h}$ on R_+^h, defined, for $f \in C_c([0,\infty))$, by

$$\int_{[0,\infty)^h} f d\mu_{2,h} = \frac{1}{N-h-1} \sum_{j=1}^{N-h-1} f(x_{j+2}-x_j, x_{j+3}-x_{j+1}, \ldots, x_{j+h+1}-x_{j+h-1}).$$

In our case, we normalize $\mathfrak{F}_Q(\mathcal{I})$ to get the sequence $x_j = N(Q,\mathcal{I})\gamma_j/|\mathcal{I}|$, $1 \leq j \leq N(Q,\mathcal{I})$ with mean spacing equal to one. Accordingly, we get the sequence $(\mu_{2,h}^{Q,\mathcal{I}})_{Q\geq 1}$ of the h-th level of intervals of a third probabilities on $[0,\infty)^h$. We show that this sequence converges, as $Q \to \infty$, to a probability measure $\mu_{2,h}$, which is independent of $\mathcal{I}$, and can be expressed explicitly.

For any $\gamma_i = a_i/q_i$ and $\gamma_j = a_j/q_j$ in $\mathfrak{F}_Q$, we set $\Delta(\gamma_i,\gamma_j) = \Delta(i,j) = -\left|\begin{smallmatrix} a_i & a_j \\ q_i & q_j \end{smallmatrix}\right|$. This is the numerator of the difference $\gamma_j - \gamma_i$. It is well known that $\Delta(\gamma',\gamma'') = -1$ for any consecutive elements of $\mathfrak{F}_Q$, and it turns out that this equality is responsible for the existence of the h-spacing distribution of the Farey sequence. Though, this relation is no longer true for larger intervals, but there is a convenient replacement. To see this, let us note that a Farey fraction can be uniquely determined by its two predecessors. Indeed, if $\frac{a'}{q'} < \frac{a''}{q''} < \frac{a'''}{q'''}$ are consecutive fractions of $\mathfrak{F}_Q$, we have $a''' = ka'' - a'$ and $q''' = kq'' - q'$, where $k = \Delta(\gamma',\gamma''') = \left[\frac{q'+Q}{q''}\right]$.

The basic idea of our procedure is to parametrize the set of h-tuples of intervals of a third in terms of just two variables that run over a completely described

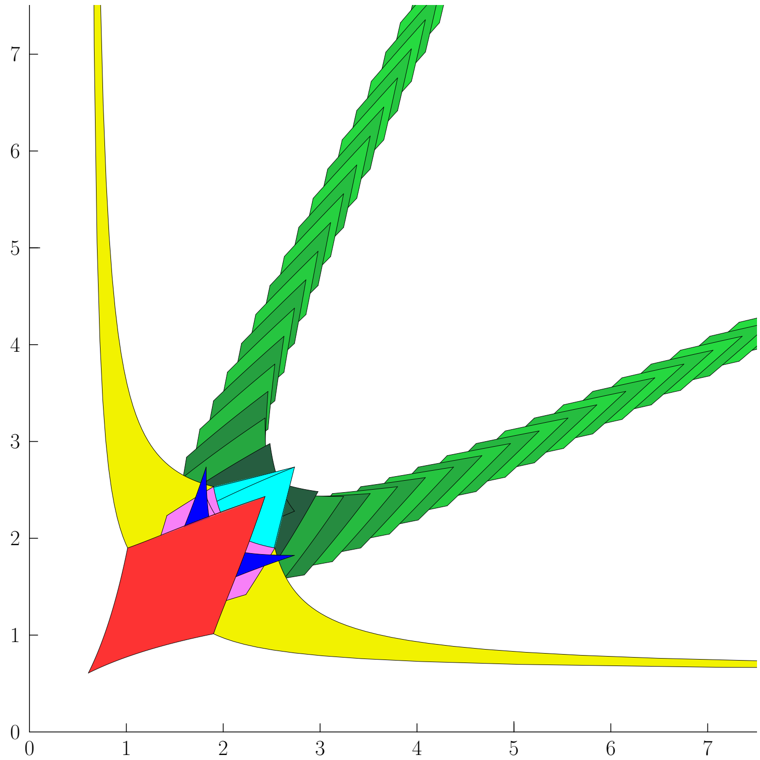

Figure 1: The support of $\mu_{2,2}$.

domain. The set of pairs of consecutive denominators of fractions in $\mathfrak{F}_Q$ are exactly the elements of

$$\left\{(q', q''):\ 1 \le q', q'' \le Q,\ q' + q'' > Q \text{ and } (q', q'') = 1\right\}.$$

