Some latticial properties of Hilbert algebras
by
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Abstract

The aim of this paper is to study some properties of Hilbert algebras relative to the natural order. Connections between Hilbert algebras with infimum and Hertz algebras are made. For the case of Hilbert algebras with supremum is offered an equational characterization. Afterwards, some rules of calculus for Hilbert algebras with a latticeal structure are provided and a result from [14] relative to implicative semilattices is improved and generalized to the case of Hilbert algebras with infimum.

Key Words: Hilbert algebra, Heyting algebra, Hilbert algebra with infimum, Hilbert algebra with supremum, Hertz algebra, semi-Boolean lattices, semi-Boolean Hilbert algebra.

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1 Introduction

The concept of Hilbert algebras was introduced in the 50’s by L. Henkin ([11]) and T. Skolem for investigations in intuitionistic and other non-classical logics, as an algebraic counterpart of Hilbert’s positive implicative propositional calculus ([17]). Hilbert algebras were intensively studied by A. Diego ([7]) and this theory was further developed by D. Busneag ([2], [3]). This paper is structured in six sections. The first two describe the general background of our work and include some elementary aspects of Hilbert algebras that will be necessary.

Hilbert algebras will be treated from a latticial point of view relative to the natural order. Many results related to the natural order of a Hilbert algebra were found. Most of them treat the case in which such algebras become meet-semilattices (Hilbert algebras with infimum ([9])). Some of these results will be remembered in this paper and will be further developed. Connections among Hilbert algebras with infimum, Hertz algebras and Heyting algebras will be made. All of these can be found in section 3.
The case of Hilbert algebras with the property that are join-semilattices relative to the natural order has not been intensively studied. Some results can be found in [21] and [13] but no equational characterization for such algebras was made. Such result is offered in section 4.

In section 5, we try to assemble all the above results from sections 3 and 4 and study Hilbert algebras with a latticial structure relative to the natural order. Although such results are included also in section 3 as a connection between bounded Hilbert algebras and Heyting algebras, in this section we try to examine the unbounded case of Hilbert algebras with latticial structure.

Finally, in section 6, all the rules of calculus relative to different kinds of Hilbert algebras (with infimum, with supremum or with a latticeal structure) are used to improve a result from [14] relative to semi-Boolean lattices.

In this paper the symbols $\Rightarrow$ and $\Leftrightarrow$ will be used for logical implication and respectively logical equivalence.

2 Preliminaries

We include some elementary properties of Hilbert algebras that are necessary for this paper; for more details we refer to [2], [3] and [7].

**Definition 1.** ([7]) A Hilbert algebra is an algebra $(A, \to, 1)$ of type $(2, 0)$ such that the following axioms are verified for every $a, b, c \in A$:

- $(a_1)$ $a \to (b \to a) = 1$;
- $(a_2)$ $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$;
- $(a_3)$ If $a \to b = b \to a = 1$, then $a = b$.

In [7] it is proved that the system of axioms $\{a_1, a_2, a_3\}$ is equivalent to the system $\{a_4, a_5, a_6, a_7\}$, where:

- $(a_4)$ $a \to a = 1$;
- $(a_5)$ $1 \to a = a$;
- $(a_6)$ $a \to (b \to c) = (a \to b) \to (a \to c)$;
- $(a_7)$ $(a \to b) \to ((b \to a) \to a) = (b \to a) \to ((a \to b) \to b)$.

The relation $a \leq b$ defined by $a \leq b \iff a \to b = 1$ is a partial order on $A$ (called the natural order on $A$); with respect to this order, 1 is the largest element of $A$. $A$ will be called bounded if $A$ has a smallest element 0; in this case, for $a \in A$ we denote $a^* = a \to 0$.

Following [2] and [7] in a Hilbert algebra $A$ we have the following rules of calculus, for $a, b, c \in A$:

- $(c_1)$ $a \leq b \to a$;
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\((c_2)\) \(a \leq (a \to b) \to b;\)

\((c_3)\) \(a \to 1 = 1;\)

\((c_4)\) \(a \to (b \to c) = b \to (a \to c);\)

\((c_5)\) \(((a \to b) \to b) \to b = a \to b;\)

\((c_6)\) \(a \to b \leq (b \to c) \to (a \to c);\)

\((c_7)\) If \(a \leq b,\) then \(c \to a \leq c \to b\) and \(b \to c \leq a \to c;\)

\((c_8)\) \((a \to b) \to (b \to a) = b \to a;\)

\((c_9)\) \(((a \to b) \to a) \to b = a \to b.\)

For two elements \(a, b\) of a Hilbert algebra \(A\) we denote

\[a \sqcup b = (a \to b) \to b;\]

\[a \dot\lor b = (a \to b) \to ((b \to a) \to a).\]

Clearly \(a \sqcup b \leq a \dot\lor b.\)

Following [2], [3] and [7] in a Hilbert algebra \(A\) we have the following rules of calculus relative to \(\sqcup\) and \(\dot\lor\), for every \(a, b, c \in A:\)

\((c_{10})\) \(a, b \leq a \sqcup b, (a \to b) \sqcup (b \to a) = 1;\)

\((c_{11})\) \(a \sqcup a = a;\)

\((c_{12})\) \(a \sqcup 1 = 1 \sqcup a = 1;\)

\((c_{13})\) \(a \leq b\) iff \(a \sqcup b = b;\)

\((c_{14})\) \(a \sqcup (b \sqcup c) = b \sqcup (a \sqcup c);\)

\((c_{15})\) \(a \to (b \to c) = (a \to c) \sqcup (b \to c);\)

\((c_{16})\) \(a \to (b \sqcup c) = (a \to b) \sqcup (a \to c);\)

\((c_{17})\) \(a, b \leq a \dot\lor b;\)

\((c_{18})\) \(a \dot\lor b = b \dot\lor a;\)

\((c_{19})\) \(a \dot\lor a = a;\)

\((c_{20})\) \(a \dot\lor 1 = 1;\)

\((c_{21})\) \(a \dot\lor (a \to b) = 1;\)

\((c_{22})\) \(a \to (b \dot\lor c) = (a \to b) \dot\lor (a \to c);\)

\((c_{23})\) \((a \to b) \dot\lor c = a \to (b \dot\lor c);\)
(c_{24}) \ (a \to b) \lor (b \to a) = 1;

(c_{25}) \ a \to (b \to c) = (a \to c) \lor (b \to c).

**Proposition 1.** Let $A$ be a Hilbert algebra and $a, b \in A$. Then

(c_{26}) \ a \leq b \iff a \sqcap c \leq b \sqcap c \text{ for every } c \in A.

**Proof:** "\Rightarrow". Clearly, by (c_{27}).

"\Leftarrow". By hypothesis, for $c = a \to b$ we have $a \sqcup (a \to b) \leq b \sqcup (a \to b)$. But $a \sqcup (a \to b) = (a \to (a \to b)) \to (a \to b) = (a \to b) \to (a \to b) = 1$ and $b \sqcup (a \to b) = (b \to (a \to b)) \to (a \to b) = 1 \to (a \to b) = a \to b$. So we obtain that $1 = a \to b$, that is, $a \leq b$.

3 Hilbert algebras with infimum

An interesting case of Hilbert algebras are the one that become meet-semilattices relative to the natural order. An equational characterization for such algebras is offered by Figallo in [9]. For a long time it was considered that this kind of algebras was identical with Hertz algebras (see Definition 4). H. Porta was the first author who made that assertion ([16]), but it was wrong. Figallo and Zillani [8] eliminate this confusion with some counterexamples and show that Hilbert algebras with infimum become Hertz algebras only if they verify a supplementary condition which is noted in this paper with $(P)$ (see Theorem 3). In this section are offered some equivalent conditions with $(P)$, which mean new rules of calculus. We will see that these rules are inherited from Heyting algebras.

