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An application of Dirichlet L-series to the computation of certain integrals *

by
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Abstract

In this article we compute integrals of the form

$$\int_0^1 \frac{\sum_{n=1}^{q-1} \chi_{-q}(n) x^{n-1}}{1-x^q} \ln\left(\ln\frac{1}{x}\right) dx,$$

where χ_q is the odd Dirichlet character (mod q). The argument is based on the theory of L-series due to Dirichlet. Our formulas were initially found via the PSLQ algorithm of H.R.P. Ferguson.

Key Words: Dirichlet character, L-series, PSLQ algorithm.
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1 Introduction

In the book of I. S. Gradzhteyn and I. M. Ryzhyk [11], the following two formulas may be found:

$$\int_0^1 \ln\left(\ln\frac{1}{x}\right)\frac{dx}{1+x^2} = \frac{\pi}{2}\ln\left[\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\sqrt{2\pi}\right], \quad \text{at page 532} \tag{1.1}$$

and

$$\int_0^1 \ln\left(\ln\frac{1}{x}\right)\frac{dx}{1+x+x^2} = \frac{\pi}{\sqrt{3}}\ln\left[\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}(2\pi)^{1/3}\right], \quad \text{at page 571.} \tag{1.2}$$

Other intricate integrals reduce to these ones via a change of variable, for example,

$$\int_{\pi/4}^{\pi/2} \ln(\ln\tan x)\, dx = \int_0^1 \ln\left(\ln\frac{1}{x}\right)\frac{dx}{1+x^2},$$

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or can be deduced by algebraic manipulations, as it is the case of

$$\int_0^1 \ln\left(\ln\frac{1}{x}\right)\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\left[\frac{5}{6}\ln 2\pi - \ln\Gamma\left(\frac{1}{6}\right)\right]. \tag{1.3}$$

See [11], page 572.

The right way to approach these formulas comes from the Dirichlet L-series theory. In fact, both are of the form

$$I_{-q} = \int_0^1 \frac{\sum_{n=1}^{q-1} \chi_{-q}(n)\, x^{n-1}}{1-x^q} \ln\left(\ln\frac{1}{x}\right) dx, \tag{1.4}$$

where χ_{-q} is the odd Dirichlet character $(\operatorname{mod} q)$. More precisely, the integral (1.1) corresponds to the odd Dirichlet character (mod 4),

$$\chi_{-4} : \mathbb{Z} \to \{-1,0,1\}, \quad \chi_{-4}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \pmod 4 \\ -1 & \text{if} \quad n \equiv 3 \pmod 4 \\ 0 & \text{if} \quad n \equiv 0,2 \pmod 4 \end{cases}$$

while the integral (1.2) corresponds to the odd Dirichlet character (mod 3),

$$\chi_{-3} : \mathbb{Z} \to \{-1,0,1\}, \quad \chi_{-3}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \pmod 3 \\ -1 & \text{if} \quad n \equiv 2 \pmod 3 \\ 0 & \text{if} \quad n \equiv 0 \pmod 3 \end{cases} .$$

The aim of this paper is to compute in compact form all integrals of the type (1.4). Similar integrals were computed by V. Adamchik (see [1] or [2]), but our approach is different.

2 Background on Dirichlet characters

Let $m \in \mathbb{N}^\star$ and f a multiplicative morphism from $(\mathbb{Z}/m\mathbb{Z})^*$ into $\{-1,1\}$. The function

$$\chi : \mathbb{Z} \to \{-1,0,1\}\,;\; \chi(n) = \begin{cases} f(n \operatorname{mod} m) & \text{if} \quad (n,m)=1 \\ 0 & \text{if} \quad (n,m) \neq 1 \end{cases} \tag{2.1}$$

is called a *Dirichlet character* (mod m).

A character is *even* if $\chi(-n) = \chi(n)$ for all n, and *odd* if $\chi(-n) = -\chi(n)$ for all n.