Since we are mainly interested in what happens when $Q \to \infty$, we reduce the scale Q times, and consider the background triangle $\mathcal{T} = \left\{(x, y):\ 0 < x \le 1,\ x + y > 1\right\}$, called *the Farey triangle*. We split it into a series of polygons as follows. Firstly, for each $(x, y) \in \mathbb{R}^2$, we set $L_0(x, y) = x$, $L_1(x, y) = y$, and then, for $i \ge 2$, we define recursively:

$$L_i(x, y) = \left[\frac{1 + L_{i-2}(x, y)}{L_{i-1}(x, y)}\right] L_{i-1}(x, y) - L_{i-2}(x, y)\,.$$

Then, as in [3], we consider the map

$$\mathbf{k} : \mathcal{T} \to (\mathbb{N}^*)^h, \qquad \mathbf{k}(x, y) = \big(k_1(x, y), \ldots, k_h(x, y)\big),$$

where $k_i(x,y) = \left[\frac{1+L_{i-1}(x,y)}{L_i(x,y)}\right]$. The functions $k_i(x,y)$ are locally constant, and the subsets of $\mathcal{T}$ on which they are constant plays a special role. Thus, for any $\mathbf{k} \in (\mathbb{N}^*)^h$, we get the convex polygon

$$\mathcal{T}_{\mathbf{k}} = \left\{(x,y) \in \mathcal{T}\colon\ \mathbf{k}(x,y) = \mathbf{k}\right\}.$$

Notice that $\mathcal{T} = \bigcup\limits_{\mathbf{k}\in(\mathbb{N}^*)^h} \mathcal{T}_{\mathbf{k}}$ and $\mathcal{T}_{\mathbf{k}} \cap \mathcal{T}_{\mathbf{k}'} = \emptyset$ whenever $\mathbf{k} \neq \mathbf{k}'$.

Next we consider the application $\Phi_{2,h}\colon\ \mathcal{T} \to (0,\infty)^h$ defined by

$$\Phi_{2,h}(x,y) = \frac{3}{\pi^2}\left(\frac{k_1(x,y)}{L_0(x,y)L_2(x,y)}, \frac{k_2(x,y)}{L_1(x,y)L_3(x,y)}, \dots, \frac{k_h(x,y)}{L_{h-1}(x,y)L_{h+1}(x,y)}\right).$$

Our main result shows that, indeed, for $Q \to \infty$, the sequence $(\mu_{2,h}^{Q,\mathcal{I}})_{Q\geq 1}$ converges to a measure and $\Phi_{2,h}(x,y)$ is the needed tool to describe its support.

THEOREM 1. *The sequence $(\mu_{2,h}^{Q,\mathcal{I}})_{Q\geq 1}$ converges weakly to a probability measure $\mu_{2,h}$, which is independent of $\mathcal{I}$. The support $\mathcal{D}_{2,h}$ of $\mu_{2,h}$ is the closure of the range of $\Phi_{2,h}$, and*

$$\mu_{2,h}(\mathcal{C}) = 2\mathrm{Area}(\Phi_{2,h}^{-1}(\mathcal{C})),$$

for any parallelepiped $\mathcal{C} = \prod_{j=1}^{h}(\alpha_j,\beta_j) \subset (0,\infty)^h$.

In Table 1 from Section 3 we provide explicit formulae for all the pieces that form $\mathcal{D}_{2,2}$.

2 The Existence of the Limiting Measure

It is plain that in order to prove Theorem 1, it suffices to see the effect of $\mu_{2,h}$ on bounded parallelepipeds. For any $\mathcal{C} = \prod_{j=1}^{h}(\alpha_j,\beta_j) \subset (0,\infty)^h$, we define

$$\mu_{2,h}^{Q,\mathcal{I}}(\mathcal{C}) := \frac{1}{N_{\mathcal{I}}(Q)} \cdot \#\left\{\gamma_j \in \mathfrak{F}_{\mathcal{I}}(Q)\colon \begin{array}{l} \frac{\alpha_i|\mathcal{I}|}{N_{\mathcal{I}}(Q)} < \gamma_{j+i+1} - \gamma_{j+i-1} < \frac{\beta_i|\mathcal{I}|}{N_{\mathcal{I}}(Q)}, \\ \text{for } i = 1,\dots,h \end{array}\right\}.$$

We have to show that the sequence $\{\mu_{2,h}^{Q,\mathcal{I}}\}_Q$ is convergent when $Q \to \infty$ and the limit is independent of $\mathcal{I}$. In the beginning we treat the case of the complete interval $\mathcal{I} = [0,1]$.

2.1 The case $\mathcal{I} = [0,1]$

In the following we write shortly $\mu_{2,h}^{Q}$ instead of $\mu_{2,h}^{Q,[0,1]}$.

