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# The Word and Generalized Word Problem for Semigroups under Wreath Products

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**Abstract**

The aim of this paper is to investigate the solvability of the word and generalized word problem for the wreath product of infinite and finite semigroups.

**Key Words**: Decision problems, Word Problems, Wreath products, Finite and Infinite Semigroups.
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## 1 Introduction and Preliminaries

In this paper we generally consider the word and generalized word problem for the wreath product of semigroups. Algorithmic problems such as the *word, conjugacy* and *isomorphism problems* have played an important role in group theory since the work of M.Dehn in early 1900's. These problems are called *decision problems* which ask for a "yes" or "no" answer to a specific question. Among these decision problems especially the word problem has been studied widely in groups and semigroups ([1]).

Let $X$ be a non-empty set. We denote by $X^+$ the free semigroup on $X$ consisting of all non-empty words over $X$. A *semigroup presentation* is an ordered pair $\mathcal{P} = [X; R]$, where $R \subseteq X^+ \times X^+$. An element $x$ of $X$ is called a *generating symbol*, while an element $(u_1, v_1)$ of $R$ is called a *defining relation*, and is usually written as $u_1 = v_1$. Also if $X = \{x_1, \cdots, x_m\}$ and $R = \{u_1 = v_1, \cdots, u_n = v_n\}$, we write $[x_1, \cdots, x_m;\ u_1 = v_1, \cdots, u_n = v_n]$ for $[X; R]$.

In order to define a semigroup associated with $\mathcal{P}$ we introduced the following elementary operation on positive words (the words which do not have negative powers) on $X$. So let $W$ be a positive word on $X$.

- If $W$ contains a subword $r_\epsilon$, where $\epsilon = \pm 1$, $r_+ = r_- \in R$, then replace it by $r_{-\epsilon}$.

Two positive words $W_1$, $W_2$ are *equivalent* (*relative to* $\mathcal{P}$) if there is a finite chain of elementary operations given above leading from $W_1$ to $W_2$. This is an equivalence relation on the set of all positive words on $X$. Let $[W]_{\mathcal{P}}$ denote the equivalence class containing $W$. A multiplication can be defined on equivalence classes by $[W_1]_{\mathcal{P}}.[W_2]_{\mathcal{P}} = [W_1W_2]_{\mathcal{P}}$. It is easy to check that this multiplication is well-defined. The set of all equivalence classes together with this multiplication form a semigroup, the *semigroup defined by* $\mathcal{P}$, denoted by $S(\mathcal{P})$. If both $X$ and $R$ are finite sets then $\mathcal{P} = [X; R]$ is said to be a *finite presentation*. In particular if a semigroup $S$ can be defined by a finite presentation, then $S$ is said to be *finitely presented*. Morever if the generating set $X$ is finite then $S$ is said to be *finitely generated*.

Let $S$ be a semigroup. $S$ is said to have a *solvable word problem with respect to a generating set* $A$ if there exists an algorithm which, for any words $u, v \in A^+$, decides whether the relation $u = v$ holds in $S$ or not. It is a well-known fact that the solvability of the word problem does not depend on the choice of the finite generating set for $S$. In other words, *a finitely generated semigroup* $S$ *has a solvable word problem* if $S$ has a solvable word problem with respect to any finite generating set. Also it is known that when $S$ is a finitely presented semigroup, the word problem for $S$ is solvable if and only if $S$ has a recursively enumerable set of unique normal forms (see [4]).

Throughout this paper, for a mapping $f : A \to B$, and for the elments $a \in A$, $b \in B$, the terminology "$f$ sends $a$ to $b$" will be written on the right by $af = b$. Also we denote by $\mathbb{N}$ the set $\{0, 1, 2, \ldots\}$ of all natural numbers and by $\mathbb{N}^*$ the set $\mathbb{N} \setminus \{0\}$.

For infinite semigroups $S$ and finite semigroups $T$, our main tools in this paper are to investigate the solvability of the word problem for the wreath product $SwrT$ (Section 2) and to examine the generalized word problem for the same wreath product by using the normal form constructions of words (Section 3).

