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Vector valued holomorphic functions

by

Elisabetta Barletta and Sorin Dragomir

To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

We survey results on holomorphic functions (of one complex variable) with values in a complex topological vector space hinting to their extension to the case of several complex variables. We give a version of the Hartogs theorem on separate analyticity for weakly holomorphic functions with values in a complex Fréchet space. The theory of α-differentiable functions (due to N. Teodorescu, [27], and extended by F-H. Vasilescu, [28], to functions with values in a Fréchet space) is briefly reviewed as related to areolar derivatives. We present a selection of results on holomorphic functions with values in a complex Banach space with an emphasis on the boundary behavior of vector-valued holomorphic functions. We announce an extension of work by M.S. Baouendi & F. Treves, [3] (on the approximation of CR functions by holomorphic functions) to the case of CR functions with values in a complex Fréchet space.

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1 Introduction

The main purpose of the present paper is to survey some of the known results on holomorphic functions $f : \Omega \to X$ where $\Omega \subset \mathbb{C}^n$ is an open set ($n \geq 1$) and X a complex topological vector space. Vector-valued holomorphic functions are useful in the theory of 1-parameter semigroups (cf. e.g. W. Arendt et al., [2]) and in analytic functional calculus (cf. e.g. F-H. Vasilescu, [29]). Except for the treatment in [29] (dealing with holomorphic functions of several complex variables, with values in a complex Fréchet space, as related to the functional calculus associated to a system of several commuting operators, cf. J.L. Taylor, [25]-[26]) the body of the present day literature is confined to functions of one complex variable mostly with values in a complex Banach space. In the spirit of functional analysis there are essentially two approaches to analyticity of vector-valued functions, through the notions of a *weakly holomorphic* and a (*strongly*) *holomorphic* function $f : \Omega \to X$ (cf. Section 2 for definitions) the first of the two being easier to check in practical examples. Holomorphic functions are always weakly holomorphic. It is then an important question (addressed by K.G. Grosse-Erdmann, [9]-[10], and W. Arendt & N. Nikolski, [1]) to determine the minimal assumptions under which a weakly holomorphic function is strongly holomorphic as well. Let $\Omega \subset \mathbb{C}$ be an open set, X a complex Banach space, and $W \subset X^*$ a subset. Let $\sigma(X, W)$ be the W-topology of X (the weak topology on X induced by W, cf. e.g. [21], p. 62). By a result in [1] a $\sigma(X, W)$-holomorphic function $f : \Omega \to X$ is holomorphic if and only if each $\sigma(X, W)$-bounded set in X is bounded. If $f : \Omega \to X$ is additionally assumed to be locally bounded then the mere knowledge that $\Lambda \circ f \in \mathcal{O}(\Omega)$ for any $\Lambda \in W$ and some separating subspace $W \subset X^*$ implies that $f : \Omega \to X$ is holomorphic (cf. Theorem 15 below). A generalization of this result (to the case where X is a locally convex space assumed to be locally complete, cf. [22]) was obtained by F.G. Grosse-Erdmann, [10] (cf. Theorem 16 below).

The exposition is organized as follows. In Section 2 we review the result that weakly holomorphic functions (of one complex variable) with values in a locally convex space X are (strongly) continuous and discuss the extension of the result to the case of functions of several complex variables (cf. Theorem 2). If additionally X is a complex Fréchet space then each weakly holomorphic function $f : \Omega \subset \mathbb{C}^n \to X$ ($n \geq 1$) may be shown to be holomorphic (cf. Theorem 4). This is a classical result by A. Grothendieck, [11]. Sections 3 to 5 describe a selection of results in the theory of holomorphic functions with values in a complex Fréchet or Banach space and report on results by W. Arendt & N. Nikolski, [1], F.G. Grosse-Erdmann, [10], F-H. Vasilescu, [28], and P. Vieten, [30]. Section 6 discusses the extension (to the case of vector-valued CR functions) of a result by M.S. Baouendi & F. Treves, [3], on uniform approximation of CR functions with holomorphic functions. Theorem 19 in Section 6 is new.

While writing the present survey the Authors kept in mind three possible developments of the theory that is i) recovering results in complex analysis in several complex variables, ii) allowing for more general complex topological vector spaces X as target spaces (e.g. Fréchet spaces instead of Banach spaces), and iii) building a theory of vector-valued CR functions (an open problem so far).

2 Holomorphic functions

Let $\Omega \subset \mathbb{C}^n$ be an open subset ($n \geq 1$) and X a complex topological vector space. A function $f : \Omega \to X$ is *weakly holomorphic* in Ω if $\Lambda \circ f \in \mathcal{O}(\Omega)$ i.e. $\Lambda \circ f : \Omega \to \mathbb{C}$ is a holomorphic function for any $\Lambda \in X^*$. Also $f : \Omega \to X$ is (*strongly*) *holomorphic* if for any $a \in \Omega$ there is a neighborhood $a \in U \subset \Omega$ and a series $\sum_{|\alpha| \geq 0} (z - a)^\alpha x_\alpha$ with $x_\alpha \in X$ such that $\sum_{|\alpha|=0}^{\infty} (z - a)^\alpha x_\alpha = f(z)$ for any $z \in U$.

As each $\Lambda \in X^*$ is continuous, strongly holomorphic functions are weakly holomorphic as well. As to the converse one may state (cf. Theorem 3.31 in [21], p. 82)

Theorem 1. *Let X be a locally convex space. Let $\Omega \subset \mathbb{C}$ be an open set. Let $f : \Omega \to X$ be a weakly holomorphic function. Then*

i) *f is strongly continuous in Ω.*

Let us additionally assume that the closed convex hull of $f(\Gamma)$ is a compact subset of X for any $\Gamma \in \mathbf{\Gamma}(\Omega)$. Then

ii) *If $\Gamma \in \mathbf{\Gamma}(\Omega)$ is a curve such that $\mathrm{Ind}_\Gamma(z) = 0$ for any $z \in \mathbb{C} \setminus \Omega$ then*

$$\int_\Gamma f(\zeta)\, d\zeta = 0, \tag{2.1}$$

$$f(z) = \frac{1}{2\pi i} \int_\Gamma (\zeta - z)^{-1} f(\zeta)\, d\zeta, \quad z \in \Omega, \ \ \mathrm{Ind}_\Gamma(z) = 1. \tag{2.2}$$

Moreover

$$\int_{\Gamma_1} f(\zeta)\, d\zeta = \int_{\Gamma_2} f(\zeta)\, d\zeta, \tag{2.3}$$

for any $\Gamma_1, \Gamma_2 \in \mathbf{\Gamma}(\Omega)$ such that $\mathrm{Ind}_{\Gamma_1}(z) = \mathrm{Ind}_{\Gamma_2}(z)$ for each $z \in \mathbb{C} \setminus \Omega$. Let us additionally assume that X is a complex Fréchet space. Then

iii) *f is $\mathbb{C}$-differentiable at each $z_0 \in \Omega$ that is the limit*

$$\lim_{z \to z_0} (z - z_0)^{-1} [f(z) - f(z_0)]$$

exists in the topology of X for any $z_0 \in \Omega$.

Here $\mathbf{\Gamma}(\Omega)$ denotes the set of all closed rectifiable curves $\Gamma = \{\gamma(t) : a \leq t \leq b\}$ in $\Omega \subset \mathbb{C}$ and

$$\mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta - z}, \quad \Gamma \in \mathbf{\Gamma}(\Omega), \quad z \in \mathbb{C}.$$

Formulae (2.1)-(2.2) are respectively Cauchy's theorem and Cauchy's formula for X-valued weakly holomorphic functions (cf. e.g. Theorem 1.5 and formula (1.59) in [24], p. 42-49, for the scalar valued counterpart). The proof of strong continuity of weakly holomorphic functions relies essentially on Cauchy's formula for ordinary (scalar valued) holomorphic functions (cf. [21], p. 83-84). The assumption that X is a locally convex space is exploited in two ways. First in a locally convex space weakly bounded subsets are (strongly) bounded (cf. Theorem 3.18 in [21], p. 70). Second for any locally convex space X the dual X^* separates points (cf. Corollary to Theorem 3.4 in [21], p. 59-60).