With the notations from the Introduction, we see that $\gamma_{j+i+1} - \gamma_{j+i-1} = k_{j+i}/q_{j+i+1}q_{j+i-1}$. Then $\mu_{2,h}^{Q}(\mathcal{C})$ can be written as

$$\mu_{2,h}^{Q}(\mathcal{C}) = \frac{1}{N(Q)} \cdot \#\left\{\gamma_j \in \mathfrak{F}(Q)\colon \begin{array}{l} \frac{N(Q)}{\beta_i} < \frac{q_{j+i+1}q_{j+i-1}}{k_{j+i}} < \frac{N(Q)}{\alpha_i}, \\ \text{for } i = 1,\dots,h \end{array}\right\}. \tag{1}$$

Knowing that $q_{j+i} = QL_i(q_j/Q, q_{j+1}/Q)$, we consider the set

$$\Omega^Q(\mathcal{C}) = \left\{ (x,y) \in Q\mathcal{T}\colon \begin{array}{l} \frac{N(Q)}{Q^2\beta_i} < \frac{L_{i-1}\left(\frac{x}{Q},\frac{y}{Q}\right)L_{i+1}\left(\frac{x}{Q},\frac{y}{Q}\right)}{k_i\left(\frac{x}{Q},\frac{y}{Q}\right)} < \frac{N(Q)}{Q^2\alpha_i}, \\ \text{for } i = 1,\dots,h \end{array} \right\}. \tag{2}$$

Since neighbor denominators in $\mathfrak{F}_Q$ are always coprime, relation (1) turns into

$$\mu^Q_{2,h}(\mathcal{C}) = \frac{1}{N(Q)} \cdot \#\left\{ (x,y) \in \Omega^Q(\mathcal{C}) \cap \mathbf{N}^2\colon \ \gcd(x,y) = 1 \right\}.$$

Next, we select the points with coprime coordinates using Möbius summation (cf. [1, Lemma 2]), and we find that

$$\mu^Q_{2,h}(\mathcal{C}) = \frac{1}{N(Q)} \Big(\frac{6}{\pi^2} \operatorname{Area}\big(\Omega^Q(\mathcal{C})\big) + O\big(\operatorname{length}\big(\partial\Omega^Q(\mathcal{C})\big) \log Q \big) \Big). \tag{3}$$

Splitting $\mathcal{T}$ into the series of polygons $\mathcal{T}_{\mathbf{k}}$, we see that the error term in (3) is $O(Q\log Q)$. In the main term, we replace $\Omega^Q(\mathcal{C})$ by the bounded set $\Omega(\mathcal{C}) = \Omega^Q(\mathcal{C})/Q$. These yield

$$\mu^Q_{2,h}(\mathcal{C}) = \frac{6Q^2}{\pi^2 N(Q)} \operatorname{Area}\big(\Omega(\mathcal{C})\big) + O_{\mathcal{C}}\Big(\frac{\log Q}{Q}\Big). \tag{4}$$

It remains to replace in (4) the set $\Omega(\mathcal{C})$ by a set as in (2), but with bounds independent of Q in the corresponding inequalities. This set is

$$\mathfrak{O}(\mathcal{C}) := \left\{ (x,y) \in Q\mathcal{T}\colon \begin{array}{l} \frac{3}{\pi^2\beta_i} < \frac{L_{i-1}(x,y)L_{i+1}(x,y)}{k_i(x,y)} < \frac{3}{\pi^2\alpha_i}, \\ \text{for } i = 1,\dots,h \end{array} \right\}. \tag{5}$$

Notice that $\mathfrak{O}(\mathcal{C})$ is exactly $\Phi^{-1}_{2,h}(\mathcal{C})$. The replacement does not change the error term because, via $N(Q) = 3Q^2/\pi^2 + O(Q\log Q)$, we have:

$$\max_{1\le i\le h} \Big\{ \Big| \frac{N(Q)}{\alpha_i Q^2} - \frac{3}{\pi^2\alpha_i} \Big|, \Big| \frac{N(Q)}{\beta_i Q^2} - \frac{3}{\pi^2\beta_i} \Big| \Big\} = O_{\mathcal{C}}\Big(\frac{\log Q}{Q}\Big), \tag{6}$$

which implies

$$\operatorname{Area}\big(\Omega(\mathcal{C}) \triangle \mathfrak{O}(\mathcal{C})\big) = O_{\mathcal{C}}\Big(\frac{\log Q}{Q}\Big). \tag{7}$$

Therefore, by (5) and (7), we get

$$\mu^Q_{2,h}(\mathcal{C}) = 2\operatorname{Area}\big(\mathfrak{O}(\mathcal{C})\big) + O_{\mathcal{C}}\Big(\frac{\log Q}{Q}\Big). \tag{8}$$

In particular, this gives $\mu_{2,h}(\mathcal{C}) = \lim_{Q\to\infty} \mu^Q_{2,h}(\mathcal{C}) = 2\operatorname{Area}\big(\mathfrak{O}(\mathcal{C})\big)$, concluding the proof of the theorem when $\mathcal{I} = [0,1]$.