## 2 The Word Problem of Semigroups under Wreath Product

Since wreath products ([10]) can also be considered as special semidirect products, firstly, let us recall the definition of the semidirect product on semigroups $S$ and $T$. So let $\theta : T \longrightarrow End(S)$ be an antimorphism of $T$ into the endomorphism semigroup of $S$. For $t \in T$, let us denote $s(t\theta)$ by ${}^{t}s$. Therefore the semidirect product $S \rtimes_\theta T$ consists of the set $S \times T$ equipped with the multiplication

$$(s, t).(s_1, t_1) = (s\ {}^{t}s_1, tt_1).$$

In addition to this, the definition of the wreath product of semigroups can be given as follows. Let $X$ be a set. Then the set $S^X$ of all mappings $X \to S$ forms a semigroup under "component-wise" multiplication of mappings; this is called the *Cartesian power* of $S$ by $X$. Now let $e$ be a fixed/distinguished element of $S$; the *support* of an $f \in S^X$ *relative to* $e$ is the set defined by

$$supp_e(f) = \{x \in X : xf \neq e\}.$$

The set

$$S^{(X)_e} = \{ f \in S^X : |\operatorname{supp}_e(f)| < \infty \}$$

is a subsemigroup of $S^X$ and is called the *direct power of $S$ relative to $e$*. Then the unrestricted wreath product $SwrT$ is the set $S^T \rtimes_\theta T$ with the multiplication

$$(f,t).(g,u) = (f\ {}^t g, tu), \tag{1}$$

where ${}^t g \in S^T$ is defined by

$$(x)\ {}^t g = (xt)g.$$

Now let us suppose that the element $e$ (in $S$) is a distinguished idempotent. The (restricted) wreath product $S_e wrT$ (with respect to $e$) is the subsemigroup of the unrestricted wreath product $SwrT$, generated by the set

$$\{(f,t) \in SwrT : \ |\operatorname{supp}_e(f)| < \infty\}.$$

In fact if $T$ is finite then, clearly, $SwrT = S_e wrT$. In [11], Robertson et al. gave necessary and sufficient conditions for $S_e wrT$ to be finitely generated and finitely presented where $T$ is finite and infinite. If $T$ is infinite then there must be some restrictions on $S$ (particularly $S$ must be a monoid) for the wreath product $SwrT$ to be finitely generated (see [11, Theorem 5.1]). This is the reason for us why we take $T$ is a finite semigroup for the main results of this paper.

It is known that whenever we have a finite presentation of a semigroup $S$ and we know that $S$ is finite, then we can compute the multiplication table of the semigroup $S$. Let us summarize the construction of multiplication table of a semigroup $S$.

For a given finite presentation $[X; R]$ of a finite semigroup $S$ we start two enumeration processes. The first process lists all relations which follow from $R$. In finite time this procedure realizes that our semigroup is finite and then decreases the upper bound for the size of the semigroup $S$. The second process lists all finite semigroups generated by $X$ and satisfying relations from $R$. This procedure increases the lower bound for the size of the semigroup $S$. So, in finite time the upper bound will be equal to the lower bound and we can construct the multiplication table of $S$ from the relations computed by the first procedure.

Therefore, whenever one speaks about a finite semigroup we expect that the semigroup is given by the multiplication table and hence the (generalized) word problem for this semigroup is trivially solvable. So if $S$ and $T$ are finite semigroups then their wreath product is finite as well, and the multiplication table of $SwrT$ can be computed from the definition of the wreath product. Consequently, $SwrT$ has solvable (generalized) word problem. Hence we will take the semigroups $S$ and $T$ are infinite and finite, respectively for the solvability of word problem of $SwrT$.

Before the main theorem of this paper, we need to give the following important result which is about the finite generation of $SwrT$.

**Theorem 2.1.** [11] *Let $S$ be an infinite semigroup whose diagonal act is finitely generated, and let $T$ be a finite non-trivial semigroup. Then $SwrT$ is finitely generated if and only if the following conditions are satisfied:*

1. *$S^2 = S$ and $T^2 = T$;*
2. *$S$ is finitely generated.*

Now we can give our main result as follows. We should note that the sufficient conditions of Theorem 2.1 are satisfied in this main result.