We start this section with some definitions and a result from [9].

**Definition 2. ([9])** A Hilbert algebra $(A, \to, 1)$ is called a Hilbert algebra with infimum when the underlying structure $(A, \leq)$ with the order induced by $\to$ is a meet-semilattice.

**Definition 3. ([9])** An $iH$-algebra is an algebra $(A, \to, \land, 1)$ of type $(2,2,0)$ in which the reduct $(A, \to, 1)$ is a Hilbert algebra and the following axioms are satisfied:

(a_8) \ a \land (b \land c) = (a \land b) \land c;

(a_9) \ a \land a = a;

(a_{10}) \ a \land (a \to b) = a \land b;

(a_{11}) \ (a \to (b \land c)) \to ((a \to c) \land (a \to b)) = 1.

**Theorem 1. ([9])** Let $(A, \to, \land, 1)$ of type $(2,2,0)$. Then the following two conditions are equivalent:

(i) $A$ is an $iH$-algebra;
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(ii) $A$ is an Hilbert algebra with infimum.

**Corollary 1.** If $(A, \rightarrow, 1)$ is a Hilbert algebra with infimum and $a, b, x \in A$, then $x \leq a \rightarrow b$ implies $a \land x \leq b$.

**Proof:** Using (a10), if $x \leq a \rightarrow b$, then $a \land x \leq a \land (a \rightarrow b) = a \land b \leq b$.

In this paper, various rules of calculus for different kinds of Hilbert algebras will be proved. We will see that all these rules are inherited from a more restrictive class of algebras, Heyting algebras.

Suppose that $(L, \land, \lor)$ is a lattice and $a, b \in L$. If there is a largest element $x \in L$ such that $a \land x \leq b$, then this element is denoted by $a \rightarrow b$ and is called the relative pseudocomplement of $a$ with respect to $b$. The definition of relative pseudocomplement is equivalent to the existence of an element $a \rightarrow b$ such that $a \land x \leq b \iff x \leq a \rightarrow b$. A Heyting algebra ([1]) is a lattice with 0 in which $a \rightarrow b$ exists for each $a, b \in L$.

In [1] the following rules of calculus in a Heyting algebra are being proved, rules which we will see that are also true in some particular cases of Hilbert algebras:

**Theorem 2.** ([4]) Let $L$ a Heyting algebra and $a, b, c \in L$. Then:

- $(c_{27}) a \land (a \rightarrow b) = a \land b$;
- $(c_{28}) a \rightarrow (b \rightarrow c) = (a \land b) \rightarrow c$;
- $(c_{29}) a \land (b \rightarrow c) = a \land ((a \land b) \rightarrow (a \land c))$;
- $(c_{30}) (a \lor b) \rightarrow c = (a \rightarrow c) \land (b \rightarrow c)$;
- $(c_{31}) a \rightarrow (b \land c) = (a \rightarrow b) \land (a \rightarrow c)$.

We say that a Hilbert algebra with infimum has the property $(P)$ if for every $a, b \in A$,

$$(P) : a \rightarrow (b \rightarrow (a \land b)) = 1$$

(that is, $a \land b$ is in the deductive system of $A$ generated by $a$ and $b$, see [2],[7]).

**Remark 1.** Following $(c_{28})$ we deduce that Heyting algebras are examples of bounded Hilbert algebras with the property $(P)$.

Another particular case of Hilbert algebras are Hertz algebras:

**Definition 4.** ([9], [15]) A Hertz algebra is an algebra $(A, \rightarrow, \land, 1)$ of type $(2, 2, 0)$ which satisfies the following axioms:

- $(a_{12}) a \rightarrow a = 1$;
- $(a_{13}) (a \rightarrow b) \land b = b$;
\((a_{14})\) \(a \land (a \to b) = a \land b;\)

\((a_{15})\) \(a \to (b \land c) = (a \to c) \land (a \to b).\)

In [8] it is proved:

**Theorem 3.** For a Hilbert algebra \(A\), the following assertions are equivalent:

(i) \(A\) is a Hertz algebra;

(ii) \(A\) is a Hilbert algebra with infimum which verifies property \((P)\).

The following lemma offers an equivalent condition with \((P)\) for a Hilbert algebra with infimum to become a Hertz algebra. In the case of a Hilbert algebra with a latticeal structure, this condition will lead to the existence of the relative pseudocomplement for every two elements.

**Lemma 1.** Let \(A\) be a Hilbert algebra with infimum. The following assertions are equivalent:

(i) \(A\) has the property \((P)\);

(ii) For every \(a, b, x \in A\), \(a \land x \leq b\) iff \(x \leq a \to b\).

**Proof:** (i) \(\Rightarrow\) (ii). Let \(a, b, x \in A\) such that \(a \land x \leq b\). Then \(a \to (a \land x) \leq a \to b\), hence \(x \to (a \to (a \land x)) \leq x \to (a \to b)\). But by the property \((P)\), \(x \to (a \to (a \land x)) = 1\), hence \(x \to (a \to b) = 1\), therefore \(x \leq a \to b\). The converse implication results from Corrolary 1.

(ii) \(\Rightarrow\) (i). Let \(a, b \in A\). Since \(a \land b \leq a \land b\), we deduce that \(a \leq b \to (a \land b)\), hence \(a \to (b \to (a \land b)) = 1\), that is, \(A\) has the property \((P)\).

**Corollary 2.** Suppose \((A, \to, 1)\) is a bounded Hilbert algebra such that \(A\) is a lattice relative to the natural order. Then the following assertions are equivalent:

(i) \(A\) has the property \((P)\);

(ii) \(A\) is a Heyting algebra.

**Remark 2.** For an example of Hilbert algebra with infimum which is not a Heyting algebra, see [9]. In what follows, other conditions equivalent with \((P)\) are offered. We will see that the rules \((c_{28})\) and \((c_{29})\) are also true in every Hertz algebra.

**Lemma 2.** If \(A\) is a Hilbert algebra with infimum, then for every \(a, b, c \in A\), we have

\((c_{32})\) \(a \to (b \to c) \leq (a \land b) \to c.\)
Proof: We have:

\[ (a \rightarrow (b \rightarrow c)) \rightarrow ((a \wedge b) \rightarrow c) = \]
\[ (a \wedge b) \rightarrow [(a \rightarrow (b \rightarrow c)) \rightarrow c] (\text{a}_2) \]
\[ [(a \wedge b) \rightarrow (a \rightarrow (b \rightarrow c))] \rightarrow [(a \wedge b) \rightarrow c] (\text{a}_2) \]
\[ [((a \wedge b) \rightarrow a) \rightarrow ((a \wedge b) \rightarrow (a \wedge b) \rightarrow c)] \rightarrow [(a \wedge b) \rightarrow c] = \]
\[ [1 \rightarrow (1 \rightarrow ((a \wedge b) \rightarrow c))] \rightarrow [(a \wedge b) \rightarrow c] = \]
\[ [(a \wedge b) \rightarrow c] \rightarrow ((a \wedge b) \rightarrow c) = 1, \]
hence \( a \rightarrow (b \rightarrow c) \leq (a \wedge b) \rightarrow c. \)

Proposition 2. Let \( A \) be a Hilbert algebra with infimum. The following assertions are equivalent:

(i) \( A \) has the property (P);

(ii) For every \( a, b, c \in A \), \( a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c; \)