When $n \geq 2$ Cauchy's formula (for holomorphic functions of one complex variable) plays a similar role. Indeed let $\Omega \subset \mathbb{C}^n$ be an open set ($n \geq 2$) and let $a \in \Omega$. Let $\rho = (\rho_1, \cdots, \rho_n)$ be a polyradius ($\rho_j > 0$) such that the polydisc $\overline{P}(a, 2\rho) = \{z \in \mathbb{C}^n : |z_j - a_j| \leq 2\rho_j, \ 1 \leq j \leq n\}$ is contained in Ω. Let $f : \Omega \to X$ be a weakly holomorphic function and $\Lambda \in X^*$. Let us set $a^{(j)} = (a_1, \cdots, a_j) \in \mathbb{C}^j$ for $1 \leq j \leq n$. As $\Lambda \circ f : \Omega \to \mathbb{C}$ is holomorphic for each $z \in P(a, 2\rho)$ (by applying twice Cauchy's formula)

$$\Lambda[f(z)] - \Lambda[f(a)] =$$

$$= \frac{1}{2\pi i} \sum_{j=1}^n (z_j - a_j) \int_{|\zeta_j - a_j| = 2\rho_j} \frac{\Lambda\left[f\left(a^{(j-1)}, \zeta_j, z_{j+1}, \cdots, z_n\right)\right]}{(\zeta_j - z_j)(\zeta_j - a_j)} d\zeta_j .$$

Let $M(\Lambda) = \sup\left\{|\Lambda[f(\zeta)]| : \zeta \in \overline{P}(a, 2\rho)\right\}$. If $z \in \overline{P}(a, \rho) \setminus \{a\}$ then

$$|\Lambda[f(z)] - \Lambda[f(a)]| \leq M(\Lambda)\, |z - a| \sum_{j=1}^n \frac{1}{\rho_j} .$$

Consequently the set

$$\left\{|z - a|^{-1} [f(z) - f(a)] : z \in \overline{P}(a, \rho) \setminus \{a\}\right\} \tag{2.4}$$

is weakly bounded in X hence strongly bounded, as well. Let $V \subset X$ be an open neighborhood of the origin $0 \in X$. As (2.4) is bounded there is $s > 0$ such that

$$f(z) - f(a) \in t|z - a|V, \quad z \in \overline{P}(a, \rho) \setminus \{a\}, \quad t \geq s .$$

As every topological vector space has a balanced local base of neighborhoods of the origin (cf. e.g. [21], p. 13) it follows that f is (strongly) continuous in a. We have shown that

Theorem 2. *Let X be a locally convex space and $\Omega \subset \mathbb{C}^n$ an open set ($n \geq 2$). Any weakly holomorphic holomorphic function $f : \Omega \to X$ is strongly continuous.*

Once the continuity of weakly holomorphic functions $f : \Omega \to X$ is proved one may use Theorem 3.27 in [21], p. 78, to conclude that under the assumptions of Theorem 1 above the integrals $\int_\Gamma f(\zeta)\, d\zeta$ and $\int_\Gamma (\zeta - z)^{-1} f(\zeta)\, d\zeta$ are well defined elements of X (as Bochner integrals i.e. in the sense of Definition 3.26 in [21], p. 77). Then (2.1)-(2.3) hold by the classical Cauchy's formula and Cauchy's theorems applied to the holomorphic function $\Lambda \circ f$ for any $\Lambda \in X$. Similarly for functions of $n \geq 2$ complex variables one has

Theorem 3. *Let X be a locally convex space and $\Omega \subset \mathbb{C}^n$ an open set ($n \geq 2$). Let $a \in \Omega$ and $\rho = (\rho_1, \cdots, \rho_n)$ a polyradius such that $\overline{P}(a, \rho) \subset \Omega$. Let $f : \Omega \to X$ be a weakly holomorphic function such that the closed convex hull of $f\,[\partial_0 P(a, \rho)]$ is a compact subset of X. Then for each $z \in P(a, \rho)$*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 P(a,\rho)} f(\zeta_1, \cdots, \zeta_n) \prod_{j=1}^{n} (\zeta_j - z_j)^{-1}\, d\zeta_1 \cdots d\zeta_n\,. \tag{2.5}$$

Here $\partial_0 P(a, \rho) = \prod_{j=1}^n S^1(a_j, \rho_j)$ is the essential boundary of $P(a, \rho)$ and $S^1(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$ with $z_0 \in \mathbb{C}$ and $r > 0$.

Part (iii) in Theorem 1 is stated under the assumption that X is a Fréchet space. This guarantees that $\overline{\mathrm{co}}[F(\Gamma)]$ is a compact set, where $F(z) = z^{-2} f(z)$, $z \in \Gamma = \{\zeta \in \mathbb{C} : |\zeta| = 2r\}$, hence both Cauchy's theorem (2.1) and Cauchy's integral formula (2.2) hold for the holomorphic function F (and the proof in [21], p. 84, applies).

Let X be a complex Fréchet space and $f : \Omega \to X$ a weakly holomorphic function. For $a \in \mathbb{C}^n$ we set

$$\Omega_{j,a} = \{z \in \mathbb{C} : (a_1, \cdots, a_{j-1}, z, a_{j+1}, \cdots, a_n) \in \Omega\}$$

and $f_{j,a}(z) = f(a_1, \cdots, a_{j-1}, z, a_{j+1}, \cdots, a_n)$ for any $z \in \Omega_{j,a}$. Each $f_{a,j}$ is weakly holomorphic in $\Omega_{j,a}$ hence (by Theorem 1) strongly holomorphic in $\Omega_{j,a}$. Is then $f : \Omega \to X$ holomorphic? Equivalently does Hartogs' theorem hold for X-valued functions possessing this property (cf. [17], p. 43, for $X = \mathbb{C}$)? We may state

Theorem 4. *Let X be a complex Fréchet space, $\Omega \subset \mathbb{C}^n$ an open set ($n \geq 1$), and $f : \Omega \to X$ a weakly holomophic function. Then f is strongly holomorphic.*

To establish Theorem 4 one follows the arguments in the proof of the classical Hartogs' theorem (cf. e.g. [17], p. 43-44). The proof is considerably easier because weakly holomorphic functions are readily continuous (while Hartogs' theorem assumes but separate analyticity to start with). It actually suffices to show

Theorem 5. *Let X be a complex Fréchet space. Let Ω be the polydisc $\{z \in \mathbb{C}^n : |z_j| < R,\ 1 \leq j \leq n\}$ with $R > 0$. Let $f : \Omega \to X$ be a weakly holomorphic function. Then there is $0 < r < R$ and a power series $\sum_{|\alpha| \geq 0} z^\alpha x_\alpha$ with $x_\alpha \in X$ converging uniformly on $P(0, \mathbf{r})$ such that $f(z) = \sum_{|\alpha|=0}^\infty z^\alpha x_\alpha$ for any $z \in P(0, \mathbf{r})$ (here $\mathbf{r} = (r, \cdots, r)$).*

Proof: Let $D_r(0) = \{z \in \mathbb{C} : |z| < r\}$. As $f : \Omega \to X$ is continuous (cf. Theorem 2) it is Bochner integrable on the product of the circles $T_\rho = \prod_{j=1}^n \{\zeta_j \in \mathbb{C} : |\zeta_j| = \rho\}$, $0 < \rho < R$, i.e. $f \in L^1(T_\rho, X, d\zeta)$. Let $z' = (z_1, \cdots, z_{n-1})$ such that $|z_j| < R$ for any $1 \leq j \leq n-1$ and note that $D_\rho(0) \subset \Omega_{n,z'}$. As argued above $f_{n,z'}$ is holomorphic in $\Omega_{n,z'}$ and in particular in $D_\rho(0)$ hence (by Theorem 1) for $|z_n| < \rho$

$$f(z', z_n) = \frac{1}{2\pi i} \int_{|\zeta_n| = \rho} (\zeta_n - z_n)^{-1} f(z', \zeta_n)\, d\zeta_n.$$

For fixed $z_1, \cdots, z_{n-2}$ with $|z_j| < R$, $1 \leq j \leq n-2$, and fixed $\zeta_n \in D_\rho(0)$ the function $f(z_1, \cdots, z_{n-1}, \zeta_n)$ is holomorphic in the disc $|z_{n-1}| < \rho$ hence we may repeat the procedure above. In the end for any $|z_j| < \rho$, $1 \leq j \leq n$, one has

$$(2\pi i)^n f(z_1, \cdots, z_n) = \tag{2.6}$$

$$= \int_{|\zeta_n| = \rho} d\zeta_n \int_{|\zeta_{n-1}| = \rho} d\zeta_{n-1} \cdots \int_{|\zeta_1| = \rho} \prod_{j=1}^n (\zeta_j - z_j)^{-1} f(\zeta_1, \cdots, \zeta_n)\, d\zeta_1.$$

Let $\overline{P}(0, \mathbf{r}) = \{z \in \mathbb{C}^n : |z_j| \leq r\}$ where $\mathbf{r} = (r, \cdots, r)$ and $0 < r < \rho$. Let $z \in \overline{P}(0, \mathbf{r})$ and $\zeta \in T_\rho$. Then $\prod_{j=1}^n (\zeta_j - z_j)^{-1} = \sum_{|\alpha|=0}^\infty z^\alpha / \zeta^{\alpha + \mathbf{1}}$ where $\alpha + \mathbf{1} = (\alpha_1 + 1, \cdots, \alpha_n + 1)$ and the series converges uniformly for $\zeta \in T_\rho$ and $z \in \overline{P}(0, \mathbf{r})$. Let $f_\alpha(z, \zeta) = \left(z^\alpha / \zeta^{\alpha + \mathbf{1}}\right) f(\zeta)$. Then

Lemma 1. *For any $z \in \overline{P}(0,\mathbf{r})$ the series $\sum_{|\alpha|\geq 0} f_\alpha(z,\zeta)$ is convergent in the topology of X uniformly with respect to $\zeta \in T_\rho$.*

By Lemma 1 we may integrate $\sum_{|\alpha|\geq 0} f_\alpha(z,\zeta)$ term-by-term (with respect to $\zeta \in T_\rho$) and obtain (by (2.6))

$$f(z) = \sum_{|\alpha|=0}^{\infty} z^\alpha x_\alpha\,, \quad z \in \overline{P}(0,\mathbf{r}), \;\; 0 < r \leq \rho,$$

$$x_\alpha = \frac{1}{(2\pi i)^n} \int_{|\zeta_n|=\rho} d\zeta_n \int_{|\zeta_{n-1}|=\rho} d\zeta_{n-1} \cdots \int_{|\zeta_1|=\rho} \left(1/\zeta^{\alpha+\mathbf{1}}\right) f(\zeta)\, d\zeta_1 \in X.$$

Finally Theorem 5 follows from

Lemma 2. *The series $\sum_{|\alpha|\geq 0} z^\alpha x_\alpha$ converges in X uniformly for $z \in \overline{P}(0,r)$.*

□

Theorem 4 goes back to A. Grothendieck, [11]. The following Liouville type theorem holds (the proof is independent of Theorem 1).