2.2 The short interval case

Suppose now that $\mathcal{I} \subset [0,1]$ is fixed. In order to impose the condition that only the fractions from $\mathcal{I}$ are involved in the calculations, we employ the fundamental property of neighbor fractions in $\mathfrak{F}_Q$. This says that if $\gamma' = a'/q'$ and $\gamma'' = a''/q''$ are consecutive then $a''q' - a'q'' = 1$. Consequently, $a'' \equiv (q')^{-1} \pmod{q''}$, and this allows us to write the fraction a''/q'' in terms of q' and q''. Thus

$$a''/q'' \in \mathcal{I} \iff (q')^{-1} \pmod{q''} \in q''\mathcal{I}\,.$$

This time we have to estimate

$$\mu_{2,h}^{Q,\mathcal{I}}(\mathcal{C}) = \frac{1}{N_{\mathcal{I}}(Q)} \cdot \#\,\Omega_{\mathcal{I}}^{Q}\,, \tag{9}$$

where

$$\Omega_{\mathcal{I}}^{Q} = \left\{ (q',q'') \in Q\mathcal{T}\colon \begin{array}{l} \frac{N_{\mathcal{I}}(Q)}{|\mathcal{I}|Q^2\beta_i} < \frac{L_{i-1}\left(\frac{q'}{Q},\frac{q''}{Q}\right)L_{i+1}\left(\frac{q'}{Q},\frac{q''}{Q}\right)}{k_i\left(\frac{q'}{Q},\frac{q''}{Q}\right)} < \frac{N_{\mathcal{I}}(Q)}{|\mathcal{I}|Q^2\alpha_i}, \\ \text{for } i=1,\dots,h; \quad (q')^{-1} \pmod{q''} \in q''\mathcal{I} \end{array} \right\}.$$

We may write (9) as

$$\mu_{2,h}^{Q,\mathcal{I}}(\mathcal{C}) = \frac{1}{N_{\mathcal{I}}(Q)} \sum_{q=1}^{Q} N_q(\mathcal{J}_{\mathcal{C}}^{Q}(q), q\mathcal{I})\,, \tag{10}$$

where

$$N_q(\mathcal{J}_1,\mathcal{J}_2) = \#\big\{(m,n) \in \mathcal{J}_1 \times \mathcal{J}_2\colon\ mn \equiv 1 \pmod{q}\big\}\,,$$

for any $\mathcal{J}_1, \mathcal{J}_2 \subset [0, Q-1]$ and

$$\mathcal{J}_{\mathcal{C}}^{Q}(q) = \left\{ x \in (Q-q, Q]\colon \begin{array}{l} \frac{N_{\mathcal{I}}(Q)}{|\mathcal{I}|Q^2\beta_i} < \frac{L_{i-1}\left(\frac{q'}{Q},\frac{q''}{Q}\right)L_{i+1}\left(\frac{q'}{Q},\frac{q''}{Q}\right)}{k_i\left(\frac{q'}{Q},\frac{q''}{Q}\right)} < \frac{N_{\mathcal{I}}(Q)}{|\mathcal{I}|Q^2\alpha_i}, \\ \text{for } i=1,\dots,h \end{array} \right\}.$$

For the best available technique to estimate the size of $N_q(\mathcal{J}_1,\mathcal{J}_2)$ one requires bounds for Kloosterman sums (cf. [2]). This is done when $\mathcal{J}_1$ and $\mathcal{J}_2$ are intervals, but it may be easily extended for finite unions of subintervals of $[0, q-1]$ (as the set $\mathcal{J}_{\mathcal{C}}^{Q}(q)$ is), even with the same formula. For our needs here, it suffices a version with a slightly weaker term:

$$N_q(\mathcal{J}_{\mathcal{C}}^{Q}(q), q\mathcal{I}) = \frac{\varphi(q)|\mathcal{J}_{\mathcal{C}}^{Q}(q)| \cdot |\mathcal{I}|}{q} + O_{\mathcal{C},\varepsilon}\big(q^{1/2+\varepsilon}\big)\,. \tag{11}$$

Inserting (11) into (10), we get

$$\mu_{2,h}^{Q,\mathcal{I}}(\mathcal{C}) = \frac{|\mathcal{I}|}{N_{\mathcal{I}}(Q)} \sum_{q=1}^{Q} \frac{\varphi(q)|\mathcal{J}_{\mathcal{C}}^{Q}(q)|}{q} + O_{\mathcal{C},\varepsilon}\big(Q^{-1/2+\varepsilon}\big)\,. \tag{12}$$

To calculate the sum in (12), we employ the Euler-MacLaurin formula, noticing the fact that $|\mathcal{J}_{\mathcal{C}}^{Q}(q)|$, as a function of q, is piecewise continuous differentiable on $[0,1]$. We obtain

$$\begin{aligned}\sum_{q=1}^{Q}\frac{\varphi(q)|\mathcal{J}_{\mathcal{C}}^{Q}(q)|}{q} =& \frac{1}{\zeta(2)}\int\limits_{1}^{Q}|\mathcal{J}_{\mathcal{C}}^{Q}(q)|\,dq \\ &+O\biggl(\Bigl(\sup_{1\le q\le Q}|\mathcal{J}_{\mathcal{C}}^{Q}(q)|+\int\limits_{1}^{Q}\frac{\partial}{\partial q}|\mathcal{J}_{\mathcal{C}}^{Q}(q)|\,dq\Bigr)\log Q\biggr)\,.\end{aligned} \tag{13}$$

