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Boundedness analysis for certain two-dimensional differential systems via a Lyapunov approach

by
Cemil Tunç

Abstract

In this paper, the problem of boundedness of solutions of a two-dimensional differential system is considered. Based on the Lyapunov function approach, a new boundedness criterion is derived in terms of this system. An example is given to show the effectiveness of our result.

Key Words: Boundedness, two-dimensional differential system.
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1 Introduction

In 1980, Sinha [1] discussed asymptotic stability of null solution of the following two-dimensional differential system:

$$x' = f(x,y) + p_1(t)x + r(t)y,$$

$$y' = g(x,y) + s(t)x + p_2(t)y.$$

In this paper, instead of the preceding system, we consider the following two-dimensional differential system:

$$\begin{aligned} x' &= f(x,y) + p_1(t)x + r(t)y + p_3(t,x,y), \\ y' &= g(x,y) + s(t)x + p_2(t)y + p_4(t,x,y), \end{aligned} \tag{1}$$

where the prime denotes differentiation with respect to t, $t \in \mathbf{R}^+ = [0,\infty)$; $f,\ g,\ p_1,\ p_2,\ p_3,\ p_4,\ r$ and s are continuous functions in their respective arguments on $\mathbf{R}^2$, $\mathbf{R}^2$, $\mathbf{R}^+$, $\mathbf{R}^+$, $\mathbf{R}^+ \times \mathbf{R}^2$, $\mathbf{R}^+ \times \mathbf{R}^2$, $\mathbf{R}^+$ and $\mathbf{R}^+$, respectively; $r(t)$ and $s(t)$ are bounded functions, $f(0,y) = g(x,0) = 0$, and it is also assumed that the derivatives $f_x(x,y) \equiv \frac{\partial f}{\partial x}(x,y)$ and $g_y(x,y) \equiv \frac{\partial g}{\partial y}(x,y)$ exist and are continuous. We assume further that

$$p_1(t) \neq 0, p_2(t) \neq 0,$$

$$2\int_0^t p_1(s)ds = R_1(t) + Q_1(t),$$

$$2\int_0^t p_2(s)ds = R_2(t) + Q_2(t),$$

where the functions $R_i(.)$ and $Q_i(.)$, $(i = 1,\ 2)$, are defined on $\mathbf{R}^+ = [0, \infty)$, and $|Q_1(t)| < c_1$, $|Q_2(t)| < c_2$, $R_1'(t) < 0$, $R_2'(t) < 0$, in which c_1 and c_2 are some positive constants.

The motivation of this paper comes from the paper of Sinha [1]. Our aim is to improve the result established in [1] to the system (1) for boundedness of the solutions. We also give an example to illustrate the effectiveness of our result. In particular, one can refer to the papers of Tunç ([2, 3, 4]), C. Tunç and E. Tunç [5], Tunç and Şevli [6] and the references cited in these papers for some works performed on boundedness of the solutions. It is worth mentioning that our result is new and original.

2 Problem Description

We establish the following theorem.

Theorem. In addition to the basic assumptions imposed on the functions $f,\ g,\ p_1\ , p_2, p_3\ , p_4, r$ and s that appearing in the system (1), we assume that there exist two positive constants b_1 and b_2 such that the following conditions hold:

(i)

$$-b_1 e^{-Q_1(t)} R_1'(t) > 0, b_1 b_2 e^{-Q_1(t) - Q_2(t)} R_1'(t) R_2'(t) - K^2(t) > 0 \text{ for all } t \in \mathbf{R}^+,$$

where $K(t) = r(t) b_1 e^{-Q_1(t)} + s(t) b_2 e^{-Q_2(t)}$,

$$f_x(x, y) \leq 0 \text{ and } g_y(x, y) \leq 0 \text{ for all } t \in \mathbf{R}^+ \text{ and } x, y \in \mathbf{R},$$

(ii)

$$|p_3(t, x, y)| \leq q_1(t), |p_4(t, x, y)| \leq q_2(t), q_1(t) \leq q(t) \text{ and } q_2(t) \leq q(t) \text{ forall } t \in \mathbf{R}^+$$

and $x,\ y \in \mathbf{R}$,
where $q_1,\ q_2,\ q \in L^1(0, \infty)$, in which $L^1(0, \infty)$ is the space of Lebesgue integrable functions.Then, there exists a positive constant M such that the solution $(x(.),\ y(.))$ of the system (1) satisfies the inequalities

$$|x(t)| \leq M, |y(t)| \leq M$$

for all $t \geq t_0 \geq 0$.