**Theorem 2.2.** *$SwrT$ has solvable word problem if and only if both $S$ and $T$ have solvable word problem.*

**Proof**: Suppose that $S$ and $T$ have solvable word problem and let

$$w_1 = (f_1, d_1).(f_2, d_2) \cdots (f_n, d_n) \tag{2}$$

be an arbitrary word in $SwrT$. We recall that the first components $f_1, f_2, \cdots, f_n$ of factors of $w_1$ are elements of $S^T$ (that means, $f_1, f_2, \cdots, f_n$ are all mappings from $T$ to $S$) and the second components $d_1, d_2, \cdots, d_n$ of factors of $w_1$ are elements of $T$. By (1), we can write

$$w_1 = (f_1, d_1).(f_2, d_2) \cdots (f_n, d_n) = (f_1 \ {}^{d_1}f_2 \cdots \ {}^{d_1 d_2 \cdots d_{n-1}}f_n, d_1 d_2 \cdots d_n). \tag{3}$$

For any word

$$w_2 = (g_1, h_1).(g_2, h_2) \cdots (g_r, h_r) \tag{4}$$

in $SwrT$, we must examine whether $w_1$ is equivalent to $w_2$ to get the solvability of the word problem for $SwrT$. By the assumption on $T$, it is clear that

$$d_1 d_2 \cdots d_n = h_1 h_2 \cdots h_r.$$

Thus we must just check whether the first components of the words $w_1$ and $w_2$ are equal. Now, by considering (3), when we evaluate the first component of the word $w_1$ by an arbitrary word $w \in T$, we then obtain

$$\begin{aligned} (w)f_1 \ {}^{d_1}f_2 \cdots \ {}^{d_1 d_2 \cdots d_{n-1}}f_n &= (w)f_1 \, (w) \, {}^{d_1}f_2 \ \cdots \ (w) \, {}^{d_1 d_2 \cdots d_{n-1}}f_n \\ &= (w)f_1 \, (wd_1)f_2 \ \cdots \ (wd_1 d_2 \cdots d_{n-1})f_n \\ &= s_1 s_2 \cdots s_n, \end{aligned}$$

where the words $s_1, s_2, \cdots, s_n$ in $S$ are actually the images of $f_1, f_2, \cdots, f_n$ at

$$w, \, wd_1, \, \cdots, \, wd_1 d_2 \cdots d_{n-1},$$

respectively. In fact the assumption on $T$ also gives that

- the word $w$ in $T$ is equivalent to some word $w'$ in $T$,

- the word $wd_1$ in $T$ is equivalent to some word $w'd'_1$ in $T$,

and, by iterating this procedure,

- the word $wd_1d_2\cdots d_{n-1}$ in $T$ is equivalent to $w'd'_1\cdots d'_{n-1}$ in $T$.

Thus, for all $w', w'd'_1, \cdots, w'd'_1\cdots d'_{n-1} \in T$, we obtain $s'_1, s'_2, \cdots, s'_n$ in $S$. It follows that

- $s_1s_2\cdots s_n$ in $S$ is equivalent to some word $s'_1s'_2\cdots s'_n$ in $S$

since $S$ has a solvable word problem. Similarly each of the words $s'_1, s'_2, \cdots, s'_n$ in $S$ is the image of the mappings $g_1, g_2, \cdots, g_r$ at $w', w'd'_1, \cdots, w'd'_1\cdots d'_{n-1}$, respectively. Since $w \in T$ is arbitrary, this implies that

$$f_1 \ {}^{d_1}f_2 \cdots \ {}^{d_1d_2\cdots d_{n-1}}f_n = g_1 \ {}^{h_1}g_2 \cdots \ {}^{h_1h_2\cdots h_{r-1}}g_r.$$

For the sufficient part of the proof, let us suppose that $SwrT$ has solvable word problem. Also let $w_1$ and $w_2$ be some words in $SwrT$ as defined in (2) and (4), respectively. By the meaning of the word problem, the words $w_1$ and $w_2$ are equal to each other and this implies that we obtain

$$f_1 \ {}^{d_1}f_2 \cdots \ {}^{d_1d_2\cdots d_{n-1}}f_n = g_1 \ {}^{h_1}g_2 \cdots \ {}^{h_1h_2\cdots h_{r-1}}g_r \tag{5}$$

for the first components and

$$d_1d_2\cdots d_n = h_1h_2\cdots h_r \tag{6}$$

for the second components. In fact equations (5) and (6) give the solvability of the word problem for $SwrT$, as required.

Hence the result. □

## 3 Solvability of Generalized Word problem

In this section we will discuss solvability of the generalized word problem for the wreath product constructed by free abelian and finite monogenic (cyclic) semigroups. We note that although some of the fundamental facts about *monogenic semigroups* (or monoids) can be found in [3, 5] and [6, Section 1.2], we can give a brief introduction about this special semigroups as in the following paragraph.