The size of the error term is estimated observing, firstly, that $|\mathcal{J}_{\mathcal{C}}^{Q}(q)|\le Q$. Secondly, by the definition of $\mathcal{J}_{\mathcal{C}}^{Q}(q)$ it follows that there exists a partition of $[1,Q]$ in finitely many intervals with the property that the cardinality of $\mathcal{J}_{\mathcal{C}}^{Q}(q)$ is monotonic on each of them. Therefore

$$\int\limits_{1}^{Q}\frac{\partial}{\partial q}|\mathcal{J}_{\mathcal{C}}^{Q}(q)|\,dq = O_{\mathcal{C}}\bigl(Q\bigr)\,. \tag{14}$$

Then, forgathering (13), (14), (6) in (12) and using again the estimate $N_{\mathcal{I}}(Q)=3|\mathcal{I}|Q^2/\pi^2+O(Q\log Q)$, we obtain

$$\begin{aligned}\mu_{2,h}^{Q,\mathcal{I}}(\mathcal{C}) &= \frac{6|\mathcal{I}|}{\pi^2 N_{\mathcal{I}}(Q)}\int\limits_{1}^{Q}|\mathcal{J}_{\mathcal{C}}^{Q}(q)|\,dq + O_{\mathcal{C},\varepsilon}\bigl(Q^{-1/2+\varepsilon}\bigr) \\ &= 2\,\mathrm{Area}\,\bigl(\mathfrak{O}(Q)\bigr)+O_{\mathcal{C},\varepsilon}\bigl(Q^{-1/2+\varepsilon}\bigr)\,.\end{aligned}$$

This concludes the proof of the theorem.

3 The Support of the Limiting Measure

For $h=1$, we have $\mathcal{D}_{2,1}=[6/\pi^2,\infty)$. For $h\ge 2$, by Theorem 1, it follows that $\mathcal{D}_{2,h}$ is a countable union of hyper-surfaces in $[6/\pi^2,\infty)^h$.

The support $\mathcal{D}_{2,h}$ has some striking features. Let us see them in the case $h=2$. We write $\mathbf{k}=(k,l)$ and observe that

$$\mathcal{T}_{k,l}=\Bigl\{(x,y)\in\mathcal{T}_k\colon\ \frac{1+(l+1)x}{k(l+1)-1}<y\le\frac{1+lx}{kl-1}\Bigr\}\,.$$

Roughly speaking, by definition we find that $\mathcal{T}_k$ corresponds to the set of 3-tuples $(\gamma',\gamma'',\gamma''')$ of consecutive elements of $\mathfrak{F}_Q$ with the property that $\Delta(\gamma',\gamma''')=k$. Similarly, $\mathcal{T}_{k,l}$ corresponds to the set of 4-tuples $(\gamma',\gamma'',\gamma''',\gamma^{iv})$ of consecutive elements of $\mathfrak{F}_Q$ with the property that $\Delta(\gamma',\gamma''')=k$ and $\Delta(\gamma'',\gamma^{iv})=l$. We remark that $\mathcal{T}_{1,1}=\emptyset$, and also $\mathcal{T}_{k,l}=\emptyset$ whenever both k and l are ≥ 2, except when $(k,l)\in\{(2,2);(2,3);(2,4);(3,2);(4,2)\}$. Notice that the symmetry of the Farey sequence of order Q with respect to $1/2$ produces a sort of balance between the polygons $\mathcal{T}_{k,l}$ and $\mathcal{T}_{l,k}$.

Then the support $\mathcal{D}_{2,h}$ is the closure of the image of the function $\Phi_{2,2}$, which can be written as

$$\Phi_{2,2}(x,y)=\frac{3}{\pi^2}\Big(\frac{k}{xz},\,\frac{l}{yt}\Big)\,,$$

in which $z=x-ky$, $t=y-lt$, for $(x,y)\in\mathcal{T}_{k,l}$. A tedious, but elementary, computation allows us to find precisely the boundaries of $\Phi_{2,2}(\mathcal{T}_{k,l})$. The image obtained is shown in Figure 1 and the equations are listed in Table 1. All the functions that produce the equations of the boundaries of $\Phi_{2,2}(\mathcal{T}_{k,l})$ are either of the form $\frac{3}{\pi^2}\cdot\frac{et}{a+bt+c\sqrt{t(t-d)}}$, with t in a certain interval that might be unbounded, or the symmetric with respect to $x=y$ of such a curve. Here a,b,c,d,e are integers.