Proof: We employ a Lyapunov function $V = V(t, x, y)$ defined by:

$$V(t, x, y) = b_1 e^{-Q_1(t)} x^2 + b_2 e^{-Q_2(t)} y^2, \tag{2}$$

in which b_1 and b_2 are some positive constants.

It is clear that $V(t, 0, 0) = 0$, and $b_1 e^{-Q_1(t)}$ and $b_2 e^{-Q_2(t)}$ are bounded since $|Q_1(t)| < c_1$ and $|Q_2(t)| < c_2$. Hence, it is seen that the Lyapunov function V is positive definite.

Let $(x(t), y(t))$ be an arbitrary solution of the system (1). Differentiating the function V along the system (1), we have

$$\frac{d}{dt} V(t, x(t), y(t)) =$$

$$= -b_1 Q_1'(t) e^{-Q_1(t)} x^2(t) - b_2 Q_2'(t) e^{-Q_2(t)} y^2(t)$$

$$+2b_1 e^{-Q_1(t)} x(t) \frac{dx(t)}{dt} + 2b_2 e^{-Q_2(t)} y(t) \frac{dy(t)}{dt}$$

$$= -b_1 Q_1'(t) e^{-Q_1(t)} x^2(t) - b_2 Q_2'(t) e^{-Q_2(t)} y^2(t)$$

$$+2b_1 e^{-Q_1(t)} x(t) \{f(x(t), y(t)) + p_1(t) x(t) + r(t) y(t) + p_3(t, x(t), y(t))\}$$

$$+2b_2 e^{-Q_2(t)} y(t) \{g(x(t), y(t)) + s(t) x(t) + p_2(t) y(t) + p_4(t, x(t), y(t))\}.$$

In view of the assumptions

$$2 \int_0^t p_1(s) ds = R_1(t) + Q_1(t)$$

and

$$2 \int_0^t p_2(s) ds = R_2(t) + Q_2(t),$$

the preceding equality leads that

$$\begin{aligned} &\frac{d}{dt} V(t, x(t), y(t)) = \\ &= 2b_1 e^{-Q_1(t)} f(x(t), y(t)) x(t) + 2b_2 e^{-Q_2(t)} g(x(t), y(t)) y(t) \\ &+2b_1 e^{-Q_1(t)} x(t) p_3(t, x(t), y(t)) \\ &+2b_2 e^{-Q_2(t)} y(t) p_4(t, x(t), y(t)) - W_1(t), \end{aligned} \tag{3}$$

where

$$W_1 = -b_1 e^{-Q_1(t)} R_1'(t) x^2(t) -$$

$$-2\{r(t)b_1e^{-Q_1(t)} + s(t)b_2e^{-Q_2(t)}\}x(t)y(t) - b_2e^{-Q_2(t)}R_2'(t)y^2(t).$$

It follows that the expression $W_1(t)$ represents a quadratic form, and one can arrange $W_1(t)$ as the following:

$$W_1(t) = [x(t), y(t)]\, A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

where

$$A = \begin{bmatrix} -b_1e^{-Q_1(t)}R_1'(t) & -r(t)b_1e^{-Q_1(t)} - s(t)b_2e^{-Q_2(t)} \\ -r(t)b_1e^{-Q_1(t)} - s(t)b_2e^{-Q_2(t)} & -b_2e^{-Q_2(t)}R_2'(t) \end{bmatrix}$$

Now, by noting the basic information related to the positive definiteness of a quadratic form, we can conclude that $W_1(t) \geq 0$ provided that

$$-b_1e^{-Q_1(t)}R_1'(t) > 0$$

and

$$b_1b_2e^{-Q_1(t)-Q_2(t)}R_1'(t)R_2'(t) - [r(t)b_1e^{-Q_1(t)} + s(t)b_2e^{-Q_2(t)}]^2$$

$$= b_1b_2e^{-Q_1(t)-Q_2(t)}R_1'(t)R_2'(t) - K^2(t) > 0.$$

Hence, we have

$$\begin{aligned} &\tfrac{d}{dt}V(t, x(t), y(t)) \leq \\ &2b_1e^{-Q_1(t)}f(x(t), y(t))x(t) + 2b_2e^{-Q_2(t)}g(x(t), y(t))y(t) \qquad (4) \\ &+2b_1e^{-Q_1(t)}x(t)p_3(t, x(t), y(t)) + 2b_2e^{-Q_2(t)}y(t)p_4(t, x(t), y(t)). \end{aligned}$$