The *trivial* (or the *main*) *character* is given by

$$\chi_{0,m}(n) = \begin{cases} 1 & \text{if} \quad (n,m)=1 \\ 0 & \text{if} \quad (n,m) \neq 1. \end{cases} .$$

Some properties of the Dirichlet character (mod m):

$$\begin{aligned} \chi(1) &= 1; \\ \chi(n+m) &= \chi(n), \text{ for all } n; \\ \chi(n) &= 0, \text{ if } (n,m) \neq 1; \\ \chi(n_1 \cdot n_2) &= \chi(n_1) \cdot \chi(n_2), \text{ for all } n_1, n_2. \end{aligned}$$

If m_1 is a multiple of m we define the *induced character* (mod m_1):

$$\tilde{\chi} : \mathbb{Z} \to \{-1, 0, 1\}, \quad \tilde{\chi}(n) = \begin{cases} \chi(n) & \text{if } (m_1, n) = 1 \\ 0 & \text{if } (m_1, n) \neq 1. \end{cases}.$$

A *primitive character* (mod m) is a character that cannot be induced by any other character.

Examples of primitive characters (see [9], [16], [13] for details):

- $\chi_1(n) = 1$,
- $\chi_{-3}(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3 \\ -1 & \text{if } n \equiv 2 \pmod 3 \\ 0 & \text{if } n \equiv 0 \pmod 3, \end{cases}$
- $\chi_{-4}(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 3 \pmod 4 \\ 0 & \text{if } n \equiv 0, 2 \pmod 4, \end{cases}$
- $\chi_{5}(n) = \begin{cases} 1 & \text{if } n \equiv 1, 4 \pmod 5 \\ -1 & \text{if } n \equiv 2, 3 \pmod 5 \\ 0 & \text{if } n \equiv 0 \pmod 5, \end{cases}$
- $\chi_{-7}(n) = \begin{cases} 1 & \text{if } n \equiv 1, 2, 4 \pmod 7 \\ -1 & \text{if } n \equiv 3, 5, 6 \pmod 7 \\ 0 & \text{if } n \equiv 0 \pmod 7, \end{cases}$
- $\chi_{-8}(n) = \begin{cases} 1 & \text{if } n \equiv 1, 3 \pmod 8 \\ -1 & \text{if } n \equiv 5, 7 \pmod 8 \\ 0 & \text{if } n \equiv 0, 2, 4, 6 \pmod 8, \end{cases}$
- $\chi_{8}(n) = \begin{cases} 1 & \text{if } n \equiv 1, 7 \pmod 8 \\ -1 & \text{if } n \equiv 3, 5 \pmod 8 \\ 0 & \text{if } n \equiv 0, 2, 4, 6 \pmod 8, \end{cases}$
- $\chi_{-11}(n) = \begin{cases} 1 & \text{if } n \equiv 1, 3, 4, 5, 9 \pmod{11} \\ -1 & \text{if } n \equiv 2, 6, 7, 8, 10 \pmod{11} \\ 0 & \text{if } n \equiv 0 \pmod{11}, \end{cases}$

- $\chi_{12}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1, 11 \pmod{12} \\ -1 & \text{if} \quad n \equiv 5, 7 \pmod{12} \\ 0 & \text{if} \quad n \equiv 0, 2, 3, 4, 6, 8, 9, 10 \pmod{12}. \end{cases}$

Notice that $\chi_1, \chi_5, \chi_8, \chi_{12}$ are even primitive characters while $\chi_{-3}, \chi_{-4}, \chi_{-7}, \chi_{-8}, \chi_{-11}$ are odd primitive characters.