(iii) For every \( a, b, c \in A \), \( a \wedge (b \rightarrow c) = a \wedge [(a \wedge b) \rightarrow (a \wedge c)]. \)

Proof: (i) \( \Rightarrow \) (ii). By \( (c_{32}) \) we have \( a \rightarrow (b \rightarrow c) \leq (a \wedge b) \rightarrow c. \)
On the other hand,

\[ ((a \wedge b) \rightarrow c) \rightarrow (a \rightarrow (b \rightarrow c)) \] (by \( (c_{32}) \))
\[ a \rightarrow [(a \wedge b) \rightarrow (b \rightarrow c)] \] (by \( (c_{32}) \))
\[ a \rightarrow [b \rightarrow ((a \wedge b) \rightarrow c)] \] (by \( (a_2) \))
\[ [(a \rightarrow (b \rightarrow (a \wedge b))) \rightarrow (a \rightarrow (b \rightarrow c))] \rightarrow (a \rightarrow (b \rightarrow c)) \] (by \( P \))
\[ [1 \rightarrow (a \rightarrow (b \rightarrow c))] \rightarrow (a \rightarrow (b \rightarrow c)) = \]
\[ (a \rightarrow (b \rightarrow c)) \rightarrow (a \rightarrow (b \rightarrow c)) = 1, \]
hence \( (a \wedge b) \rightarrow c \leq a \rightarrow (b \rightarrow c) \), that is, \( a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c. \)

(ii) \( \Rightarrow \) (i). For \( a, b \in A \) and \( c = a \wedge c \), we have \( a \rightarrow (b \rightarrow (a \wedge b)) = (a \wedge b) \rightarrow (a \wedge c) = 1, \) hence \( A \) has the property (P).

(i) \( \Rightarrow \) (iii). Since \( (b \rightarrow c) \wedge (a \wedge b) = (b \wedge (b \rightarrow c)) \wedge a \) \( (a_2) \) \( (b \wedge c) \wedge a \leq a \wedge c, \)
we deduce \( b \rightarrow c \leq (a \wedge b) \rightarrow (a \wedge c), \) hence \( a \wedge (b \rightarrow c) \leq a \wedge [(a \wedge b) \rightarrow (a \wedge c)]. \)

On the other hand, we observe that \( (a \wedge b) \rightarrow (a \wedge c) \) \( (a_2) \) \( (a \wedge b) \rightarrow c \) \( (i) \equiv (ii) \)
\( a \rightarrow (b \rightarrow c) \) \( (a_2) \) \( a \rightarrow (a \wedge (b \rightarrow c)) \), so \( a \wedge [(a \wedge b) \rightarrow (a \wedge c)] \leq a \wedge (b \rightarrow c), \)
hence \( a \wedge (b \rightarrow c) = a \wedge [(a \wedge b) \rightarrow (a \wedge c)]. \)

(iii) \( \Rightarrow \) (i). For \( a, b \in A \) and \( c = a \wedge b \) we obtain \( a \wedge (b \rightarrow (a \wedge b)) = a \wedge [(a \wedge b) \rightarrow (a \wedge b)] = a \wedge 1 = a, \) hence \( a \leq b \rightarrow (a \wedge b), \) therefore \( a \rightarrow (b \rightarrow (a \wedge b)) = 1, \) that is, \( A \) has the property (P).
Remark 3. For the equivalence (i) $\Leftrightarrow$ (ii) see also [6], p.269.

Corollary 3. Let $A$ be a bounded Hilbert algebra with infimum. Then for $a, b \in A$ we have:

$$(c_{31}) \quad (a \rightarrow b)^* = a^{**} \land b^*.$$  

Proof: From $b \leq a \rightarrow b$, we obtain $(a \rightarrow b)^* \leq b^*$. Since $a^* = a \rightarrow 0 \leq a \rightarrow b$, by $(c_7)$ we obtain $(a \rightarrow b)^* \leq a^{**}$, hence $(a \rightarrow b)^* \leq a^{**} \land b^*$.

Also, by $(c_6)$ we have $a \rightarrow b \leq b^* \rightarrow a^*$, hence $b^* \leq (a \rightarrow b) \rightarrow a^* \leq a^{**} \rightarrow (a \rightarrow b)^*$. By Corollary 1 we obtain that $a^{**} \land b^* \leq (a \rightarrow b)^*$, hence $(a \rightarrow b)^* = a^{**} \land b^*$. \hfill \Box

4 Hilbert algebras with supremum

We have seen that in the case of Hilbert algebras that are meet-semilattices relative to the natural order it is possible to give an equational characterization (Theorem 1). This section treats the Hilbert algebras that are join-semilattices relative to the natural order and it gives an axiomatic characterization for them.

Definition 5. A Hilbert algebra $(A, \rightarrow, 1)$ is called a Hilbert algebra with supremum when the underlying structure $(A, \leq)$ with the natural order induced by $\rightarrow$ is a join-semilattice.

Clearly, Heyting and Boolean algebras are examples of Hilbert algebras with supremum. For some examples of Hilbert algebras which are not Hilbert algebras with supremum see [7], p. 46.

A particular case of Hilbert algebras with supremum are Abbott algebras (called also implication algebras or commutative Hilbert algebras). An Abbott algebra is a Hilbert algebra $(A, \rightarrow, 1)$ in which the operator $\sqcup$ is commutative, this means for every $a, b \in A$, $(a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a$. In [13], it is proved the following result.

Proposition 3. A Hilbert algebra $A$ is commutative if and only if it is a join-semilattice with respect to $\sqcup$.

The polynomials $(X \rightarrow Y) \rightarrow Y$ have been studied by Torrens in [21]. Another interesting result for Hilbert algebras with supremum in terms of such polynomials can be found there.

Proposition 4. ([21]) Let $A$ be a Hilbert algebra with supremum. Then the following conditions are equivalent:

(i) For any $a, b \in A$, $(a \rightarrow b) \lor (b \rightarrow a) = 1$;

(ii) For any $a, b \in A$, $((a \rightarrow b) \rightarrow b) \land ((b \rightarrow a) \rightarrow a)$ exists and is equal to $a \lor b$.  

Lemma 3. Let \( A \) be a Hilbert algebra and \( a, b \in A \). If there exists \( a \lor b \), then for every \( c \in A \), there exists \( (a \rightarrow c) \land (b \rightarrow c) \) and
\[
(a \rightarrow c) \land (b \rightarrow c) = (a \lor b) \rightarrow c.
\]

Proof: We denote \( d = (a \lor b) \rightarrow c \). Since \( a, b \leq a \lor b \), by \((c7)\) we deduce that \( d \leq a \rightarrow c \) and \( d \leq b \rightarrow c \). Let now \( t \in A \) such that \( t \leq a \rightarrow c, b \rightarrow c \). By \((c4)\) we deduce that \( a, b \leq t \rightarrow c \), then \( a \lor b \leq t \rightarrow c \), hence \( t \leq (a \lor b) \rightarrow c = d \), that is, \( d = (a \rightarrow c) \land (b \rightarrow c) \).

Corollary 4. Let \( A \) be a Hilbert algebra with supremum. Then, for every \( a, b, c \in A \), there exists \( (a \rightarrow c) \land (b \rightarrow c) \), and
\[
(a \rightarrow c) \land (b \rightarrow c) = (a \lor b) \rightarrow c.
\]

Corollary 5. If \( A \) is a bounded Hilbert with supremum, then for every \( a, b \in A \) there exists \( a^* \land b^* \) and \( a^* \land b^* = (a \lor b)^* \).

Remark 4. We deduce that the rule \((c30)\) which is true for Heyting algebras, is also true in a Hilbert algebra with supremum.