Theorem 6. *Let X be a complex topological vector space such that X^* separates points. If $f : \mathbb{C} \to X$ is weakly holomorphic and $f(\mathbb{C})$ is a weakly bounded subset of X then f is constant.*

3 α-Differentiability *versus* areolar derivatives

The scope of this section is to discuss α-differentiability of functions $f : \Omega \subset \mathbb{C} \to X$ with values in a Fréchet space (cf. F-H. Vasilescu, [28]) as related to areolar derivatives (cf. D. Pompeiu, [19]-[20], N. Teodorescu, [27]). Let $\alpha : [0,2\pi] \times [0,+\infty) \to \mathbb{C}$ be a continuous function such that

$$\lim_{r\to 0} \frac{1}{2\pi r} \int_0^{2\pi} \alpha(\theta,r)d\,\theta = 0 \tag{3.1}$$

Let $z_0 \in \mathbb{C}$ and $\Omega \subset \mathbb{C}$ an open neighborhood of z_0. Let X be a complex topological vector space such that X^* separates points and let $f : \Omega \to X$ be a continuous function. For $r > 0$ we consider the function $F_r : [0,2\pi] \to X$ given by $F_r(\theta) = \alpha(\theta,r) f(z_0 + re^{i\theta})$ for any $0 \leq \theta \leq 2\pi$. If i) there is $r_0 > 0$ such that $\overline{\text{co}}\,[F_r([0,2\pi])]$ is a compact subset of X for any $0 < r \leq r_0$ and ii) the limit

$$\lim_{r\to 0} \frac{1}{2\pi r} \int_0^{2\pi} \alpha(\theta,r) f(z_0 + re^{i\theta})\, d\theta \tag{3.2}$$

exists in the topology of X then f is said to be α-*differentiable* in z_0 and the limit (3.2) is denoted by $(\partial_\alpha f)(z_0)$.

Let $\alpha : [0,2\pi] \times [0,+\infty) \to \mathbb{C}$ be a continuous function with the property (3.1). Let X be a Fréchet space. Let $\Omega \subset \mathbb{C}$ be an open set and $f \in C^\infty(\Omega, X)$. Then

$$(\partial_\alpha f)(z_0) = \alpha_1 \frac{\partial f}{\partial x}(z_0) + \alpha_2 \frac{\partial f}{\partial y}(z_0) \tag{3.3}$$

where $2\pi\alpha_1 = \int_0^{2\pi} \cos\theta\; \alpha(\theta,0)\, d\theta$ and $2\pi\alpha_2 = \int_0^{2\pi} \sin\theta\; \alpha(\theta,0)\, d\theta$. Consequently the restriction $\partial_{\alpha,\infty} : C^\infty(\Omega,X) \subset C(\Omega,X) \to C(\Omega,X)$ of ∂_α to $C^\infty(\Omega,X)$ is a preclosed operator (cf. F-H. Vasilescu, [28], Lemma 4.1, p. 1030) hence it admits a unique minimal closed extension $\overline{\partial}_\alpha = (\partial_{\alpha,\infty})^- : \mathcal{D}\left[(\partial_{\alpha,\infty})^-\right] \subset C(\Omega,X) \to C(\Omega,X)$ (the closure of $\partial_{\alpha,\infty}$). We set $B^1_\alpha(\Omega,X) = \mathcal{D}\left(\overline{\partial}_\alpha\right)$. Let μ be the Lebesgue measure on $\mathbb{R}^2$. We may state (cf. Lemma 4.3 in [28], p. 1031).

Theorem 7. *Let X be a Fréchet space and $\Omega \subset \mathbb{C}$ an open subset. Let $f,\ g \in C(\Omega, X)$. The following statements are equivalent* a) $f \in B^1_\alpha(\Omega, X)$ *and* $\overline{\partial}_\alpha f = g$, *and* b) *For any* $\varphi \in C^\infty_0(\Omega)$

$$\int_\Omega (\partial_\alpha \varphi)(z)\, f(z)\, d\mu(z) = -\int_\Omega \varphi(z)\, g(z)\, d\mu(z). \tag{3.4}$$

Theorem 7 describes the weak solutions to $\overline{\partial}_\alpha f = g$ with $g \in C(\Omega, X)$. The statement in [28] is however weaker than the proved result: there one assumes *a priori* that $f \in B^1_\alpha(\Omega, X)$ while one actually shows that any continuous function f satisfying (3.4) is of class B^1_α. An ingredient in the proof is the fact that $\overline{\partial}_\alpha : B^1_\alpha(\Omega, X) \subset C(\Omega, X) \to C(\Omega, X)$ is a closed operator (motivating the need to work with the closure of $\partial_{\alpha,\infty}$).

Note that the function $\alpha(\theta) = 2(\alpha_1 \cos\theta + \alpha_2 \sin\theta)$, $0 \le \theta \le 2\pi$ obeys to (3.1) and leads to the same expression (3.3) for $(\partial_\alpha f)(z_0)$. We work with this choice of α from now on. We may state (cf. Theorem 6.5 in [29], p. 24)

Theorem 8. *Let X be a complex Fréchet space, $\Omega \subset \mathbb{C}$ an open set and $f \in C(\Omega, X)$. Then* i) *if $(2\pi r)^{-1}\int_0^{2\pi} \alpha(\theta) f(z + re^{i\theta})\, d\theta$ converges uniformly on the compact subsets of Ω as $r \to 0$ then $f \in B^1_\alpha(\Omega, X)$. Also* ii) *for any $f \in B^1_\alpha(\Omega, X)$*

$$\left(\overline{\partial}_\alpha f\right)(z) = \lim_{r\to 0} \frac{1}{2\pi r} \int_0^{2\pi} \alpha(\theta) f(z + re^{i\theta})\, d\theta, \quad z \in \Omega. \tag{3.5}$$

The notion of areolar derivative may be extended to vector valued functions (cf. L-J. Nicolescu, [18], I. Cioränescu, [6], F-H. Vasilescu, [28]) as follows. Let $\Omega \subset \mathbb{C}$ be a domain and $z_0 \in \Omega$ a point. Let X be a topological vector space such that X^* separates points. Let $f \in C(\Omega, X)$ be a continuous function such that $\overline{\text{co}}[f(\partial\omega)]$ is a compact subset of X for any domain $\omega \subset \mathbb{C}$ with simple rectifiable boundary and such that $z_0 \in \omega$ and $\overline{\omega} \subset \Omega$. We set

$$F(\omega) = \frac{1}{2i} \int_{\partial\omega} f(z)\, dz. \tag{3.6}$$

We adopt the following definition. Let X be a locally convex space. If the following limit exists

$$\varphi(z_0) = \lim_{\text{diam}(\omega)\to 0} \frac{F(\omega)}{|\omega|} \in X \tag{3.7}$$

then f is said to be (*strongly*) *monogeneous* at z_0 and $\varphi(z_0)$ is referred to as the (*strong*) *areolar derivative* of f at z_0. Here $|\omega| = \mu(\omega)$ is the Lebesgue measure of ω. One adopts the traditional notation $\varphi(z_0) = (Df/D\omega)(z_0)$. Let $\mathcal{P}$ be a separating family of seminorms inducing the topology of X as a locally convex space. By the existence of $\varphi(z_0) = \lim_{\text{diam}(\omega)\to 0} F(\omega)/|\omega|$ one means that for any $p \in \mathcal{P}$ and any positive integer $k \ge 1$ there is $r > 0$ such that $p\left(F(\omega)/|\omega| - \varphi(z_0)\right) < 1/k$ for any domain $\omega \subset \mathbb{C}$ such that $\partial\omega$ is a simple rectifiable curve, $z_0 \in \omega$ and $\overline{\omega} \subset D_r(z_0)$. Let $f \in C(\Omega, X)$ such that the integral (3.6) is well defined. A vector $\varphi(x_0) \in X$ is the *weak areolar derivative* of f at z_0 if (3.7) holds in the weak sense i.e. for any $\Lambda \in X^*$ and any $\epsilon > 0$ there is $r > 0$ such that $|\Lambda\left(F(\omega)/|\omega| - \varphi(z_0)\right)| < \epsilon$ for any domain $\omega \subset \mathbb{C}$ with simple rectifiable boundary $\partial\omega$ such that $z_0 \in \omega$ and $\overline{\omega} \subset D_r(z_0)$. A function $f : \Omega \to X$ admitting a weak areolar derivative at $z_0 \in \Omega$ is referred to as *weakly monogeneous* at z_0. As $\Lambda[F(\omega)] = (2i)^{-1}\int_{\partial\omega} \Lambda[f(z)]\, dz$ a function $f : \Omega \to X$ admitting a (strong) areolar derivative at z_0 has a weak areolar derivative at that point as well.