Let $S$ be a finite monogenic semigroup of order $k > 1$ generated by $s$. Then $s, s^2, s^3, \cdots, s^k$ all belong to $S$. Due to the definition of a semigroup, the elements $s^{k+1}, s^{k+2}, \cdots$ must be in $S$, but since the order is a finite number $k$, the element $s^{k+1}$ must be equal to an element $s^n$, where $1 \leq n \leq k$. In addition for this finite monogenic semigroup $S$ (which has order $k$ and generated by $s$), two well defined natural numbers are defined, namely the *index* $r$ and the *period* $m$ of $s$; they are related by the formula $r + m = k + 1$.

The following lemma is an immediate consequence of the result Theorem 1.9 in [3].

**Lemma 3.1.** *If $s^p = s^q$ in $S$ with $1 \leq p < q \leq k+1$ then $q = k+1$.*

Besides that a presentation about finite monogenic semigroups can be defined as in the following lemma.

**Lemma 3.2.** *Let $S$ be a finite monogenic semigroup of order $k$ and let $l$ be the index of $S$. Then a presentation of $S$ is*

$$\mathcal{P}_{k+1,l} = [x;\ x^{k+1} = x^l],$$

*where $l < k+1$ and $l, k \in \mathbb{N}^*$.*

**Proof**: Let $S(\mathcal{P}_{k+1,l})$ be a semigroup defined by $\mathcal{P}_{k+1,l}$ and let $[x]_{\mathcal{P}_{k+1,l}}$ denote the equivalence class containing $x$ (as introduced in the first section).

Let us consider the mapping $\psi : X = \{x\} \to S$ sending $x$ to $s \in S$. Since $\psi(x^{k+1}) = \psi(x^l)$, we get an induced homomorphism

$$\psi^{'} : S(\mathcal{P}_{k+1,l}) \to S, \quad [x]_{\mathcal{P}_{k+1,l}} \longmapsto s,$$

from semigroup $S(\mathcal{P}_{k+1,l})$ to the semigroup $S$. Note that $\psi^{'}$ is onto since $s \in Im\psi^{'}$. Clearly $\mathcal{P}_{k+1,l}$ is a complete rewriting presentation (see [2]), and the irreducible elements (elements that can not be applied the relation $x^{k+1} = x^l$ for reduction any more) are

$$x, x^2, \cdots, x^k.$$

Hence the distinct elements of $S(\mathcal{P}_{k+1,l})$ are $[x]_{\mathcal{P}_{k+1,l}}$, $[x^2]_{\mathcal{P}_{k+1,l}}$, $\cdots$, $[x^k]_{\mathcal{P}_{k+1,l}}$ and then $|S(\mathcal{P}_{k+1,l})| = k$. Now if $\psi^{'}$ were not injective then $\left|Im\psi^{'}\right| < |S(\mathcal{P}_{k+1,l})| = k$. But this gives a contradiction. So $\psi^{'}$ is injective, and is thus an isomorphism. □

We have proved that any monogenic semigroup of order $k$ is isomorphic to $S(\mathcal{P}_{k+1,l})$ for some $1 \leq l < k$. Now, for any $1 \leq l < k$, the semigroup $S(\mathcal{P}_{k+1,l})$ is monogenic of order $k$, generated by $[x]_{\mathcal{P}_{k+1,l}}$. We then deduce, up to isomorphism, the monogenic semigroups of order $k$ are

$$S(\mathcal{P}_{k+1,l}), \quad \text{where } l = 1, 2, \cdots, k-1.$$

Hence, since $l, k \in \mathbb{N}^*$ and $l < k+1$, we have the following lemma.

**Lemma 3.3.** *If $l \neq l'$ then $S(\mathcal{P}_{k+1,l}) \ncong S(\mathcal{P}_{k+1,l'})$.*

**Proof**: Let us assume that $l < l'$ and consider the cyclic group $C$ of order $k-l+1$, generated by $c$. By [6, 9], there is a homomorphism $\gamma$ from $S(\mathcal{P}_{k+1,l})$ onto $C$, given by $[x]_{\mathcal{P}_{k+1,l}} \overset{\gamma}{\longmapsto} c$.