Corollary 6. If \( A \) is a Hilbert algebra with supremum, then for every \( a, b \in A \) we have:
\[
(c_{34}) \quad (a \lor b) \rightarrow a = b \rightarrow a;
\]
\[
(c_{35}) \quad (a \rightarrow b) \rightarrow (a \lor b) = (a \rightarrow b) \rightarrow b.
\]

Proof: \((c_{34})\). From Corollary 4, if consider \( c = a \).
\[
(c_{35}) \quad (a \rightarrow b) \rightarrow (a \lor b) = (a \rightarrow b) \rightarrow b.
\]

Definition 6. A \( sH \)-algebra is an algebra \((A, \rightarrow, \lor, 1)\) of type \((2, 2, 0)\) such that the following axioms are verified in \( A \):
\[
(a_{16}) \quad a \rightarrow a = 1;
\]
\[
(a_{17}) \quad 1 \rightarrow a = a;
\]
\[
(a_{18}) \quad a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c);
\]
\[
(a_{19}) \quad a \lor (b \lor c) = (a \lor b) \lor c;
\]
\[
(a_{20}) \quad a \lor b = b \lor a;
\]
\[
(a_{21}) \quad (a \lor b) \rightarrow b = a \rightarrow b;
\]
\((a_{22})\) \((a \to b) \to (a \lor b) = (a \to b) \to b.\)

**Proposition 5.** If \((A, \to, \lor, 1)\) is a \(sH\)-algebra, then for every \(a, b, c \in A\) we have:

(i) \(a \to 1 = 1;\)

(ii) \(a \lor 1 = 1;\)

(iii) \(a \to b = 1\) if and only if \(a \lor b = b;\)

(iv) \(a \to b = b \to a = 1,\) then \(a = b;\)

(v) \(a \lor a = a;\)

(vi) \(a \to (b \to a) = 1.\)

**Proof:** (i) If in \((a_{18})\) we put \(c = b\) we obtain \(a \to (b \to b) = (a \to b) \to (a \to b),\)

(hence \(a \to 1 = 1.\)

(ii) From (i) and \((a_{22})\) we deduce \((a \to 1) \to (a \lor 1) = (a \to 1) \to 1,\)

(hence \(1 \to (a \lor 1) = 1 \to 1,\) therefore \(a \lor 1 = 1.\)

(iii) If \(a \to b = 1,\) by \((a_{22})\) we deduce \(1 \to (a \lor b) = 1 \to b\) and by \((a_{17})\) we obtain \(a \lor b = b.\) Conversely, if \(a \lor b = b,\) by \((a_{21})\) we deduce \(b \to b = a \to b,\)

(hence \(1 = a \to b.\)

(iv) If \(a \to b = b \to a = 1,\) then \(a \lor b = b\) and \(b \lor a = a;\) since \(a \lor b = 1\) \((a_{20})\) \(b \lor a,\)

we obtain \(a = b.\)

(v) Since \(a \to a = 1,\) from \((a_{22}),\) we deduce that \(a \lor a = a.\)

(vi) By \((a_{18})\) we obtain \(a \to (b \to a) = (a \to b) \to (a \to a) = (a \to b) \to 1 = 1.\)

**Corollary 7.** If \((A, \to, \lor, 1)\) is a \(sH\)-algebra, then \((A, \to, 1)\) is a Hilbert algebra.

**Proof:** With the notations from the above proposition, \((a_1)\) follows from (vi),

\((a_2)\) follows from \((a_{18})\) and \((a_3)\) follows from (iv).

**Proposition 6.** If \((A, \to, \lor, 1)\) is a \(sH\)-algebra, then \((A, \lor, 1)\) is a join-semilattice.

**Proof:** Since \((A, \to, 1)\) is a Hilbert algebra, following Proposition 5, (iii), the order on \(A\) is given by \(a \leq b \iff a \to b = 1 \iff a \lor b = b.\)

Since \(a \lor (a \lor b) = (a \lor a) \lor b = a \lor b,\) we deduce that \(a \leq a \lor b\) and analogously \(b \leq a \lor b.\) Let now \(c \in A\) such that \(a, b \leq c.\) Then \(a \lor c = b \lor c = c,\)

(hence \((a \lor b) \lor c = a \lor (b \lor c) = a \lor c = c,\) that is, \(a \lor b \leq c.\)

By summing the above results, we obtain:

**Theorem 4.** Let \((A, \to, 1)\) be a Hilbert algebra. The following conditions are equivalent:

(i) \((A, \to, 1)\) is a Hilbert algebra with supremum;

(ii) \((A, \to, \lor, 1)\) is a \(sH\)-algebra.
5 Hilbert algebras with latticial structure

We have studied Hilbert algebras which are meet-semilattices or join-semilattices relative to the natural order. In this section we combine these two properties and we study Hilbert algebras with latticial structure. A first result intends to study a case in which such algebras are distributive lattices.

Remark 5. Let \((A, \to, 1)\) be a Hilbert algebra. If relative to the natural order \(A\) is a lattice with property \((P)\), then \((A, \lor, \land)\) is a distributive lattice.

Indeed, by Corollary 2, \(A\) is a Heyting algebra, hence the lattice \((A, \lor, \land)\) is distributive (see [1], p.174).

Corollary 8. Let \(A\) be a Hilbert algebra such that \(A\) is a lattice relative to the natural order and has the property \((P)\). Then for every \(a, b, c, d \in A\) we have:

\[(c_{36}) \ (a \to b) \land (c \to d) \leq (a \land c) \to (b \land d);\]
\[(c_{37}) \ (a \to b) \land (c \to d) \leq (a \lor c) \to (b \lor d).\]

Proof: \((c_{36})\). We have \((a \to b) \land (c \to d) \land (a \land c) = [(a \to b) \land a] \land [(c \to d) \land c] \leq b \land d\), hence \((a \to b) \land (c \to d) \leq (a \land c) \to (b \land d)\) (by Lemma 1).

\[(c_{37})\). If denote \(e = (a \to b) \land (c \to d)\), then

\[
e \land (a \lor c) \Leftrightarrow \land (e \land c) =\]
\[
[(a \to b) \land (c \to d) \land a] \lor [(a \to b) \land (c \to d) \land c] \leq\]
\[
[(a \to b) \land a] \lor [(c \to d) \land c] \leq b \lor d,
\]

hence \(e \leq (a \lor c) \to (b \lor d)\).

Lemma 4. Let \(A\) be a Hilbert algebra which is a lattice relative to the natural order and has the property \((P)\). Then for every \(a, b, c \in A\) we have:

\[(c_{38}) \ a \land (b \to c) \leq b \to (a \land c);\]
\[(c_{39}) \ (b \to c) \land (b \lor c) \leq c.\]

Proof: \((c_{38})\). We have: \(a \land (b \to c) \leq b \to (a \land c) \Leftrightarrow a \leq (b \to c) \to (b \to (a \land c)) \Leftrightarrow a \leq b \to (c \to (a \land c)) \Leftrightarrow b \leq a \to (c \to (a \land c)) \Leftrightarrow 1.\)

\[(c_{39})\). We have \((b \to c) \land (b \lor c) \leq c \Leftrightarrow b \lor c \leq (b \to c) \to c\), which is clear by \((c_{10})\).