Let $\omega = D_r(z_0)$ be a ball in $\mathbb{C}$ such that $\overline{\omega} \subset \Omega$. Let X be a Fréchet space and $f \in C(\Omega, X)$. If the areolar derivative of f at z_0 exists then

$$\frac{Df}{D\omega}(z_0) = \lim_{r\to 0} \frac{F(D_r(z_0))}{|D_r(z_0)|} = \lim_{r\to 0} \frac{1}{2\pi r} \int_0^{2\pi} e^{i\theta} f(z_0 + re^{i\theta})\, d\theta.$$

Then f is α-differentiable with $\alpha(\theta) = e^{i\theta}$ and $(\partial_\alpha f)(z_0) = (Df/D\omega)(z_0)$. On the other hand to the choice $\alpha(\theta) = e^{i\theta}$ there corresponds (cf. (3.3)) the operator $\partial_{\alpha,\infty} = \frac{1}{2}(\partial/\partial x + i\,\partial/\partial y)$. We maintain this choice of α for the reminder of this section and drop the index α everywhere i.e. we adopt the notations $\overline{\partial} = \overline{\partial}_\alpha$ and $B^1(\Omega, X) = B^1_\alpha(\Omega, X)$. We say that $f \in B^1(\Omega, X)$ is *analytic* in Ω if $\overline{\partial} f = 0$. Let $\mathcal{O}(\Omega, X)$ be the set of all analytic functions $f \in B^1(\Omega, X)$. We may state (cf. [29])

Theorem 9. *Let X be a complex Fréchet space and $\Omega \subset \mathbb{C}$ an open set. Let $\{f_\nu\}_{\nu \geq 1} \subset \mathcal{O}(\Omega, X)$ converge in $C(\Omega, X)$ as $\nu \to \infty$ to $f \in C(\Omega, X)$. Then $f \in \mathcal{O}(\Omega, X)$.*

The following version of the Cauchy-Pompeiu formula (for vector-valued functions of class B^1) holds (cf. Theorem 7.1 in [29], p. 26)

Theorem 10. *Let X be a complex Fréchet space, $\Omega \subset \mathbb{C}$ an open set and $f \in B^1(\Omega, X)$. Let $\omega \subset \mathbb{C}$ be a relatively compact open set such that $\overline{\omega} \subset \Omega$ and $\partial\omega$ is a finite union of Jordan rectifiable curves. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\omega} (\zeta - z)^{-1} f(\zeta) d\zeta + \frac{1}{2\pi i} \int_\omega (\zeta - z)^{-1} \left(\overline{\partial} f\right)(\zeta)\, d\zeta \wedge d\overline{\zeta} \tag{3.8}$$

for any $z \in \omega$.

Corollary 1. *Let $f \in \mathcal{O}(\Omega, X)$ and let $\omega \subset \mathbb{C}$ be a relatively compact open subset such that $\overline{\omega} \subset \Omega$ and $\partial\omega$ is a finite union of Jordan rectifiable curves. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\omega} (\zeta - z)^{-1} f(\zeta) d\zeta \tag{3.9}$$

for any $z \in \omega$.

Corollary 2. *Let X be a complex Fréchet space and $\Omega \subset \mathbb{C}$ an open set. Then $\mathcal{O}(\Omega, X) \subset C^\infty(\Omega, X)$.*

In particular any analytic function $f \in \mathcal{O}(\Omega, X)$ satisfies the Cauchy-Riemann equation $f_{\overline{z}} = 0$ in Ω (equivalently f is $\mathbb{C}$-differentiable in Ω, cf. also Remark 1 in Appendix A). Then (cf. Lemma 8.6 in [29], p. 29)

Proposition 1. *Let X be a complex Fréchet space and $\Omega \subset \mathbb{C}$ an open set. Let $f \in \mathcal{O}(\Omega, X)$. Let $z \in \Omega$ and $r > 0$ such that $\overline{D}_r(z) \subset \Omega$. Then*

$$f(\zeta) = \sum_{\nu=0}^{\infty} \frac{(\zeta - z)^\nu}{\nu!} (\partial^\nu f)(z), \quad \zeta \in D_r(z), \tag{3.10}$$

and the series in the right hand side of (3.10) *is uniformly convergent on any compact subset of $D_r(z)$. In particular each $f \in \mathcal{O}(\Omega, X)$ is* (*strongly*) *holomorphic in Ω.*

The basics on power series in Fréchet spaces are given in Appendix A. Let $f \in \mathcal{O}(\Omega, X)$. A point $z_0 \in \mathbb{C}$ is an *isolated singularity* of f if $z_0 \notin \Omega$ and there is $r > 0$ such that $D_r(z_0) \setminus \{z_0\} \subset \Omega$. An isolated singularity z_0 of $f \in \mathcal{O}(\Omega, X)$ is *removable* if there is $F \in \mathcal{O}(\Omega \cup \{z_0\}, X)$ such that $F|_\Omega = f$. As an application of the Cauchy integral formula (3.9) we may prove

Theorem 11. *Let X be a complex Fréchet space and $\Omega \subset \mathbb{C}$ an open set. Let $z_0 \in \mathbb{C}$ be an isolated singularity of $f \in \mathcal{O}(\Omega, X)$. Then z_0 is removable if and only if f is bounded in some neighborhood of z_0.*

Further details on the theory of monogeneous functions (following [18] and [6]) compared to the presentation in Section 3 are given in Appendix B.

4 Holomorphic functions with values in Banach spaces

Let X be a complex Banach space and $\Omega \subset \mathbb{C}$ an open set. If $f : \Omega \to X$ is $\mathbb{C}$-differentiable then $f : \Omega \to X$ is weakly holomorphic hence Cauchy's integral formula (2.2) holds (cf. Theorem 1 above) so that (by the proof of Theorem 4 for $n = 1$) given $z_0 \in \Omega$ and $r > 0$ such that $\overline{D}_r(z_0) \subset \Omega$

$$f(z) = \sum_{\nu=0}^{\infty} (z - z_0)^\nu x_\nu \, , \quad x_\nu = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{\nu+1}} \, d\zeta,$$

converges absolutely for $z \in D_r(z_0)$ [i.e. $f : \Omega \to X$ is (strongly) holomorphic]. The following version of the identity theorem for holomorphic functions with values in a Banach space is known (cf. e.g. [2], p. 456)

Theorem 12. *Let X be a complex Banach space and $Y \subset X$ a closed subspace. Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to X$ a holomorphic function. Let us assume that there is a convergent sequence $\{z_\nu\}_{\nu \geq 1} \subset \Omega$ such that $\lim_{\nu\to\infty} z_n \in \Omega$ and $f(z_\nu) \in Y$ for any $\nu \geq 1$. Then $f(z) \in Y$ for any $z \in \Omega$.*

A subset $N \subset X^*$ is *norming* if $\|x\|_1 = \sup_{\Lambda \in N} |\Lambda(x)|$, $x \in X$, is a norm on X equivalent to the original norm $\| \cdot \|$. A function $f : \Omega \to X$ is *locally bounded* if $\sup_{z \in K} \|f(z)\| < \infty$ for every compact set $K \subset \Omega$. Then (cf. Proposition A.3 in [2])

Theorem 13. *Let $\Omega \subset \mathbb{C}$ be an open set and $N \subset X^*$ a norming set. Let $f : \Omega \to X$ be a locally bounded function such that*

$$\Lambda \circ f \in \mathcal{O}(\Omega), \quad \Lambda \in N. \tag{4.1}$$

Then $f : \Omega \to X$ is holomorphic.

Then statement (iii) in Theorem 1 (rephrased for a Banach space X) is of course less general than Theorem 13 (the requirement of weak holomorphy has been weakened down to (4.1)).