Now if there were an isomorphism

$$\omega : S(\mathcal{P}_{k+1,l'}) \to S(\mathcal{P}_{k+1,l}),$$

then the composition $\gamma\omega$, say $\omega'$, would give a homomorphism from $S(\mathcal{P}_{k+1,l'})$ onto $C$. Hence $\gamma'([x]_{\mathcal{P}_{k+1,l'}})$ would have to be a generator, say $\widehat{c}$ of $C$. But since $[x]^{k+1}_{\mathcal{P}_{k+1,l'}} = [x]^{l'}_{\mathcal{P}_{k+1,l'}}$ in $S(\mathcal{P}_{k+1,l'})$, we would have

$$\widehat{c}^{\,k+1} = \gamma'([x]^{k+1}_{\mathcal{P}_{k+1,l'}}) = \gamma'([x]^{l'}_{\mathcal{P}_{k+1,l'}}) = \widehat{c}^{\,l'},$$

so $\widehat{c}^{\,(k-l'+1)} = 1$ in $C$. But since $k - l' < k - l$, this contradicts the fact that the order of $\widehat{c}$ must be $k - l + 1$.

Hence the result. □

For simplicity, let us denote $S(\mathcal{P}_{k+1,l})$ by $S_{k+1,l}$. Summarizing all above material and lemmas, we have the following result for finite monogenic semigroups.

**Theorem 3.4.** *For a fixed $k + 1 > 2$, the semigroups $S_{k+1,l}$ $(1 \leq l \leq k)$ are monogenic of order $k$ and pairwise non-isomorphic. Any monogenic semigroup of order $k$ is isomorphic to $S_{k+1,l}$ for some $l$.*

By using Lemma 3.2 and adapting the proof of the result in [7, Theorem 2.2] to the case of the wreath product of semigroups $S$ and $T$, it is easy to see the proof of the following proposition. (We should note that the key point in this adaptation is ignoring the identity element of monoids).

**Proposition 3.5.** *Let $S$ and $T$ be finite monogenic semigroups with presentations*

$$\mathcal{P}_S = [y;\ y^{k+1} = y^l \ \ (l < k+1)], \qquad \mathcal{P}_T = [x;\ x^{m+1} = x^n \ \ (n < m+1)],$$

*respectively. Then*

$$\begin{aligned}\mathcal{P}_{SwrT} = [y^{(1)}, y^{(2)}, \cdots, y^{(m)}, x \quad ; \quad & x^{m+1} = x^n,\ (y^{(i)})^{k+1} = (y^{(i)})^l, \\ & y^{(i)}y^{(j)} = y^{(j)}y^{(i)} \ \ (1 \leq i < j \leq m), \\ & xy^{(i)} = y^{(i-1)}x \ \ (2 \leq i \leq m),\ xy^{(n)} = y^{(m)}x]\end{aligned}$$

*is a presentation for the wreath product of $S$ by $T$.*

In fact, as a quite special case, we can obtain solvability of the generalized word problem for wreath products by taking $S$ infinite and $T$ finite monogenic. So let us consider the presentation of the wreath product of the free abelian semigroup by finite monogenic semigroup. This presentation can be obtained similarly as in (7) with respect to relators on the generators $y_1, y_2$ of $S$ and generator $x$ of $T$, as in the following.

**Proposition 3.6.** *Let $S$ be a free abelian semigroup and $T$ be a finite monogenic semigroup with presentations*

$$\mathcal{P}_S = [y_1,\ y_2;\ \ y_1y_2 = y_2y_1]\,, \qquad \mathcal{P}_T = \left[x;\ \ x^{m+1} = x^n\ \ (n < m+1)\right],$$

*respectively. Then*

$$\begin{aligned} \mathcal{P}_{SwrT} = [y_1^{(a)},\, y_2^{(a)}\ (1 \leq a \leq m),\, x \quad &; \\ x^{m+1} = x^n\ \ (n < m+1)&, \\ y_i^{(a)}y_j^{(b)} = y_j^{(b)}y_i^{(a)}\ \ (i,j \in \{1,2\},\ i \leq j,\ 1 \leq a,b \leq m)&, \\ xy_1^{(a+1)} = y_1^{(a)}x\ \ (1 \leq a \leq m)&, \\ xy_2^{(b+1)} = y_2^{(b)}x\ \ (1 \leq b \leq m)&, \\ xy_1^{(n)} = y_1^{(m)}x,\ \ xy_2^{(n)} = y_2^{(m)}x&] \end{aligned} \tag{7}$$

*is a presentation for the wreath product of $S$ by $T$.*

Now we can give our attention to the main goal of this section.

Let $S_1$ be an arbitrary semigroup generated by $X$ and let $S_2$ be a subsemigroup of $S_1$. The ***generalized word problem*** for $S_2$ in $S_1$ asks *if there exists an algorithm that decides whether an arbitrary word over $X$ represents an element in the semigroup $S_2$*. For an example of the study on this subject, we can refer the paper [8] which has been examined the generalized word problem for Baumslag-Solitar semigroups.