The following Theorem is an extension of Torrens’s result (Proposition 4) to Hilbert algebras with latticial structure.
Theorem 5. Let \( A \) be a Hilbert algebra which is a lattice relative to the natural order and has property (P). Then the following conditions are equivalent for any \( a, b, c \in A \):

(i) \( (a \to b) \lor (b \to a) = 1; \)

(ii) \( a \to (b \lor c) = (a \to b) \lor (a \to c); \)

(iii) \( (a \land b) \to c = (a \to c) \lor (b \to c); \)

(iv) \( (a \lor b) \land (b \lor a) = a \lor b. \)

**Proof:** (i) \( \Rightarrow \) (ii). We have:

\[
a \to (b \lor c) = 1 \land [a \to (b \lor c)] = \]

\[
[(b \to c) \lor (c \to b)] \land [a \to (b \lor c)] \stackrel{\text{Remark 5}}{=} \\
[(b \to c) \land (a \to (b \lor c))] \lor [(c \to b) \land (a \to (b \lor c))] \stackrel{(ca)}{\leq} \\
(a \to [(b \to c) \land (b \lor c)] \land (a \to [(c \to b) \land (b \lor c)]) \stackrel{(ca)}{\leq} \\
(a \to c) \lor (a \to b).
\]

Since \( (a \to c) \lor (a \to b) \leq a \to (b \lor c) \), we deduce that \( a \to (b \lor c) = (a \to b) \lor (a \to c) \).

(ii) \( \Rightarrow \) (i). We have: \( (b \to c) \lor (c \to b) = [(c \to c) \land (b \to c)] \lor [(b \to b) \land (c \to b)] \stackrel{\text{Corollary 4}}{=} [(c \lor b) \to c] \lor [(c \lor b) \to b] \stackrel{(ii)}{=} (c \lor b) \to (c \lor b) = 1. \)

(i) \( \Rightarrow \) (iii). Clearly \( (a \to c) \lor (b \to c) \leq (a \land b) \to c \). On the other hand, we have:

\[
(a \land b) \to c = \\
[(a \land b) \to c] \land [(b \to a) \lor (a \to b)] = \\
((b \to a) \land [(a \land b) \to c]) \lor ((a \to b) \land [(a \land b) \to c]) \stackrel{\text{Prop. 2(ii)}}{=} \\
((b \to a) \land [a \to (b \to c)]) \lor ((a \to b) \land [a \to (b \to c)]) = \\
((b \to a) \land [b \to (a \to c)]) \lor ((a \to b) \land [a \to (b \to c)]) = \\
((b \to a) \land [(b \to a) \to (b \to c)]) \lor ((a \to b) \land [(a \to b) \to (a \to c)]) = \\
((b \to a) \land (b \to c)) \lor ((a \to b) \land (a \to c)) \leq \\
(b \to c) \lor (a \to c),
\]

that is, \( (a \land b) \to c = (a \to c) \lor (b \to c) \).

(iii) \( \Rightarrow \) (i). Since \( a \to (a \land b) = (a \to a) \land (a \to b) = 1 \land (a \to b) = a \to b \), we deduce \( (a \to b) \lor (b \to a) = (a \to (a \land b)) \lor (b \to (a \land b)) = (a \land b) \to (a \land b) = 1. \)
(i) $\Rightarrow$ (iv) Since $a, b \leq a \sqcup b, b \sqcup a$, we have $a \lor b \leq (a \sqcup b) \land (b \sqcup a)$. On the other hand we have:

\[
(a \sqcup b) \land (b \sqcup a) \leq a \lor b \quad \text{(Lemma 1)}
\]

\[
a \sqcup b \leq (b \sqcup a) \rightarrow (a \lor b) \quad \text{(i)}
\]

\[
a \sqcup b \leq ((b \sqcup a) \rightarrow a) \lor ((b \sqcup a) \rightarrow b) \quad \text{(ii)}
\]

\[
a \sqcup b \leq (b \rightarrow a) \lor ((b \sqcup a) \rightarrow b) \quad \text{\iff}
\]

\[
(a \sqcup b) \rightarrow [(b \rightarrow a) \lor ((b \sqcup a) \rightarrow b)] = 1 \quad \text{(iii)}
\]

\[
[(a \sqcup b) \rightarrow (b \rightarrow a)] \lor [(a \sqcup b) \rightarrow ((b \sqcup a) \rightarrow b)] = 1.
\]

Since $(a \sqcup b) \rightarrow (b \rightarrow a) = ((a \rightarrow b) \rightarrow b) \rightarrow (b \rightarrow a) = b \rightarrow [(a \rightarrow b) \rightarrow b] = 1 \rightarrow (b \rightarrow a) = b \rightarrow a$, and $(a \sqcup b) \rightarrow ((b \sqcup a) \rightarrow b) = (b \sqcup a) \rightarrow ((a \sqcup b) \rightarrow b) = (b \sqcup a) \rightarrow [(a \rightarrow b) \rightarrow b] = (b \sqcup a) \rightarrow (a \rightarrow b) = a \rightarrow ((b \sqcup a) \rightarrow b) = 1 \rightarrow (a \rightarrow b) = a \rightarrow b$, we obtain that

\[
(a \sqcup b) \land (b \sqcup a) \leq a \lor b \quad \text{(Lemma 1)}
\]

\[
[(a \sqcup b) \rightarrow (b \rightarrow a)] \lor [(a \sqcup b) \rightarrow ((b \sqcup a) \rightarrow b)] = 1 \quad \text{\iff}
\]

\[
(a \rightarrow b) \lor (b \rightarrow a) = 1,
\]

which is true by hypothesis.

(iv) $\Rightarrow$ (i) We have $(a \rightarrow b) \lor (b \rightarrow a) = ((a \rightarrow b) \lor (b \rightarrow a)) \land ((b \rightarrow a) \lor (a \rightarrow b)) = 1 \land 1 = 1$. 

\[\square\]

**Remark 6.** Theorem 5 is also true for the case of BCK algebras with property $(P)$ (see [12]; Theorem 3.1.3).

**Corollary 9.** Let $A$ be a bounded Hilbert algebra with latticeal structure and has property $(P)$. If $(a \rightarrow b) \lor (b \rightarrow a) = 1$ for every $a, b \in A$, then

\[
(a \land b)^* = a^* \lor b^* \quad \text{for every } a, b \in A.
\]

**Proof:** By Theorem 5, for every $a, b, c \in A$, we have $(a \land b) \rightarrow c = (a \rightarrow c) \lor (b \rightarrow c)$. If we put $c = 0$, we obtain $(a \land b)^* = a^* \lor b^*$. 

\[\square\]

### 6 Semi-Boolean Hilbert algebras

In this section we are trying to generalize a result from [14] regarding semi-Boolean lattices. In [14], Nemitz uses the notion of implicative semilattice. An *implicative semilattice* is an algebra $(L, \land, \rightarrow, 1)$ of type $(2, 2, 0)$, such that $(L, \land)$ is a meet-semilattice, 1 is the greatest element in $L$ and $\rightarrow$ is an operation of
pseudocomplementation defined by $a \rightarrow b = \max \{ x \in L : a \land x \leq b \}$, which is equivalent with the following: for any $a, b \in L$ there exists an element $a \rightarrow b \in L$ such that for any $x \in L$, $a \land x \leq b \iff x \leq a \rightarrow b$. Following Lemma 1 we deduce that every Hertz algebra is an implicative semilattice. Conversely, it is a simple exercise to see that every implicative semilattice is a Hertz algebra. Hence, implicative semilattices are exactly Hertz algebras, which, by Theorem 3, are Hilbert algebras with infimum and property $(P)$.

In such algebras, Nemitz defines the pseudo-join of two elements, but this notion can be introduced in the more general case of Hilbert algebras with infimum.