Corollary 3. *Let $\Omega \subset \mathbb{C}$ be a domain and $\omega \subset \Omega$ an open subset. Let $f : \omega \to X$ be a holomorphic function. Let us assume that there is a norming set $N \subset X^*$ such that for each $\Lambda \in N$ there is a holomorphic extension $F_\Lambda : \Omega \to \mathbb{C}$ of $\Lambda \circ f : \omega \to \mathbb{C}$. If $\sup_{\Lambda \in N,\, z \in \Omega} |F_\Lambda(z)| < \infty$ then $f : \omega \to X$ admits a unique holomorphic extension $F : \Omega \to X$.*

Let $\Omega \subset \mathbb{C}$ be a domain and let $A \subset \Omega$ be a subset which contains an accumulation point in Ω. Vitali's theorem asserts that given a locally bounded sequence $\{f_\nu\}_{\nu \geq 1}$ of holomorphic functions on Ω such that $\{f_\nu(z)\}_{\nu \geq 1}$ converges for each $z \in A$ there is a holomorphic function $f \in \mathcal{O}(\Omega)$ such that $\{f_\nu\}_{\nu \geq 1}$ converges to f in the compact open topology. Montel's theorem states that each locally bounded sequence $\{f_\nu\}_{\nu \geq 1}$ of holomorphic functions on Ω admits a subsequence which converges in the compact open topology (cf. e.g. [24]). In the language of functional analysis Montel's theorem asserts that each bounded subset of $\mathcal{O}(\Omega)$ (endowed with the compact-open topology) is relatively compact. Vitali's and Montel's theorems are known to be equivalent. The picture is rather different for vector-valued functions and, in contrast with the scalar case, Montel's theorem doesn't hold for holomorphic functions with values in a Banach space. Nevertheless, a version of Vitali's theorem was proved by W. Arendt & N. Nikolski, [1].

Theorem 14. *Let X be a complex Banach space. Let $\Omega \subset \mathbb{C}$ be a domain and $f_\nu : \Omega \to X$ a sequence of holomorphic functions such that $\sup_{\nu \geq 1,\, z \in D_r(z_0)} \|f_\nu(z)\| < \infty$, $\overline{D}_r(z_0) \subset \Omega$. Let us assume that the set $\Omega_0 = \{z \in \Omega : \lim_{\nu\to\infty} f_\nu(z)$ exists$\}$ has an accumulation point in Ω. Then there is a holomorphic function $f : \Omega \to X$ such that $f^{(k)}(z) = \lim_{\nu\to\infty} f_\nu^{(k)}(z)$ uniformly on the compact subsets of Ω for any $k \in \mathbb{Z}$, $k \geq 0$.*

Another generalization of Vitali's theorem to the vector-valued case (where holomorphic functions are replaced by an appropriate sheaf of smooth functions on $\Omega \subset \mathbb{R}^n$) was produced by E. Jordá Mora, [14].

Let X be a topological vector space. A subset $W \subset X^*$ is *separating* if $\Lambda(x) = 0$ for each $\Lambda \in W$ implies $x = 0$. We may state (cf. again [1]) the following criterion of analyticity

Theorem 15. *Let X be a complex Banach space. Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to X$ a locally bounded function. Let $W \subset X^*$ be a separating subspace such that $\Lambda \circ f \in \mathcal{O}(\Omega)$ for any $\Lambda \in W$. Then $f : \Omega \to X$ is holomorphic.*

The proof follows from Vitali's theorem (Theorem 14 above), Krein-Šmulian's theorem (cf. Theorem 2.7.11 in [16]) and Theorem 13.

A more general version of Theorem 15 was established by F.G. Grosse-Erdmann, [10]. Let X be a locally convex space. Let $B \subset X$ be a closed, absolutely convex and bounded subset. Let X_B denote the linear span over $\mathbb{C}$ of B in X. The Minkowski functional of B is $\mu_B(x) = \inf\{t > 0 : t^{-1}x \in B\}$ fr any $x \in X$. The space X is *locally complete* if $(X_B\,,\, \mu_B)$ is a Banach space for any B as above. It may be shown (cf. S.A. Saxon & L.M. Sánchez Ruiz, [22]) that a locally convex space X is locally complete if and only if $\sum_{\nu \geq 1} a_\nu x_\nu$ converges in X for every bounded sequence $\{x_\nu\}_{\nu \geq 1} \subset X$ and every sequence $\{a_\nu\}_{\nu \geq 1} \subset \mathbb{C}$ with $\sum_{\nu=1}^{\infty} |a_\nu| < \infty$.

Let X be a topological vector space and $\Omega \subset \mathbb{C}$ be an open set. A function $f : \Omega \to X$ is *locally bounded* if for each $z \in \Omega$ there is a neighborhood $z \in U \subset \Omega$ such that f is bounded on U. We may state (cf. Theorem 1 in [10], p. 399)

Theorem 16. *Let X be a locally complete space. Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to X$ a function. If* i) *there is a separating subset $W \subset X^*$ such that $\Lambda \circ f \in \mathcal{O}(\Omega)$ for any $\Lambda \in W$, and* ii) *f is locally bounded, then $f : \Omega \to X$ is holomorphic.*

It should be mentioned that Theorem 16 has already been applied in an array of situations e.g. to the summability of power series (cf. K.G. Grosse-Erdmann, [9]), to weighted spaces of vector-valued holomorphic functions (cf. K.D. Bierstedt & S. Holtmanns, [4]), to Tauberian convergence theorems (cf. [1]) and to the extension of vector-valued meromorphic functions (cf. E. Jordá Mora, [13]).

Let X be a complex Banach space and $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Let $H_p(\mathbb{C}_+, X)$ consist of all holomorphic functions $f : \mathbb{C}_+ \to X$ such that $\|f\|_{H_p(\mathbb{C}_+,X)} = \sup_{x>0} \left(\int_{-\infty}^{+\infty} \|f(x+iy)\|^p \, dy \right)^{1/p} < \infty$. Next we consider $\Sigma_\alpha = \{re^{i\theta} \in \mathbb{C} : r > 0, \ |\theta| < \alpha\}$ and the space $H_p(\Sigma_\alpha, X)$ consisting of all holomorphic functions $f : \Sigma_\alpha \to X$ such that $\|f\|_{H_p(\Sigma_\alpha,X)} = \sup_{|\theta|<\alpha} \left(\int_0^\infty \|f(re^{i\theta})\|^p \, dr \right)^{1/p} < \infty$. By a result of A.M. Sedlecki, [23], the spaces $H_p(\mathbb{C}_+, \mathbb{C})$ and $H_p(\Sigma_{\pi/2}, \mathbb{C})$ are isomorphic for all $0 < p < \infty$. In the vector-valued case, as shown by P. Vieten, [30], the spaces $H_p(\mathbb{C}_+, X)$ and $H_p(\Sigma_{\pi/2}, X)$ are isomorphic for any $1 \leq p < \infty$ and any complex Banach space X. As an application P. Vieten studied (cf. *op. cit.*) the boundary behavior of holomorphic functions $f \in H_p(\Sigma_\alpha, X)$. To report on his findings we need some preparation. Let $L^p(\mathbb{R}, X, dt)$ be the Bochner L^p-space (cf. e.g. [2], p. 14). A function $u \in L^p(\mathbb{R}, X, dt)$ is the *boundary values* of f if f is the Poisson integral of u i.e. $f(z) = \int_{-\infty}^{+\infty} P_z(t) u(t) dt$, $z \in \mathbb{C}_+$ where P_z is the Poisson kernel $P_z(t) = (1/\pi) x [x^2 + (y-t)^2]^{-1}$, $t \in \mathbb{R}$, $z = x + iy \in \mathbb{C}_+$. Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. Let $Y_q(\mathbb{R}) = L^q(\mathbb{R}, \mathbb{C})$ for any $1 < q \leq \infty$ and let $Y_\infty(\mathbb{R})$ be the space of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ vanishing at infinity. Let $1 < p \leq \infty$. An operator $T : Y_q(\mathbb{R}) \to X$ is *p-bounded* if there is $u \in L^p(\mathbb{R}, \mathbb{C})$ such that $\|Tv\| \leq \int_{-\infty}^{+\infty} |v(t)| u(t) dt$ for any $v \in Y_q(\mathbb{R})$. Cf. J. Diestel & J.J. Uhl, [7], for the description of p-bounded operators. An operator $T : Y_\infty(\mathbb{R}) \to X$ is 1-*bounded* if there is a function of bounded variation $\phi : \mathbb{R} \to \mathbb{R}$ such that $\|Tv\| \leq \int_{-\infty}^{+\infty} |v(t)| d\phi(t)$ for any $v \in Y_\infty(\mathbb{R})$. Cf. [2], p. 49-50, for the Riemann-Stieltjes integral of a X-valued function. Let $h_p(\mathbb{C}_+, X)$ consist of all harmonic functions $f : \mathbb{C}_+ \to X$ such that $\|f\|_{h_p(\mathbb{C}_+,X)} < \infty$ where $\|f\|_{h_p(\mathbb{C}_+,X)}$

equals $\sup_{x>0}\left(\int_{-\infty}^{+\infty}\|f(x+iy)\|^p dy\right)^{1/p}$ if $1 \le p < \infty$ and $\sup_{z\in\mathbb{C}_+}\|f(z)\|$ if $p=\infty$. A complex Banach space has the *Radon-Nikodym property* if each Lipschitz function $f : [0,+\infty) \to X$ is differentiable a.e. in $[0,+\infty)$ (cf. Proposition 1.2.4 in [2], p. 19, for an equivalent description of spaces with the Radon-Nikodym property). We may state (cf. Theorem 2 in [30])