Now, we give another main result of this paper by the following theorem for solvability of the generalized word problem under wreath product of free abelian semigroup of rank two by finite monogenic semigroup.

**Theorem 3.7.** *The generalized word problem is solvable for $SwrT$, where $S$ is free abelian of rank two and $T$ is finite monogenic semigroup.*

**Proof**: For the proof, firstly, we need to construct the normal forms of the words that belong to $SwrT$. Clearly these words consist of the generators $y_1^{(a)}$, $y_2^{(a)}$ $(1 \leq a \leq m)$ and $x$. Since we actually work on $S^T \rtimes_\theta T$ we can take the *base normal forms* as

$$\left(y_2^{(a_1)}\right)^{p_1}\left(y_2^{(a_2)}\right)^{p_2}\cdots\left(y_2^{(a_r)}\right)^{p_r}\left(y_1^{(b_1)}\right)^{q_1}\left(y_1^{(b_2)}\right)^{q_2}\cdots\left(y_1^{(b_s)}\right)^{q_s}x^c\,, \tag{8}$$

where $1 \leq a_i, b_j \leq m$ , $p_i, q_j \in \mathbb{N}$ $(1 \leq i \leq r,\ 1 \leq j \leq s)$, $1 \leq c \leq m$. In fact other words in $SwrT$ (or, more specially, in $S^T \rtimes_\theta T$) such as

$$\left(y_2^{(a_i+p)}\right)^{c_1}\left(y_1^{(b_j+q)}\right)^{c_2}x^d,$$

where $p \neq q$, $0 \leq p,q \leq m-2$, $c_1, c_2 \in \mathbb{N}$ and $1 \leq d \leq m$ turn into the words of the form given in (8), by using the relators $xy_1^{(a+1)} = y_1^{(a)}x$ $(1 \leq a \leq m)$,

$xy_2^{(b+1)} = y_2^{(b)}x$ $(1 \leq b \leq m)$, $xy_1^{(n)} = y_1^{(m)}x$ and $xy_2^{(n)} = y_2^{(m)}x$ in presentation (7).

Now let $v$ be a word on the set $\left\{y_1^{(a)}, y_2^{(a)}\ (1 \leq a \leq m),\ x\right\}$ and represent some arbitrary element of $SwrT$. Suppose $U = \{u_1, u_2, \cdots, u_t\}$ is a set of words representing the generators for some finitely generated subsemigroup of $SwrT$. By using the last four relators of the presentation in (7), we need to exhibit an algorithm for deciding whether or not $v$ is equivalent to some products of elements of $U$. To do that let us write the set $U$ by a general form such as $\left\{(y_1^{(i)})^p x^c, (y_2^{(i)})^p x^c\right\}$, where $1 \leq i \leq m$, $p \in \mathbb{N}$, $1 \leq c \leq m$. Since the elements can be obtained by a product of the finite number of elements of $U$ and any words taken in $SwrT$ can be formed to the base normal forms given in (8), there is no any word outside of the set $U = \left\{(y_1^{(i)})^p x^c, (y_2^{(i)})^p x^c\right\}$. Therefore the generalized word problem is solvable for the presentation $\mathcal{P}_{SwrT}$ in (7). □

**Remark 3.8.** It is apparent that solvability of the generalized word problem requires solvability of the word problem in groups. This case can be easily seen by taking the subgroup (in definition of generalized word problem) as trivial subgroup. But we form this position by considering the meaning of the word problem in semigroups. As we did in the proof of Theorem 3.7, we can rewrite the set $U$ in an explicit form as follows:

$$\begin{aligned}&\{u_1 = (y_1^{(1)})^p x^c,\ u_2 = (y_2^{(1)})^p x^c, u_3 = (y_1^{(2)})^p x^c,\ u_4 = (y_2^{(2)})^p x^c, \cdots,\\ &u_{2m-1} = (y_1^{(m)})^p x^c,\ u_{2m} = (y_2^{(m)})^p x^c\}, \qquad (9)\end{aligned}$$

where $p \in \mathbb{N}$ and $1 \leq c \leq m$. Any word $w_1$ which is taken from $SwrT$ is actually equivalent to some product of words in the set (9). Hence this gives us solvability of the word problem for wreath product of free abelian semigroup of rank two by finite monogenic semigroup.

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