**Definition 7.** Let $A$ be a Hilbert algebra with infimum. By the pseudo-join $a \sqcup b$ of two elements $a$ and $b$ of $A$, we shall mean the element

$$a \sqcup b = [(a \rightarrow b) \land (b \rightarrow a)].$$

**Remark 7.** By Theorem 5, (iv), if $A$ is a lattice relative to the natural order and it has the property $(P)$, then for every $a, b \in A$, $a \sqcup b = a \lor b$.

Nemitz([14]) provides some elementary rules of calculus relative to the pseudo-join of two elements. Some of these rules are also true in our case of Hilbert algebras with infimum.

**Proposition 7.** Let $A$ be a Hilbert algebra with infimum and $a, b \in A$. Then:

$(c_{40})$ \quad $a \sqcup b = b \sqcup a$, $(a \rightarrow b) \lor (b \rightarrow a) = 1$;  

$(c_{41})$ \quad $a, b \leq a \sqcup b$ and $a \sqcup b \leq a \lor b$;  

$(c_{42})$ \quad $a \leq b$ if and only if $a \sqcup b = b$;  

$(c_{43})$ \quad $a \sqcup a = a$ and $a \sqcup 1 = 1$;  

$(c_{44})$ \quad $a \rightarrow (b \rightarrow c) = (a \rightarrow c) \lor (b \rightarrow c)$.

**Proof:** $(c_{40}) - (c_{43})$. Obvious.

$(c_{44})$. We have $((a \rightarrow c) \rightarrow (b \rightarrow c)) \rightarrow (b \rightarrow c) = (b \rightarrow ((a \rightarrow c) \rightarrow c) \rightarrow (b \rightarrow c) = b \rightarrow ((a \rightarrow c) \rightarrow c) = b \rightarrow (a \rightarrow c)$ and analogously $((b \rightarrow c) \rightarrow (a \rightarrow c)) \rightarrow (a \rightarrow c) = a \rightarrow (b \rightarrow c)$. Since, by $(c_{41})$, $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$, we obtain $(a \rightarrow c) \lor (b \rightarrow c) = (b \rightarrow (a \rightarrow c)) \land (a \rightarrow (b \rightarrow c)) = a \rightarrow (b \rightarrow c)$.

**Remark 8.** There are Hilbert algebras with infimum in which the pseudo-join is not the join (relative to the natural order): in Example 1.1 from [9], we consider the elements $b$ and $c$. Clearly $b \lor c = d$, but $b \lor c = 1$.

After that, Nemitz introduces the notion of semi-Boolean lattices, which are implicative semilattices where the pseudo-join of every two elements is a join. In what follows we will work with a similar notion, but in the more general case of Hilbert algebras with infimum.
Remark 9. Let $A$ be a semi-Boolean lattice (according to Nemitz). Then $A$ is a Hilbert algebra with a latticeal structure which verifies property $(P)$ and in which for any two elements $a, b \in A$, $a \lor b = a \lor b = [(a \rightarrow b) \rightarrow b] \land [(b \rightarrow a) \rightarrow a]$. Then, by Theorem 4, in $A$ we have $(a \rightarrow b) \lor (b \rightarrow a) = 1$ for any $a, b \in A$. This means that Theorem 5 and Corollary 9 provide new rules of calculus in a semi-Boolean lattice.

Definition 8. If in a Hilbert algebra $A$ with infimum, the pseudo-join of any two elements is a join (relative to the natural order), then we say that $A$ is a semi-Boolean Hilbert algebra.

Remark 10. An example of a semi-Boolean Hilbert algebra can be found in [9], example 2.3. It is a simple exercise to see that for every two elements their pseudo-join is the join relative to the natural order. That algebra is not a Heyting algebra and since it is a Hilbert algebra with infimum, we conclude that it is neither a Hertz algebra hence it does not verify property $(P)$ (see Lemma 1).

Theorem 6. Let $A$ be a Hilbert algebra with infimum. Then the following conditions are equivalent:

(i) $A$ is a semi-Boolean Hilbert algebra;

(ii) $(a \lor b) \rightarrow c = (a \rightarrow c) \land (b \rightarrow c)$ for every $a, b, c \in A$;

(iii) $a \lor (b \land c) = (a \lor b) \land (a \lor c)$ for every $a, b, c \in A$;

(iv) If $a, b \in A$ such that $a \leq b$, then $a \lor c \leq b \land c$ for any element $c$ of $A$.

Proof: (i) $\Rightarrow$ (ii). See Corollary 4.

(iii) $\Rightarrow$ (i). Let $a, b \in A$ and $t \in A$ such that $a, b \leq t$. Then $(a \lor b) \rightarrow t = (a \rightarrow t) \land (b \rightarrow t) = 1 \land 1 = 1$, hence $a \lor b \leq t$, so $a \lor b = a \lor b$.

(ii) $\Rightarrow$ (iii). Clearly.

(iii) $\Rightarrow$ (i). Let $a, b \in A$ and $t \in A$ such that $a, b \leq t$. Then $(a \lor b) \lor t = a \lor (b \lor t) = a \lor (b \lor t)$, hence $a \lor b \leq t$, so $a \lor b = a \lor b$.

(i) $\Rightarrow$ (iv). Clearly.

(iv) $\Rightarrow$ (i). Let $a, b \in A$ and $t \in A$ such that $a, b \leq t$. From $a \leq t$, we deduce that $a \lor b \leq b \lor t = t$, hence $a \lor b \leq t$, so $a \lor b = a \lor b$. $\square$

In [14], a theorem is proved which offers a characterization of semi-Boolean lattices. We will give a similar result for the more general case of semi-Boolean Hilbert algebras. We start by defining the notion of regular and dense elements of a Hilbert algebra.

Let $A$ be a bounded Hilbert algebra. We recall that for $a \in A$ we denote $a^* = a \rightarrow 0$. An element $a \in A$ will be called regular if $a^{**} = a$ and dense if $a^* = 0$. We denote by $R(A)$ the set of regular elements of $A$ (clearly $R(A) = \{ a^* : a \in A \}$) and by $D(A)$ the set of dense elements of $A$. 
Theorem 7. ([4],p.179)If $A$ is a bounded Hilbert algebra, then $R(A)$ becomes a Boolean algebra, where for $a,b \in R(A)$,

\[
\begin{align*}
& a \land b = (a \rightarrow b^*)^* \in R(A), \\
& a \lor b = (a^* \land b^*)^* \in R(A), \\
& a' = a^* \in R(A).
\end{align*}
\]

The map $\varphi_A : A \rightarrow R(A)$, $\varphi_A(a) = a^{**}$ for every $a \in A$ is an onto morphism of bounded Hilbert algebras.

Remark 11. The above theorem is known in literature under the name of Glivenko’s Theorem. More recently, Cignoli and Torrens ([5]) generalized the theorem to a class of bounded BCK-algebras, while Rump ([19]) gives an analysis of Glivenko’s theorem for the case of KL-algebras. A similar result for the case of unbounded Hilbert algebras can be found in [18].

Remark 12. In [4] it is proved that in a bounded Hilbert algebra $A$, we have the following identity:

\[ (c_{45}) (a \rightarrow b)^{**} = a^{**} \rightarrow b^{**} \text{ for every } a, b \in A. \]

This means that if $a, b \in R(A)$, then $a \rightarrow b = (a \rightarrow b)^{**}$, so $a \rightarrow b \in R(A)$. We conclude that $R(A)$ is a Hilbert subalgebra of the Hilbert algebra $A$. Also, we must observe that for every $a, b \in R(A)$, $a \rightarrow_{R(A)} b = a' \lor_{R(A)} b = a^* \lor_{R(A)} b = a^{**} \rightarrow_A b = a \rightarrow_A b$.