We conclude by making a few remarks. Firstly, we mention that $\Phi_{2,2}$ has a symmetrization effect, namely, it makes $\Phi_{2,2}(\mathcal{T}_{n,m})$ and to $\Phi_{2,2}(\mathcal{T}_{m,n})$ to be symmetric with respect to the first diagonal $y=x$, for any $m,n\geq 1$. The diamond[1] $\Phi_{2,2}(\mathcal{T}_{2,2})$ is the single nonempty domain $\Phi_{2,2}(\mathcal{T}_{k,l})$ that has $y=x$ as axis of symmetry. The top of the beak of the swallow $\mathcal{D}_{2,h}$ has coordinates $(6/\pi^2,6/\pi^2)$. The asymptotes of the wings are $y=6/\pi^2$ and $x=6/\pi^2$. The highest density is on a region situated in the neck, where many components of the swallow overlap partially or completely.

Table 1 below lists all the equations of the boundaries of $\Phi_{2,2}(\mathcal{T}_{k,l})$. In the head of the table MN represents an edge of $\mathcal{T}_{k,l}$ (listed in counterclockwise order, starting either from the East or from the North side) and $g_{MN}(t)$ is a parametrization of $\Phi_{2,2}(MN)$.

Table 1: The edges of $\mathcal{D}_{2,2}$.

k,l	MN	$\frac{\pi^2}{3}g_{MN}(t)$	the domain of t
$1,2$	$(\frac{1}{3},1);\ (0,1)$	$\frac{2t}{\sqrt{t(t-4)}}$	$\frac{9}{2}\leq t\leq\infty$
$1,2$	$(0,1);\ (\frac{1}{5},\frac{4}{5})$	$\frac{16t}{-12+3t+5\sqrt{t(t-8)}}$	$\frac{25}{3}\leq t\leq\infty$
$1,2$	$(\frac{1}{5},\frac{4}{5});\ (\frac{1}{3},1)$	$\frac{16t}{-12-3t+5\sqrt{t(t+8)}}$	$\frac{9}{2}\leq t\leq\frac{25}{3}$
$1,3$	$(\frac{1}{2},1);\ (\frac{1}{3},1)$	$\frac{6t}{t+3\sqrt{t(t-4)}}$	$4\leq t\leq\frac{9}{2}$

continued on next page

[1]Remark that the edges of $\Phi_{2,2}(\mathcal{T}_{2,2})$ are close to being, but are not exactly straight lines. The same applies for the edges of the diamonds in the tail.