Theorem 17. *Let $1 \le p \le \infty$. A harmonic function $f : \mathbb{C}_+ \to X$ belongs to $h_p(\mathbb{C}_+, X)$ if and only if there is a p-bounded operator $T : Y_q(\mathbb{R}) \to X$ such that $f(z) = T(P_z) = \lim_{s\to 0^+}\int_{-\infty}^{+\infty} P_z(t)f(s+it)dt$ for any $z \in \mathbb{C}_+$. If $1 < p \le \infty$ and X has the Radon-Nikodym property there exists $u \in L^p(\mathbb{R}, X)$ which is the boundary values of f.*

The statements about elements in $h_p(\mathbb{C}_+, X)$ apply to those in $H_p(\mathbb{C}_+, X)$ as well. To state a similar result on functions $f \in H_p(\Sigma_\alpha, X)$ let us set $f_\eta(t) = f\,[|t|\,\exp(i\,\mathrm{sign}(t)\,\eta)]$, $t \in \mathbb{R}$, $0 < \eta < \alpha$. A function $u \in L^p(\mathbb{R}, X)$ is the *boundary values* of $f \in H_p(\Sigma_\alpha, X)$ if

$$\lim_{\eta\to\alpha^-}\int_{-\infty}^{+\infty} v(t)f_\eta(t)dt = \int_{-\infty}^{+\infty} v(t)u(t)dt, \quad v \in Y_q(\mathbb{R}).$$

Then (cf. Theorem 3 in [30])

Theorem 18. *Let $1 < p \le \infty$ and $0 < \alpha < \pi$. For each $f \in H_p(\Sigma_\alpha, X)$ there is a p-bounded operator $T : Y_q(\mathbb{R}) \to X$ such that*

$$T(v) = \lim_{\eta\to\alpha^-}\int_{-\infty}^{+\infty} v(t)f_\eta(t)dt, \quad v \in Y_q(\mathbb{R}).$$

If $1 < p \le \infty$ and X has the Radon-Nikodym property there exists a function $v \in L^p(\mathbb{R}, X)$ which is the boundary values of f.

Boundary values of vector-valued holomorphic functions defined on a half plane in $\mathbb{C}$ were previously studied by M. Itano, [12], by solving inhomogeneous Cauchy-Riemann equations (building on the ideas of A. Martineau, [15]). The recent work by P. Domański & M. Langenbruch, [8], develops the theory of hyperfunctions with values in a locally convex (not necessarily metrizable) space X and looks at the natural limits to such a theory i.e. characterizes the locally convex spaces X for which a reasonable theory[1] of X-valued hyperfunctions exists. Vector-valued hyperfunctions can be interpreted as boundary values of vector-valued harmonic or holomorphic functions. The existence of X-valued hyperfunctions is closely related to the solvability of the Laplace equation. A locally convex space X is (*weakly*) *N-admissible* ($N \in \mathbb{Z}$, $N \ge 1$) if for any (bounded) open set $\Omega \subset \mathbb{R}^N$ the N-dimensional Laplace operator $\Delta : C^\infty(\Omega, X) \to C^\infty(\Omega, X)$ is surjective. By a result in [8] if X is $(N+1)$-admissible then a reasonable theory of N-dimensional X-valued hyperfunctions may be built.

5 Vector valued CR functions

Let $(M, T_{1,0}(M))$ be a CR manifold and $\Omega \subset M$ an open subset. Let X be a complex topological vector space. A function $f \in C^1(\Omega, X)$ is said to be a *CR function* if $\overline{Z}(f) = 0$ for any $Z \in \Gamma^\infty(\Omega, T_{1,0}(M))$. Let $\mathrm{CR}^1(\Omega, X)$ denote the set of all CR functions $f : \Omega \to X$. Let $M \subset \mathbb{C}^n$ be a CR submanifold. Given an open subset $\Omega \subset M$ the *CR extension problem* is to look for an open set $D \subset \mathbb{C}^n$ such that $\Omega \subset D$ and the sequence $\mathcal{O}(D, X) \xrightarrow{r} \mathrm{CR}^1(\Omega, X) \to 0$ is exact, where $\mathcal{O}(D, X)$ is the space of all

[1] One that produces a flabby sheaf whose set of sections supported by a compact set $K \subset \mathbb{R}^N$ equals the space $L(\mathcal{A}(K), X)$ of all X-valued linear continuous operators on the space of germs of analytic functions on K (cf. [8], p. 1098).

holomorphic functions $F : D \to X$ and r is the restriction morphism. The known approaches [i.e. the analytic disc (cf. [5], p. 206-221) and the Fourier transform (cf. [5], p. 229-244) techniques] to the solution to the CR extension problem in the scalar case (i.e. $X = \mathbb{C}$) make use of a fundamental result by M.S. Baouendi & F. Treves, [3]. It is our purpose in the present section to announce an extension of the quoted result (cf. also Theorem 1 in [5], p. 191) to the case of CR functions with values in a given complex topological vector space X, as a first step towards the solution to the CR extension problem (in the vector valued case). We may state

Theorem 19. *Let $M \subset \mathbb{C}^n$ be a real hypersurface of class C^2 and $p \in M$. Let X be a complex topological vector space such that X^* separates points. For any open neighborhood $\Omega \subset M$ of p there is an open set $\omega \subset M$ with $p \in \omega \subset\subset \Omega$ such that for each CR function $f \in C^1(\Omega, X)$ with $\overline{\mathrm{co}}\,[f(\overline{\omega})]$ a compact subset of X there is a sequence $\{F_k\}_{k\geq 1}$ of holomorphic functions $F_k \in \mathcal{O}(\mathbb{C}^n, X)$ such that $F_k \to f$ uniformly on ω as $k \to \infty$.*

Our motivation springs from analytic functional calculus (cf. e.g. [26]). The analytic functional calculus is an algebra homomorphism from the algebra of germs of holomorphic functions on a neighborhood of the joint spectrum of a commutative system of operators, with values in a Banach algebra (cf. also [29]). We expect that the accomplishment of our program (as to the solution to the CR extension problem in the vector valued case) will allow for the construction of a CR functional calculus.

Let M be a real hypersurface of $\mathbb{C}^n$ of class C^2. Let $p \in M$. By a result in [5] (cf. Lemma 1, p. 103) there is a system of holomorphic coordinates for $\mathbb{C}^n$ such that p is the origin and $M = \{(z = x + iy, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : y = h(x, w)\}$ where $h : \mathbb{R} \times \mathbb{C}^{n-1} \to \mathbb{R}$ is a function of class C^2 such that $h(0) = 0$ and $Dh(0) = 0$. We set $w = u + iv \in \mathbb{C}^{n-1}$, $t = (x, u) \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n$ and $s = (y, v) \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n$. Also let $\zeta = t + is \in \mathbb{C}^n$. Next we consider $H : \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{R} \times \mathbb{R}^{n-1}$ given by $H(t, v) = (h(x, u + iv), v)$, $t = (x, u)$, so that $H(0) = 0$ and $(\partial H/\partial t)(0) = 0$. Let $\delta > 0$ and $\eta \in C_0^\infty(\mathbb{R})$ such that $0 \leq \eta(t) \leq 1$, $\eta(t) = 1$ for $|t| < \delta$, and $\eta(t) = 0$ for $|t| \geq 2\delta$. Let $\varphi(z) = \eta(|z|)$ for any $z \in \mathbb{C}^n$. We are only interested in the geometry of $M \cap D$ for a small neighborhood $D \subset \mathbb{C}^n$ of the origin hence from now we replace M by $M_\varphi = \{(x + iy, w) : y = h_\varphi(x, w)\}$ where $h_\varphi = \varphi h$. To keep notation simple we drop φ yet we may assume that $\mathrm{supp}(h) \subset T \times V$ for some neighborhoods of the origin $T \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^{n-1}$. One has $M \cap D = \{t + iH(t, v) : t \in T, \ v \in V\}$ hence $M \cap D$ carries a foliation $\mathcal{F}$ such that $(M \cap D)/\mathcal{F} = \{M_v(T) : v \in V\}$ where $M_v(T) = \{t + iH(t, v) : t \in T\}$.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\Gamma = \mathrm{supp}(\varphi) \subset T$ and $\varphi(t) = 1$ for any $t \in T'$ and some open set $T' \subset \mathbb{R}^n$ with $0 \in T' \subset\subset T$. Let $g(t + is) = \varphi(t)$ for any $t + is \in \mathbb{C}^n$. Also we set $\langle \zeta \rangle = \sum_{j=1}^n \zeta_j^2$ for each $\zeta \in \mathbb{C}^n$. Let us consider the map $\zeta : \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{C}^n$ given by $\zeta(t, v) = t + iH(t, v)$, $t \in \mathbb{R}^n$, $v \in \mathbb{R}^{n-1}$.

Lemma 3. *Let X be a complex topological vector space such that X^* separates points and $\Omega = M \cap D$. Let $K_v = \zeta(\Gamma, v)$ and $f \in C(\Omega, X)$ such that $\overline{\mathrm{co}}[f(K_v)]$ is a compact subset of X. Then for any $\zeta \in M_v(T')$, $v \in V$*

$$\lim_{\epsilon \to 0^+} \frac{\epsilon^{-n}}{\pi^{n/2}} \int_{\xi \in M_v(T)} g(\xi) \exp\left\{-\epsilon^{-2}\langle \zeta - \xi \rangle\right\} f(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n = f(\zeta).$$

The limit is uniform in $v \in V$ and $\zeta \in M_v(T')$.