Let $W_2(t)$ represent the first two terms included in (4):

$$W_2(t) = 2b_1e^{-Q_1(t)}f(x(t), y(t))x(t) + 2b_2e^{-Q_1(t)}g(x(t), y(t))y(t).$$

By the fact $f(0, y) = 0$, $g(x, 0) = 0$ and the generalized mean value theorem for the derivative, we obtain that there exist $\theta_1(t)$ and $\theta_2(t) \in [0, 1]$ such that

$$\begin{aligned} W_2(t) \; &= 2b_1e^{-Q_1(t)}\tfrac{f(x(t),y(t))-f(0,y(t))}{x(t)}x^2(t) \\ &+2b_2e^{-Q_2(t)}\tfrac{g(x(t),y(t))-g(x(t),0)}{y(t)}y^2(t) \\ &= 2b_1e^{-Q_1(t)}f_x(\theta_1x(t), y(t))x^2(t) + 2b_2e^{-Q_2(t)}g_y(x(t), \theta_2y(t))y^2(t). \end{aligned}$$

Making use of the assumptions $f_x(x, y) \leq 0$ and $g_y(x, y) \leq 0$, it follows that $W_2(t) \leq 0$. This fact now yields to the following inequality:

$$\frac{d}{dt}V(t, x(t), y(t)) \leq 2b_1e^{-Q_1(t)}x(t)p_3(t, x(t), y(t)) + 2b_2e^{-Q_2(t)}y(t)p_4(t, x(t), y(t)).$$

By noting the assumption (ii), the inequalities $|y| < 1 + y^2$, $|z| < 1 + z^2$ and the boundedness of $b_1 e^{-Q_1(t)}$ and $b_2 e^{-Q_2(t)}$, the preceding inequality implies that

$$\begin{aligned}
&\tfrac{d}{dt} V(t, x(t), y(t)) \leq \\
&\leq 2b_1 e^{-Q_1(t)} |x(t)| |p_3(t, x(t), y(t))| + 2b_2 e^{-Q_2(t)} |y(t)| |p_4(t, x(t), y(t))| \\
&\leq 2b_1 e^{-Q_1(t)} |p_3(t, x(t), y(t))| x^2 + 2b_2 e^{-Q_2(t)} |p_4(t, x(t), y(t))| y^2(t) \\
&+2b_1 e^{-Q_1(t)} |p_3(t, x(t), y(t))| + 2b_2 e^{-Q_2(t)} |p_4(t, x(t), y(t))| \\
&\leq 2b_1 e^{-Q_1(t)} q_1(t) x^2(t) + 2b_2 e^{-Q_2(t)} q_2(t) y^2(t) \qquad (5) \\
&+2b_1 e^{-Q_1(t)} q_1(t) + 2b_2 e^{-Q_2(t)} q_2(t) \\
&\leq 2\{b_1 e^{-Q_1(t)} x^2(t) + b_2 e^{-Q_2(t)} y^2(t)\} q(t) \\
&+2b_1 e^{-Q_1(t)} q_1(t) + 2b_2 e^{-Q_2(t)} q_2(t) \\
&\leq 2V(t, x(t), y(t)) q(t) + k_1 q_1(t) + k_2 q_2(t),
\end{aligned}$$

where $2b_1 e^{-Q_1(t)} \leq k_1$, $2b_2 e^{-Q_2(t)} \leq k_2$, k_1 and k_2 are some positive constants, which we now assume.

Integrating (5) from 0 to t , using the assumptions $q_1 \in L^1(0, \infty)$, $q_2 \in L^1(0, \infty)$, $q \in L^1(0, \infty)$, and Gronwall-Reid-Bellman inequality, we obtain

$$\begin{aligned}
V(t, x(t), y(t)) &\leq V(0, x(0), y(0)) + k_1 A + k_2 B + 2 \int_0^t V(s, x(s), y(s)) q(s) ds \\
&\leq \{V(0, x(0), y(0)) + k_1 A + k_2 B\} \exp[2 \int_0^t q(s) ds] \\
&= \{V(0, x(0), y(0)) + k_1 A + k_2 B\} \exp(2C) = M_1 < \infty,
\end{aligned}$$

where $M_1 > 0$ is a constant, $A = \int_0^\infty q_1(s) ds$, $B = \int_0^\infty q_2(s) ds$ and $C = \int_0^\infty q(s) ds$.