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k, l	MN	$\frac{\pi^2}{3} g_{MN}(t)$	the domain of t
$1,3$	$(\frac{1}{3},1);\ (\frac{1}{5},\frac{4}{5})$	$\frac{12t}{-t+3\sqrt{t(t+8)}}$	$\frac{9}{2} \le t \le \frac{25}{3}$
$1,3$	$(\frac{1}{5},\frac{4}{5});\ (\frac{1}{4},\frac{3}{4})$	$\frac{24t}{-20+7t+9\sqrt{t(t-8)}}$	$8 \le t \le \frac{25}{3}$
$1,3$	$(\frac{1}{4},\frac{3}{4});\ (\frac{2}{7},\frac{5}{7})$	$\frac{24t}{-20+7t-9\sqrt{t(t-8)}}$	$8 \le t \le \frac{49}{6}$
$1,3$	$(\frac{2}{7},\frac{5}{7});\ (\frac{1}{2},1)$	$\frac{54t}{-24-7t+11\sqrt{t(t+12)}}$	$4 \le t \le \frac{49}{6}$
$1,4$	$(\frac{3}{5},1);\ (\frac{1}{2},1)$	$\frac{4t}{t-2\sqrt{t(t-4)}}$	$\frac{25}{6} \le t \le 4$
$1,4$	$(\frac{1}{2},1);\ (\frac{2}{7},\frac{5}{7})$	$\frac{12t}{-t+2\sqrt{t(t+12)}}$	$4 \le t \le \frac{49}{6}$
$1,4$	$(\frac{2}{7},\frac{5}{7});\ (\frac{1}{3},\frac{2}{3})$	$\frac{32t}{-28+11t-13\sqrt{t(t-8)}}$	$\frac{49}{6} \le t \le 9$
$1,4$	$(\frac{1}{3},\frac{2}{3});\ (\frac{3}{5},1)$	$\frac{128t}{-40-13t+19\sqrt{t(t+16)}}$	$\frac{25}{6} \le t \le 9$
$1,\ l \ge 5$	$(\frac{l-1}{l+1},1);\ (\frac{l-2}{l},1)$	$\frac{2lt}{(l-2)t-l\sqrt{t(t-4)}}$	$\frac{l^2}{2(l-2)} \le t \le \frac{(l+1)^2}{2(l-1)}$
$1,\ l \ge 5$	$(\frac{l-2}{l},1);\ (\frac{l-3}{l+1},\frac{l-1}{l+1})$	$\frac{2l(l-1)t}{(2-l)t+l\sqrt{t(t+4l-4)}}$	$\frac{l^2}{2(l-2)} \le t \le \frac{(l+1)^2}{2(l-3)}$
$1,\ l \ge 7$	$(\frac{l-3}{l+1},\frac{l-1}{l+1});\ (\frac{l-2}{l+2},\frac{l}{l+2})$	$\frac{8lt}{4+4l+(l-5)t-(l+3)\sqrt{t(t-8)}}$	$\frac{(l+1)^2}{2(l-3)} \le t \le \frac{(l+2)^2}{2(l-2)}$
$1,\ l = 5,6$	$(\frac{l-3}{l+1},\frac{l-1}{l+1});\ (\frac{l-2}{l+2},\frac{l}{l+2})$	$\frac{8lt}{4+4l+(l-5)t+(l+3)\sqrt{t(t-8)}}$	$\frac{(l+1)^2}{2(l-3)} \le t \le \frac{(l+2)^2}{2(l-2)}$
$1,\ l \ge 5$	$(\frac{l-2}{l+2},\frac{l}{l+2});\ (\frac{l-1}{l+1},1)$	$\frac{4l^3t^2}{(1-2l)t+\sqrt{t(t+4l)}} \times \times \frac{1}{(l-1)t-(l+1)\sqrt{t(t+4l)}}$	$\frac{(l+1)^2}{2(l-1)} \le t \le \frac{(l+2)^2}{2(l-2)}$
$2,1$	$(1,1);\ (\frac{1}{3},\frac{2}{3})$	$\frac{4t^2}{(t+2)(t-2)}$	$2 \le t \le 6$
$2,1$	$(\frac{1}{3},\frac{2}{3});\ (\frac{2}{5},\frac{3}{5})$	$\frac{9t}{-12+4t-5\sqrt{t(t-6)}}$	$6 \le t \le \frac{25}{4}$
$2,1$	$(\frac{2}{5},\frac{3}{5});\ (1,1)$	$\frac{9t}{-12-4t+5\sqrt{t(t+6)}}$	$2 \le t \le \frac{25}{4}$
$2,2$	$(1,\frac{4}{5});\ (1,1)$	$\frac{-8t^2}{(t+2)(t-6)}$	$2 \le t \le \frac{10}{3}$
$2,2$	$(1,1);\ (\frac{2}{5},\frac{3}{5})$	$\frac{6t}{-t+2\sqrt{t(t+6)}}$	$2 \le t \le \frac{25}{4}$
$2,2$	$(\frac{2}{5},\frac{3}{5});\ (\frac{1}{2},\frac{1}{2})$	$\frac{18t}{-30+13t-14\sqrt{t(t-6)}}$	$\frac{25}{4} \le t \le 8$
$2,2$	$(\frac{1}{2},\frac{1}{2});\ (1,\frac{4}{5})$	$\frac{50t}{-30-11t+14\sqrt{t(t+10)}}$	$\frac{10}{3} \le t \le 8$
$2,3$	$(1,\frac{5}{7});\ (1,\frac{4}{5})$	$\frac{-12t^2}{(t+2)(t-10)}$	$\frac{10}{3} \le t \le \frac{14}{3}$
$2,3$	$(1,\frac{4}{5});\ (\frac{1}{2},\frac{1}{2})$	$\frac{30t}{-4t+6\sqrt{t(t+10)}}$	$\frac{10}{3} \le t \le 8$
$2,3$	$(\frac{1}{2},\frac{1}{2});\ (\frac{4}{5},\frac{3}{5})$	$\frac{27t}{24-2t+7\sqrt{t(t-6)}}$	$8 \le t \le \frac{25}{4}$