Corollary 10. If $A$ is a bounded Hilbert algebra with infimum, then for every $a, b \in R(A)$ $a \land_{R(A)} b = a \land b$, that is, $R(A)$ is a meet-sub-semilattice of $A$.

Proof: If $a, b \in R(A)$, by Theorem 7, $a \land_{R(A)} b = (a \rightarrow b)^* \leq a, b$. To prove $(a \rightarrow b)^* = a \land b$, let $t \in A$ such that $t \leq a, b$. We deduce that $t^{**} \leq a^{**} = a$ and $t^{**} \leq b^{**} = b$. Because $t^{**} \in R(A)$, we have $t^{**} \leq a \land_{R(A)} b = (a \rightarrow b)^*$. Since $t \leq t^{**}$, we deduce that $t \leq a \land_{R(A)} b$, hence $a \land_{R(A)} b = a \land b$.

The following rules of calculus will be useful to obtain a characterization for semi-Boolean Hilbert algebras.

Lemma 5. Let $A$ be a bounded Hilbert algebra with infimum. Then for every $a, b, c \in A$ we have:

(i) $a \rightarrow (b \rightarrow (a \land b)) = 1$;
(ii) $a \rightarrow (a \land b) = a \rightarrow b$;
(iii) $a \rightarrow (b \rightarrow c) = (a \land b) \rightarrow c$;
(iv) $a \lor_{R(A)} b = (a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a = a \lor b$. 
Proof: Using Theorem 7 and Remark 12 we deduce that (i), (ii) and (iv) are true because they are true in the Boolean algebra $R(A)$.

We prove (iii). Following Lemma 2 we have $a \to (b \to c) \leq (a \land b) \to c$. Also, using (ii), we have $[(a \land b) \to c] \to [a \to (b \to c)] = a \to [(a \land b) \to c] \to (b \to c) = [(a \to (a \land b)) \to (a \to c)] \to [a \to (b \to c)] = [(a \to b) \to (a \to c)] \to [a \to (b \to c)] = 1.

The next Theorem contains some properties of dense elements in a Hilbert algebra.

Theorem 8. ([4],p.179-180) If $A$ is a bounded Hilbert algebra, then:

(i) $D(A)$ is a deductive system of $A$ (hence a Hilbert subalgebra of $A$);

(ii) For every $a \in A$, $a^{**} \to a \in D(A)$;

(iii) For every $a \in A$, $a \in D(A)$ iff $a = b^{**} \to b$ for some $b \in A$;

(iv) For every $a \in A$, there exists $a^{**} \land (a^{**} \to a)$ and $a^{**} \land (a^{**} \to a) = a$;

(v) For every $a \in A$, $(a^{**} \to a) \to a = a^{**}$.

Corollary 11. ([4],p.180) Let $A$ be a bounded Hilbert algebra. Then:

(i) For every $a \in A$, there exist $r_a \in R(A)$ and $d_a \in D(A)$ such that $a = r_a \land d_a$ (namely, $r_a = a^{**}$ and $d_a = a^{**} \to a$);

(ii) For $a, b \in A$, $a^{**} = b^{**}$ iff there are $d_1, d_2 \in D(A)$ such that $d_1 \to x = d_2 \to y$.

We will also need the following results:

Lemma 6. Let $A$ be a Hilbert algebra with infimum and $x, y, z, t \in A$ such that $x \leq y$ and $x \to z \leq x \to t$. Then $x \land z \leq y \land t$.

Proof: From $x \to z \leq x \to t$, we obtain $x \land (x \to z) \leq x \land (x \to t)$ and by $(a_{10})$ we have $x \land z \leq x \land t \leq y \land t$. \hfill $\square$

Lemma 7. Let $A$ be a bounded Hilbert algebra and $a, b \in A$. Then there exists $[(a^{**} \to b^{**}) \to b^{**}] \land [b^{**} \to ((a \to b) \to b)]$ and

$$(c_{46}) \quad (a \to b) \to b = [(a^{**} \to b^{**}) \to b^{**}] \land [b^{**} \to ((a \to b) \to b)].$$

Proof: Clearly $(a \to b) \to b \leq [(a \to b) \to b]^{**} \quad (c_{46}) (a^{**} \to b^{**}) \to b^{**}$ and $(a \to b) \to b \leq b^{**} \to ((a \to b) \to b)$. To prove $(c_{46})$, let $t \in A$ such that $t \leq (a^{**} \to b^{**}) \to b^{**}$ and $t \leq b^{**} \to ((a \to b) \to b)$. Then $a^{**} \to b^{**} \leq t \to b^{**} \leq t \to ((a \to b) \to b)$, hence $a \to b \leq (a \to b)^{**} = a^{**} \to b^{**} \leq t \to ((a \to b) \to b)$, therefore $t \leq (a \to b) \to ((a \to b) \to b) = (a \to b) \to b$. This means that $(c_{46})$ holds. \hfill $\square$
Corollary 12. Let $A$ be a bounded Hilbert algebra with supremum and $a, b \in A$. Then

\[(\forall \mathbf{x}) \ a \lor b = [(a^{**} \to b^**) \to b^**] \land [(b^{**} \to a^{**}) \to a^{**}] \land (a^{**} \to (a \to b)) \land [(b \to a) \to a] \land [(a \to b) \to b].\]

Let’s consider a bounded Hilbert algebra $A$ with supremum. We saw that $R(A)$ becomes a meet-semilattice of $A$. In what follows, we shall say that $R(A)$ is a join-semilattice of $A$ if for every $a, b \in R(A)$ there exists $a \lor b \in A$ and $a \lor b \in R(A)$. It is obvious that if $R(A)$ is a join-semilattice of $A$, then for every $a, b \in R(A)$ we have $a \lor_{A} b = a \lor_{R(A)} b$ and using Lemma 5, we obtain $a \lor_{A} b = a \lor_{R(A)} b = a \lor b$.

Corollary 13. If $A$ is a bounded Hilbert algebra with supremum such that $R(A)$ is a join-semilattice of $A$, then for every $a, b \in A$ we have

\[a \lor b = (a^{**} \lor b^{**}) \land [(a^{**} \to ((b \to a) \to a)] \land [(b^{**} \to ((a \to b) \to b)].\]

Theorem 9. Let $A$ be a bounded Hilbert algebra with supremum. Then the following conditions are equivalent:

(i) $A$ is a semi-Boolean Hilbert algebra;

(ii) $R(A)$ is a join-semilattice of $A$ and for every $d_1, d_2 \in D(A)$ there exists $d_1 \lor d_2 = d_1 \lor d_2 = d_1 \lor d_2 = d_1 \lor d_2$.

Proof: (i) $\Rightarrow$ (ii). Suppose $A$ is a semi-Boolean Hilbert algebra. Then for every $a, b \in A$, $a \lor_{A} b = a \lor_{A} b = [(a \to b) \to b] \land [(b \to a) \to a]$. By Remark 12 and Corollary 10, if $a, b \in R(A)$ then $a \lor_{A} b = [(a \to b) \to b] \land_A [(b \to a) \to a] = [(a \to b) \to b] \land_{R(A)} [(b \to a) \to a] \in R(A)$, hence $R(A)$ is a join-semilattice of $A$.