Finally Theorem 19 follows from Lemma 4 below (stated under the assumptions of Lemma 3)

Lemma 4. *Let $f \in C^1(\Omega, X)$ be a CR function defined on a neighborhood Ω of the origin in M such that $\overline{\mathrm{co}}\,[f(K_0)]$ is compact in X. There is an open set $\Omega' \subset M$ with $0 \in \Omega' \subset\subset \Omega$ such that for any $\zeta \in \Omega'$*

$$\lim_{\epsilon \to 0^+} \frac{\epsilon^{-n}}{\pi^{n/2}} \int_{\xi \in M_0(T)} g(\xi) \exp\left\{-\epsilon^{-2}\langle \zeta - \xi \rangle\right\} f(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n = f(\zeta)$$

and the convergence is uniform in $\zeta \in \Omega'$.

Corollary 4. *Let $M \subset \mathbb{C}^n$ be a real hypersurface of class C^2 and $p \in M$. Let X be a complex Fréchet space. For any open neighborhood $\Omega \subset M$ of p there is an open set $\omega \subset M$ with $p \in \omega \subset \Omega$ such that each X-valued CR function of class C^1 on Ω may be uniformly approximated on ω by a sequence of X-valued holomorphic functions on $\mathbb{C}^n$.*

A. Power series arguments

1) Let X be a complex topological vector space and $\Omega \subset \mathbb{C}$ an open set. If $z = x+iy$ we set as customary $\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\,\partial/\partial y)$ and $\partial/\partial \overline{z} = \frac{1}{2}(\partial/\partial x + i\,\partial/\partial y)$. Let $f : \Omega \to X$ be $\mathbb{C}$-differentiable at $z_0 \in \Omega$. Then f admits partial derivatives at z_0. Precisely if $f'(z_0) = \lim_{z \to z_0}(z - z_0)^{-1}[f(z) - f(z_0)] \in X$ then $(\partial f/\partial x)(z_0) = f'(z_0)$ and $(\partial f/\partial y)(z_0) = i\,f'(z_0)$. In particular $(\partial f/\partial z)(z_0) = f'(z_0)$ and $(\partial f/\partial \overline{z})(z_0) = 0$. Also f is differentiable at z_0 (as a function of two real variables) and $(d_{z_0} f)h = f'(z_0)h$ for any $h \in \mathbb{C}$. Viceversa if $f : \Omega \to X$ is differentiable at z_0 and $(\partial f/\partial \overline{z})(z_0) = 0$ then f is $\mathbb{C}$-differentiable at z_0 and $\lim_{z \to z_0}(z - z_0)^{-1}[f(z) - f(z_0)] = (\partial f/\partial x)(z_0)$.

2) Let X be a topological vector space and $\{x_\nu\}_{\nu \geq 0} \subset X$. *If $\sum_{\nu \geq 0} x_\nu$ is convergent then $x_\nu \to 0$ on X as $\nu \to \infty$.*

3) Let X be a Fréchet space and $\{x_\nu\}_{\nu \geq 0}$ a sequence in X. Let $\mathcal{P}$ be a separating family of seminorms defining the topology of X. *If $\sum_{\nu=0}^{\infty} p(x_\nu) < \infty$ for each $p \in \mathcal{P}$ then the series $\sum_{\nu \geq 0} x_\nu$ is convergent in X.*

4) Let X be a complex Fréchet space and $\{x_\nu\}_{\nu \geq 0} \subset X$. *If there is $z_0 \in \mathbb{C} \setminus \{0\}$ such that $\sum_{\nu \geq 0} z_0^\nu x_\nu$ is convergent then $\sum_{\nu \geq 0} z^\nu x_\nu$ is convergent for any $z \in D_{|z_0|}(0)$. Also $\sum_{\nu \geq 0} z^\nu x_\nu$ is uniformly convergent for $z \in D_r(0)$ for any $0 < r < |z_0|$.* The *convergence radius* of $\sum_{\nu \geq 0} z^\nu x_\nu$ is taken to be $R = \sup\{|z_0| : \sum_{\nu \geq 0} z_0^\nu x_\nu$ is convergent in $X\}$. For each $p \in \mathcal{P}$ we set $\ell(p) = \limsup_{\nu \to \infty} p\,(x_\nu)^{1/\nu}$. We may state

Proposition 2. i) *If $0 < \ell(p) < a$ for some $a > 0$ and any $p \in \mathcal{P}$ we set $R = \inf\{1/\ell(p) : p \in \mathcal{P}\}$. Then $R > 0$ and the series $\sum_{\nu \geq 0} z^\nu x_\nu$ is convergent (respectively divergent) for any $z \in D_R(0)$ (respectively for any $z \in \mathbb{C} \setminus \overline{D}_R(0)$).*

ii) *If $0 < \ell(p) < \infty$ for any $p \in \mathcal{P}$ yet there is a sequence $\{p_j\}_{j \geq 1} \subset \mathcal{P}$ such that $\lim_{j \to \infty} \ell(p_j) = \infty$ or $\ell(p) = \infty$ for some $p \in \mathcal{P}$ then $\sum_{\nu \geq 0} z^\nu x_\nu$ is divergent for any $z \in \mathbb{C} \setminus \{0\}$.*

iii) *If $\ell(p) = 0$ for some $p \in \mathcal{P}$ let $\mathcal{P}_0 = \{p \in \mathcal{P} : \ell(p) = 0\}$ and $R = \inf\{1/\ell(p) : p \in \mathcal{P} \setminus \mathcal{P}_0\}$. If $\sup\{\ell(p) : p \in \mathcal{P} \setminus \mathcal{P}_0\} < \infty$ then $R > 0$ and $\sum_{\nu \geq 0} z^\nu x_\nu$ is convergent for any $z \in D_R(0)$ while if $\sup\{\ell(p) : p \in \mathcal{P} \setminus \mathcal{P}_0\} = \infty$ then $R = 0$ and $\sum_{\nu \geq 0} z^\nu x_\nu$ is divergent for any $z \in \mathbb{C} \setminus \{0\}$.*

By slightly restating Proposition 2 one has

Corollary 5. *Let X be a complex Fréchet space and $\{x_\nu\}_{\nu \geq 1} \subset X$. Let $\mathcal{P}$ be a separating family of seminorms determining the topology of X and $\ell(p) = \limsup_{\nu \to \infty} p\,(x_\nu)^{1/\nu}$ for each $p \in \mathcal{P}$. Let $\mathcal{P}_0 = \{p \in \mathcal{P} : \ell(p) = 0\}$ and $R = \inf\{1/\ell(p) : p \in \mathcal{P} \setminus \mathcal{P}_0\}$. Then R is the radius of convergence of the series $\sum_{\nu \geq 0} z^\nu x_\nu$. Precisely* a) *if $\mathcal{P}_0 = \mathcal{P}$ then $R = \infty$ and* b) *if $\mathcal{P} \setminus \mathcal{P}_0 \neq \emptyset$ then either $\sup_{p \in \mathcal{P} \setminus \mathcal{P}_0} \ell(p) = \infty$ and then $R = 0$ or $\sup_{p \in \mathcal{P} \setminus \mathcal{P}_0} \ell(p) < \infty$ and then $0 < R < \infty$.*

5) The *derivative* of $S = \sum_{\nu \geq 0} z^\nu x_\nu$ is by definition the series $S' = \sum_{\nu \geq 0}(\nu+1)z^\nu x_{\nu+1}$. If $\{a_n\}_{n \geq 1}$ is a sequence of nonnegative numbers then $\limsup_{n \to \infty}[(n+1)a_{n+1}]^{1/n} = \limsup_{n \to \infty} a_n^{1/n}$ hence S and its derivative have the same radius of convergence. Then

Proposition 3. *Let R be the radius of convergence of the series $S = \sum_{\nu \geq 0} z^\nu x_\nu$. If $R > 0$ let $f_S : D_R(0) \to X$ given by $f_S(z) = \sum_{\nu=0}^{\infty} z^\nu x_\nu$ for any $|z| < R$. Then f_S is $\mathbb{C}$-differentiable on $D_R(0)$ and $f'_S(z) = f_{S'}(z)$ for any $|z| < R$.*

An alternative approach to Proposition 3 was devised by F-H. Vasilescu (cf. Lemma 8.5 in [29], p. 29)

Proposition 4. *Let X be a complex Fréchet space and $\{p_m\}_{m\geq 1}$ a countable family of seminorms determining the topology of X. Let $\{x_\nu\}_{\nu\geq 0} \subset X$ be a sequence such that $\ell = \sup_{m\geq 1} \limsup_{\nu\to\infty} p_m(x_\nu)^{1/\nu} < \infty$. Then the function $f : D_r(z) \to X$ given by $f(z) = \sum_{\nu=0}^{\infty} z^\nu x_\nu$ for any $z \in D_{1/\ell}(0)$ is analytic that is $f \in \mathcal{O}(D_{1/\ell}(0), X)$ [with $D_{1/\ell}(0) = \mathbb{C}$ when $\ell = 0$].*

B. (α)-Holomorphic functions with values in Fréchet spaces

Let X be a complex Fréchet space and $f : \Omega \to X$ a monogeneous function defined on an open set $\Omega \subset \mathbb{C}$. Following the terminology in [27] we say f is (α)-*holomorphic* if its areolar derivative $\varphi(z) = (Df/D\omega)(z)$ is continuous at any point $z \in \Omega$. Thus the class of (α)-holomorphic functions may be seen as a generalization of $C^1(\Omega, X)$. Indeed each $f \in C^1(\Omega, X)$ is (α)-holomorphic and $Df/D\omega = f_{\overline{z}}$. Is there an analog to the Cauchy-Pompeiu formula for (α)-holomorphic functions? At this stage of the exposition of the theory, although a (α)-holomorphic function is α-differentiable with $\alpha(\theta) = e^{i\theta}$, Theorem 10 doesn't apply (as it is unknown at this point whether f is $B^1(\Omega, X)$ regular). To clear up this matter we transpose a few facts from [27] to the case of X-valued (α)-holomorphic functions. As emphasized by L-J. Nicolescu, [18], the proofs are but straightforward verifications.