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k,l	MN	$\frac{\pi^2}{3}g_{MN}(t)$	the domain of t
$2,3$	$(\frac{4}{5},\frac{3}{5});\ (1,\frac{5}{7})$	$\frac{147t}{-56-22t+27\sqrt{t(t+14)}}$	$\frac{14}{3}\le t\le\frac{25}{4}$
$2,4$	$(1,\frac{2}{3});\ (1,\frac{5}{7})$	$\frac{-16t^2}{(t+2)(t-14)}$	$\frac{14}{3}\le t\le 6$
$2,4$	$(1,\frac{5}{7});\ (\frac{4}{5},\frac{3}{5})$	$\frac{28t}{-3t+4\sqrt{t(t+14)}}$	$\frac{14}{3}\le t\le\frac{25}{4}$
$2,4$	$(\frac{4}{5},\frac{3}{5});\ (1,\frac{2}{3})$	$\frac{36t}{30-t+8\sqrt{t(t-6)}}$	$6\le t\le\frac{25}{4}$
$3,1$	$(1,\frac{3}{5});\ (1,\frac{2}{3})$	$\frac{-9t^2}{(t+3)(t-6)}$	$3\le t\le\frac{15}{4}$
$3,1$	$(1,\frac{2}{3});\ (\frac{1}{2},\frac{1}{2})$	$\frac{9t^2}{(t+3)(2t-3)}$	$3\le t\le 6$
$3,1$	$(\frac{1}{2},\frac{1}{2});\ (\frac{4}{7},\frac{3}{7})$	$\frac{32t}{-72+31t-11\sqrt{3t(3t-16)}}$	$6\le t\le\frac{147}{20}$
$3,1$	$(\frac{4}{7},\frac{3}{7});\ (1,\frac{3}{5})$	$\frac{50t}{-60-23t+9\sqrt{3t(3t+20)}}$	$\frac{15}{4}\le t\le\frac{147}{20}$
$3,2$	$(1,\frac{1}{2});\ (1,\frac{3}{5})$	$\frac{-18t^2}{(t+3)(t-15)}$	$\frac{15}{4}\le t\le 6$
$3,2$	$(1,\frac{3}{5});\ (\frac{4}{7},\frac{3}{7})$	$\frac{10t}{-2t+\sqrt{t(9t+60)}}$	$\frac{15}{4}\le t\le\frac{147}{20}$
$3,2$	$(\frac{4}{7},\frac{3}{7});\ (\frac{3}{5},\frac{2}{5})$	$\frac{64t}{-168+79t-27\sqrt{3t(3t-16)}}$	$\frac{147}{20}\le t\le\frac{25}{3}$
$3,2$	$(\frac{3}{5},\frac{2}{5});\ (1,\frac{1}{2})$	$\frac{64t}{-72-11t+7\sqrt{3t(3t-16)}}$	$6\le t\le\frac{25}{3}$
$4,1$	$(1,\frac{3}{7});\ (1,\frac{1}{2})$	$\frac{-16t^2}{(t+4)(t-12)}$	$4\le t\le\frac{28}{5}$
$4,1$	$(1,\frac{1}{2});\ (\frac{3}{5},\frac{2}{5})$	$\frac{16t^2}{(t+4)(3t-4)}$	$4\le t\le\frac{20}{3}$
$4,1$	$(\frac{3}{5},\frac{2}{5});\ (\frac{2}{3},\frac{1}{3})$	$\frac{25t}{-80+37t-38\sqrt{t(t-5)}}$	$9\le t\le\frac{20}{3}$
$4,1$	$(\frac{2}{3},\frac{1}{3});\ (1,\frac{3}{7})$	$\frac{49t}{-56-23t+26\sqrt{t(t+7)}}$	$\frac{28}{5}\le t\le 9$
$4,2$	$(1,\frac{2}{5});\ (1,\frac{3}{7})$	$\frac{-32t^2}{(t+4)(t-28)}$	$\frac{28}{5}\le t\le\frac{20}{3}$
$4,2$	$(1,\frac{3}{7});\ (\frac{2}{3},\frac{1}{3})$	$\frac{14t}{-3t+4\sqrt{t(t+7)}}$	$\frac{28}{5}\le t\le 9$
$4,2$	$(\frac{2}{3},\frac{1}{3});\ (1,\frac{2}{5})$	$\frac{50t}{60-9t+16\sqrt{t(t-5)}}$	$\frac{20}{3}\le t\le 9$
$k\ge 5,\ 1$	$(1,\frac{2}{k+1});\ (1,\frac{2}{k})$	$\frac{-k^2t^2}{(t-k^2+k)(t+k)}$	$k\le t\le\frac{k(k+1)}{k-1}$
$k\ge 5,\ 1$	$(1,\frac{2}{k});\ (\frac{k-1}{k+1},\frac{2}{k+1})$	$\frac{k^2t^2}{(t+k)((k-1)t-k)}$	$k\le t\le\frac{k(k+1)}{k-1}$
$k\ge 5,\ 1$	$(\frac{k-1}{k+1},\frac{2}{k+1});\ (\frac{k}{k+2},\frac{2}{k+2})$	$\frac{4(k+1)^2t^2}{(k+2)t-\sqrt{kt(kt-4k-4)}}\times$ $\times\frac{1}{(k^2-2)t-k\sqrt{kt(kt-4k-4)}}$	$\frac{k(k+1)}{k-1}\le t\le\frac{(k+2)^2}{k}$
$k\ge 5,\ 1$	$(\frac{k}{k+2},\frac{2}{k+2});\ (1,\frac{2}{k+1})$	$\frac{2(k+1)^2t^2}{t-\sqrt{kt(kt-4k-4)}}\times$ $\times\frac{1}{(-k-2)t+\sqrt{kt(kt-4k-4)}}$	$\frac{k(k+1)}{k-1}\le t\le\frac{(k+2)^2}{k}$

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