Now, subject to the above discussion, we arrive at the following:

$$b_1 e^{-Q_1(t)} x^2 + b_2 e^{-Q_2(t)} y^2 = V(t, x, y) \leq M_1.$$

Therefore, one can easily conclude, for some positive constant M, that

$$|x(t)| \leq M, |y(t)| \leq M,$$

for all $t \geq t_0 \geq 0$.

The proof is complete. □

Example. Consider the following two-dimensional differential system:

$$\begin{aligned} x' &= -x^3y^2 + \tfrac{a}{2}(1-\varepsilon\sin\lambda t)x + \tfrac{1}{1+t^2+\sin^2 x+y^2} \\ y' &= -x^2y^5 + \tfrac{b}{2}(1+\varepsilon\cos\lambda t)y + \tfrac{2}{1+t^2+\cos^2 x+y^2}, \end{aligned} \tag{6}$$

where ε and λ ($\neq 0$) are some arbitrary constants, and a and b are some negative constants.

It is clear that the two-dimensional differential system (6) is a special case of the two-dimensional differential system (1). By comparing (6) with (1) and taking into account the assumptions of the theorem, it follows the following:

$$f(x,y) = -x^3y^2,$$

$$f(0,y) = 0,$$

$$f_x(x,y) = -3x^2y^2 \le 0,$$

$$p_1(t)x = \frac{a}{2}(1-\varepsilon\sin\lambda t)x,$$

$$2\int_0^t p_1(s)ds = a\int_0^t (1-\varepsilon\sin\lambda s)ds = (at - \frac{a\varepsilon}{\lambda}) + \frac{a\varepsilon}{\lambda}\cos\lambda t = R_1(t) + Q_1(t),$$

$$R_1(t) = at - \frac{a\varepsilon}{\lambda},$$

$$R_1'(t) = a < 0,$$

$$Q_1(t) = \frac{a\varepsilon}{\lambda}\cos\lambda t,$$

$$|Q_1(t)| = \left|\frac{a\varepsilon}{\lambda}\cos\lambda t\right| \le \frac{|a\varepsilon|}{|\lambda|},$$

$$r(t) = 0,$$

$$p_3(t,x,y) = \frac{2}{1+t^2+\sin^2 x+y^2},$$

$$|p_3(t,x,y)| \le \frac{3}{1+t^2} = q_1(t),$$

$$\int_0^\infty q_1(s)ds = \int_0^\infty \frac{3}{1+s^2}ds = \frac{3\pi}{2} < \infty,$$

$$g(x,y) = -x^2y^5,$$

$$g(x,0) = 0,$$

$$g_y(x,y) = -5x^2y^4 \le 0,$$

$$s(t)x = 0,$$

$$p_2(t)y = \frac{b}{2}(1 + \varepsilon\cos\lambda t)y,$$

$$2\int_0^t p_2(s)ds = b\int_0^t (1 + \varepsilon\cos\lambda s)ds = bt + \frac{b\varepsilon}{\lambda}\sin\lambda t = R_2(t) + Q_2(t),$$

$$R_2(t) = bt,$$

$$R_2'(t) = b < 0$$

$$Q_2(t) = \frac{b\varepsilon}{\lambda}\sin\lambda t,$$

$$|Q_2(t)| = \left|\frac{b\varepsilon}{\lambda}\sin\lambda t\right| \leq \frac{|b\varepsilon|}{|\lambda|},$$

$$p_4(t,x,y) = \frac{1}{1 + t^2 + \cos^2 x + y^2},$$

$$|p_4(t,x,y)| \leq \frac{2}{1+t^2} = q_2(t)$$

and

$$\int_0^\infty q_2(s)ds = \int_0^\infty \frac{2}{1+s^2}ds = \pi < \infty,$$

that is, $q_2 \in L^1(0,\infty)$.

Let $q(t) = \frac{4}{1+t^2}$. Clearly, $q_1(t) \leq q(t)$, $q_2(t) \leq q(t)$ and $q \in L^1(0,\infty)$.

Thus, all the assumptions of Theorem hold. That is, all solutions of the system (6) are bounded.

Conclusion

By means of Lyapunov function approach this paper obtained a new boundedness criterion for a certain two-dimensional differential system. An example is showed to the importance and applicability of this criterion. Our criterion improves an important result obtained on the stability of the null solution of a two-dimensional differential system in the literature to boundedness of the solutions of an extended two-dimensional differential system.

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Department of Mathematics,
Faculty of Arts and Sciences
Yüzüncü Yıl University,
65080, Van, TURKEY.
E-mail: `cemtunc@yahoo.com`