(ii) $\Rightarrow$ (i). To prove that $A$ is a semi-Boolean Hilbert algebra, let $a, b$ and $c$ elements of $A$ such that $a, b \leq c$. To prove that $a \lor b = a \lor b$ it is sufficiently to prove that $a \lor b \leq c$. By Theorem 8, $c = c^{**} \lor d_c$, where $d_c = c^{**} \leq c \in D(A)$. Similarly, we denote $d_a = a^{**} \to a$ and $d_b = b^{**} \to b$, $d_a, d_b \in D(A)$ and let also be $e_a = b^{**} \to d_a$ and $e_b = a^{**} \to d_b$ (clearly $e_a, e_b \in D(A)$ since $d_a \leq e_a$ and $d_b \leq e_b$).

By Corollary 13 we have $a \lor b = (a^{**} \lor b^{**}) \land [(a^{**} \to ((b \to a) \to a)] \land [(b^{**} \to ((a \to b) \to b)].$ As in the case of Lemma 6 we denote $x = a^{**} \lor b^{**}$, $y = [(a^{**} \land b^{**}) \to y] \land [(a^{**} \land b^{**}) \to y] \land [(a^{**} \land b^{**}) \to y]$ and $z = c^{**}$ and $t = d_c$. Since $x \leq z$, to prove $a \lor b \leq c \Leftrightarrow x \land y \leq z \land t$, by Lemma 6 it is sufficient to prove that

\[(1) \ x \lor y \leq x \to t\]

Since $R(A)$ is a join-semilattice of $A$ and a Boolean algebra, we have $x = a^{**} \lor b^{**} = (a^{**} \land b^{**}) \lor (a^{**} \land b^{**}) \lor (a^{**} \land b^{**})$. Then, by Lemma 3 we have

\[(2) \ x \lor y = [(a^{**} \land b^{**}) \to y] \land [(a^{**} \land b^{**}) \to y] \land [(a^{**} \land b^{**}) \to y] \lor [(a^{**} \land b^{**}) \to y] \land [(a^{**} \land b^{**}) \to y] \land [(a^{**} \land b^{**}) \to y] \land [(a^{**} \land b^{**}) \to y] \land [(a^{**} \land b^{**}) \to y].\]

So, to prove (1), by (2) and (3) it is sufficient to prove that:

\[(4) \ (a^{**} \land b^{**}) \to y \leq (a^{**} \land b^{**}) \to d_c\]

\[(5) \ (a^{**} \land b^{**}) \to y \leq (a^{**} \land b^{**}) \to d_c\]

\[(6) \ (a^{**} \land b^{**}) \to y \leq (a^{**} \land b^{**}) \to d_c.\]
With the notations \( a \sqcup b = (a \rightarrow b) \rightarrow b \) and \( b \sqcap a = (b \rightarrow a) \rightarrow a \) we have \( y = (a^{**} \rightarrow (b \sqcup a)) \land (b^{**} \rightarrow (a \sqcup b)) \), so
\[
(a^{**} \land b^{**}) \rightarrow y = (a^{**} \land b^{**}) \rightarrow [(a^{**} \rightarrow (b \sqcup a)) \land (b^{**} \rightarrow (a \sqcup b))] \leq \\
[(a^{**} \land b^{**}) \rightarrow (a^{**} \rightarrow (b \sqcup a))] \land [(a^{**} \land b^{**}) \rightarrow (b^{**} \rightarrow (a \sqcup b))] \leq \\
[(a^{**} \land b^{**}) \rightarrow (a^{**} \rightarrow (b \sqcup a))] = \\
(a^{**} \land b^{**}) \rightarrow (b \sqcup a).
\]

Since \( 0 \leq a \), we have \( b^{*} \leq b \rightarrow a \), so \( (b \rightarrow a) \rightarrow a \leq b^{*} \rightarrow a \), hence \( b \sqcup a \leq b^{*} \rightarrow a \), therefore
\[
(a^{**} \land b^{**}) \rightarrow y \leq (a^{**} \land b^{**}) \rightarrow (b \sqcup a) \leq \\
(a^{**} \land b^{**}) \rightarrow (b^{*} \rightarrow a) = (a^{**} \land b^{**}) \rightarrow a \leq (a^{**} \land b^{**}) \rightarrow d_{c}
\]
(since \( a \leq c \leq c^{**} \rightarrow c = d_{c} \)). So we have (4). Analogously (5) can be proved.

To prove (6), from \( a \leq d_{c} \) we obtain \( b^{**} \rightarrow a \leq b^{**} \rightarrow d_{c} \), hence \( a^{**} \rightarrow (b^{**} \rightarrow a) \leq a^{**} \rightarrow (b^{**} \rightarrow d_{c}) \). Similarly, \( e_{b} \leq (a^{**} \land b^{**}) \rightarrow d_{c} \), hence \( e_{a} \sqcup e_{b} = e_{a} \lor e_{b} \leq (a^{**} \land b^{**}) \rightarrow d_{c} \).

We will prove that
\[
(a^{**} \land b^{**}) \rightarrow y = e_{a} \sqcup e_{b} \quad \text{and so (6) follows.}
\]

Since \( a \leq (b \rightarrow a) \rightarrow a \leq a^{**} \rightarrow ((b \rightarrow a) \rightarrow a) \) and similarly \( a \leq a^{**} \rightarrow ((a \rightarrow b) \rightarrow b) \). We deduce that \( a \leq y \), hence \( (a^{**} \land b^{**}) \rightarrow a \leq (a^{**} \land b^{**}) \rightarrow y \).

Since, by Lemma 5,(iii) \( (a^{**} \land b^{**}) \rightarrow a = a^{**} \rightarrow (b^{**} \rightarrow a) = e_{a} \), we conclude that \( e_{a} \leq (a^{**} \land b^{**}) \rightarrow y \). Similarly \( e_{b} \leq (a^{**} \land b^{**}) \rightarrow y \), so \( e_{a} \sqcup e_{b} = e_{a} \lor e_{b} \leq (a^{**} \land b^{**}) \rightarrow y \).

On the other hand, it is easy to see that \( (e_{a} \rightarrow e_{b}) \rightarrow e_{b} = b^{**} \rightarrow [a^{**} \rightarrow (a \sqcup b)] \) and \( (e_{b} \rightarrow e_{a}) \rightarrow e_{a} = a^{**} \rightarrow [b^{**} \rightarrow (b \sqcup a)] \), so, using Lemma 5,(iii) we have
\[
(a^{**} \land b^{**}) \rightarrow y = a^{**} \rightarrow (b^{**} \rightarrow y) = \\
a^{**} \rightarrow (b^{**} \rightarrow [(a^{**} \rightarrow (b \sqcup a)) \land (b^{**} \rightarrow (a \sqcup b))]) \leq \\
a^{**} \rightarrow (b^{**} \rightarrow (a^{**} \rightarrow (b \sqcup a)) \land (b^{**} \rightarrow (a \sqcup b))) = \\
a^{**} \rightarrow (a^{**} \rightarrow (b^{**} \rightarrow (b \sqcup a)) \land (b^{**} \rightarrow (a \sqcup b))) \leq \\
[a^{**} \rightarrow (a^{**} \rightarrow (b^{**} \rightarrow (b \sqcup a))) \land [a^{**} \rightarrow (b^{**} \rightarrow (a \sqcup b))] = \\
[a^{**} \rightarrow (b^{**} \rightarrow (b \sqcup a)) \land [a^{**} \rightarrow (b^{**} \rightarrow (a \sqcup b))] = \\
[(e_{a} \rightarrow e_{b}) \rightarrow e_{b}] \land [(e_{b} \rightarrow e_{a}) \rightarrow e_{a}] = e_{a} \sqcup e_{b},
\]
hence (7) holds.

\[\square\]

\textbf{Remark 13.} If \( A \) is a Hilbert algebra with infimum and property (P), then we obtain the main result from [14] for implicative semilattices.

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References


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