A function $f : \Omega \to X$ is *weakly* (α)-*holomorphic* if $\Lambda \circ f : \Omega \to \mathbb{C}$ is (α)-holomorphic as a scalar valued function (cf. [27], p. 8) for each $\Lambda \in X^*$. Any (α)-holomorphic function is weakly (α)-holomorphic. Let $\omega \subset \mathbb{C}$ be a domain such that $\overline{\omega} \subset \Omega$ and $\Gamma = \partial\omega$ is a closed rectifiable curve. If f is (α)-holomorphic then

$$\Lambda\left[\frac{Df}{D\omega}(z)\right] = \frac{D(\Lambda \circ f)}{D\omega}(z)\,, \quad z \in \Omega,\ \ \Lambda \in X^*, \tag{B..1}$$

hence (by (19) in [27], p. 28)

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,dz = \frac{1}{\pi}\int_\omega \frac{Df}{D\omega}(z)\,d\mu(z). \tag{B..2}$$

Cf. also [18], p. 1008. As a consequence of (B..2) the class of (α)-holomorphic functions may be seen as a generalization of $\mathcal{O}(\Omega, X)$. Indeed if $Df/D\omega = 0$ then (by (B..2)) $\int_\Gamma f(z)\,dz = 0$ for any $\Gamma \in \mathbf{\Gamma}(\Omega)$ so that $\int_\Gamma \Lambda[f(z)]\,dz = 0$ for any $\Lambda \in X^*$. Thus (by the classical Cauchy theorem) f is weakly holomorphic in Ω and we may use Theorem 1 in Section 1 to conclude that f is strongly holomorphic.

Let $f : \Omega \to X$ be a (α)-holomorphic function. Then (by (22) in [27], p. 33, and our (B..1))

$$f(\zeta) = \frac{1}{2\pi i}\int_\Gamma (z-\zeta)^{-1} f(z)\,dz - \frac{1}{\pi}\int_\omega (z-\zeta)^{-1}\varphi(z)\,d\mu(z). \tag{B..3}$$

Cf. also (2) in [18], p. 1009. This is the Cauchy-Pompeiu type formula we were seeking for. We may state

Theorem 20. *Let $f : \Omega \to X$ be a (α)-holomorphic function. Let $\omega \subset \mathbb{C}$ be a domain such that $\overline{\omega} \subset \Omega$ and $\Gamma = \partial\omega \in \mathbf{\Gamma}(\Omega)$.* i) *If the areolar derivative of f is a holomorphic function i.e. $Df/D\omega = h$ for some $h \in \mathcal{O}(\Omega, X)$ then*

$$2\pi i f(\zeta) = \int_\Gamma \frac{1}{z-\zeta}\ [f(z) - \overline{z}\,h(z)]\ dz + \int_\Gamma \frac{\overline{\zeta}}{z-\zeta}\,h(z)\,dz. \tag{B..4}$$

In particular $f \in C^\infty(\Omega, X)$. ii) *If f admits continuous areolar derivatives $D^\nu f/D\omega^\nu \in C(\Omega, X)$ of arbitrary order $\nu \geq 0$ which are equi-bounded in Ω then*

$$f(\zeta) = \frac{1}{2\pi i}\sum_{\nu=0}^{\infty}\int_\Gamma \frac{1}{\nu!}\,\frac{\left(\overline{\zeta}-\overline{z}\right)^\nu}{z-\zeta}\,\frac{D^\nu f}{D\omega^\nu}(z)\,dz \tag{B..5}$$

and the convergence is uniform in $\zeta \in \omega$.

The formulae (B..4)-(B..5) for $X = \mathbb{C}$ are due to N. Teodorescu, [27], p. 13-19. When X is a complex Banach space Theorem 20 was established by L-J. Nicolescu, [18].

Let $C^k_{\overline{z}}(\Omega, X)$ be the class of all (α)-holomorphic functions $f : \Omega \to X$ admitting continuous areolar derivatives $D^\nu f/D\omega^\nu \in C(\Omega, X)$ up to order $0 \le \nu \le k$. We may state

Theorem 21. (I. Ciorănescu, [6]) *Let* $k \ge 1$. *If* $\Lambda \circ f \in C^{k+1}_{\overline{z}}(\Omega, \mathbb{C})$ *for any* $\Lambda \in X^*$ *then* $f \in C^k_{\overline{z}}(\Omega, X)$.

In particular the following analog (where areolar derivatives replace ordinary derivatives) to a result by A. Grothendieck, [11], holds (cf. I. Ciorănescu, [6], p. 843).

Corollary 6. *Let* X *be a complex Fréchet space,* $\Omega \subset \mathbb{C}$ *an open set, and* $f : \Omega \to X$ *a continuous function. Then* $f \in C^\infty_{\overline{z}}(\Omega, X)$ *if and only if* $\Lambda \circ f \in C^\infty_{\overline{z}}(\Omega, \mathbb{C})$ *for any* $\Lambda \in X^*$.

There is yet another approach to the Cauchy-Pompeiu type formula (B..3) closing our parallel among the work in [18], [6], and the exposition in [28]. Let $f : \Omega \to X$ be a monogeneous function with the areolar derivative φ. Let $\mathcal{P}$ be a separating family of seminorms determining the topology of X. Following [27], p. 26, we say $(1/|\omega|) \int_\Gamma f(z)\, dz$ converges *uniformly* to $\varphi(z_0)$ as $\omega \to z_0 \in \Omega$ if for any $p \in \mathcal{P}$ and any integer $k \ge 1$ there is $\rho = \rho(p, k)$ (independent of z_0) such that

$$\frac{1}{2\pi i |\omega|} \int_\Gamma f(z)\, dz - \frac{1}{\pi} \varphi(z_0) \in V(p, k) \tag{B..6}$$

for any domain $\omega \subset \mathbb{C}$ such that $\overline{\omega} \subset D_\rho(z_0)$ and $\Gamma = \partial\omega \in \mathbf{\Gamma}(\Omega)$. We may show that

Theorem 22. *Let* X *be a complex Fréchet space and* $\Omega \subset \mathbb{C}$ *an open set. Let* $f \in C^1_{\overline{z}}(\Omega, X)$ *be a* (α)*-holomorphic function and* $z_0 \in \Omega$. *Then* $(1/|\omega|) \int_\Gamma f(z)\, dz$ *converges uniformly to* $\varphi(z_0)$ *as* $\omega \to z_0$.

For $X = \mathbb{C}$ Theorem 22 is due to N. Teodorescu, [27], p. 28. As a consequence of Theorem 22 it follows that $(2\pi r)^{-1} \int_0^{2\pi} e^{i\theta} f(z_0 + re^{i\theta})\, d\theta$ converges uniformly in $z_0 \in \Omega$ as $r \to 0$ hence (by Theorem 8) $f \in B^1(\Omega, X)$. Therefore Theorem 10 may be applied thus yielding (B..3). Theorem 22 follows from

Lemma 5. *Let* $f \in C^1_{\overline{z}}(\Omega, X)$ *be a* (α)*-holomorphic function and* $\varphi = Df/D\omega \in C(\Omega, X)$ *its areolar derivative. Then for any* $p \in \mathcal{P}$ *and any integer* $\ell \ge 1$ *there is* $r = r(p, \ell) > 0$ *such that for each* $z_0 \in \Omega$ *one has* $\varphi(z) - \varphi(z_0) \in V(p, \ell)$ *for any* $z \in \omega$ *and any domain* $\omega \subset \mathbb{C}$ *such that* $\overline{\omega} \subset D_r(z_0)$.

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Università degli Studi della Basilicata,
Dipartimento di Matematica e Informatica,
Via dell'Ateneo Lucano 10,
Campus di Macchia Romana, 85100 Potenza, Italy
E-mail: elisabetta.barletta@unibas.it
sorin.dragomir@unibas.it