Examples of induced characters:

- $\tilde{\chi}_2(n) = \begin{cases} \chi_1(n) & \text{if} \quad (2,n) = 1 \\ 0 & \text{if} \quad (2,n) \neq 1 \end{cases} = \begin{cases} 1 & \text{if} \quad n \equiv 1 \pmod{2} \\ 0 & \text{if} \quad n \equiv 0 \pmod{2} \end{cases},$

- $\tilde{\chi}_{-6}(n) = \begin{cases} \chi_{-3}(n) & \text{if} \quad (6,n) = 1 \\ 0 & \text{if} \quad (6,n) \neq 1 \end{cases}$

$$= \begin{cases} 1 & \text{if} \quad n \equiv 1 \pmod{6} \\ -1 & \text{if} \quad n \equiv 5 \pmod{6} \\ 0 & \text{if} \quad n \equiv 0, 2, 3, 4 \pmod{6}, \end{cases}$$

- $\tilde{\chi}_{-9}(n) = \begin{cases} \chi_{-3}(n) & \text{if} \quad (9,n) = 1 \\ 0 & \text{if} \quad (9,n) \neq 1 \end{cases} = \chi_{-3}(n),$

- $\tilde{\chi}_{10}(n) = \begin{cases} \chi_5(n) & \text{if} \quad (10,n) = 1 \\ 0 & \text{if} \quad (10,n) \neq 1 \end{cases}$

$$= \begin{cases} 1 & \text{if} \quad n \equiv 1, 9 \pmod{10} \\ -1 & \text{if} \quad n \equiv 3, 7 \pmod{10} \\ 0 & \text{if} \quad n \equiv 0, 2, 4, 5, 6, 8 \pmod{10}. \end{cases}$$

$\tilde{\chi}_2, \tilde{\chi}_{10}$ are even induced characters and $\tilde{\chi}_{-6}, \tilde{\chi}_{-9}$ are odd induced characters.

The *Dirichlet L function* is the complex function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \operatorname{Re} s > 1, \tag{2.2}$$

The function ζ of Hurwitz is the complex function

$$\zeta(s,a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^s}, \quad \operatorname{Re} s > 1,\ 0 < a \leq 1. \tag{2.3}$$

This function has an analytic extension to $\mathbb{C}\backslash\{1\}$, with a simple pole at $s = 1$. See [15], pp. 265-267.

Remark 1. *If χ is a character $(mod\ k)$ and $n = qk + r, 1 \leq r \leq k,\ q = 0, 1, 2, \ldots,$ then*

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{r=1}^{k} \sum_{q=0}^{\infty} \frac{\chi(qk+r)}{(qk+r)^s} =$$

$$= \frac{1}{k^s} \sum_{r=1}^{k} \chi(r) \sum_{q=0}^{\infty} \frac{1}{\left(q + \frac{r}{k}\right)^s} = \frac{1}{k^s} \sum_{r=1}^{k} \chi(r)\, \zeta\left(s, \frac{r}{k}\right)$$

In particular,

$$L(s,\chi_{-3}) = \frac{1}{3^s}\left(\zeta\left(s,\frac{1}{3}\right) - \zeta\left(s,\frac{2}{3}\right)\right)$$

and

$$L(s,\chi_{-4}) = \frac{1}{4^s}\left(\zeta\left(s,\frac{1}{4}\right) - \zeta\left(s,\frac{3}{4}\right)\right).$$

Lemma 1. *The* $L-\text{series}$ *of Dirichlet is absolutely convergent for* $\operatorname{Re} s > 1$ *and convergent for* $\operatorname{Re} s > 0$*. Moreover* $L(s,\chi)$ *is analytic on the entire complex plan if* $\chi \neq \chi_1$*, while* $L(s,\chi_1)$ *has a pole at* $s = 1$.

The following are known:

Lemma 2. (See [3], pp.171 and 262) *Every* χ *character* $(mod\ k)$ *has the form*

$$\chi(n) = \psi(n)\,\chi_0(n), \qquad n \in \mathbb{Z} \tag{2.4}$$

where χ_0 *is the main character* $(mod\ k)$ *and* ψ *is a primitive character* $(mod\ d)$*, for some* $d|k$*. Moreover,*

$$L(s,\chi) = L(s,\psi) \prod_{\substack{p|q \\ p\ prime}} \left(1 - \frac{\psi(p)}{p^s}\right). \tag{2.5}$$

For example, since $\chi_{-6} = \chi_{-3}\chi_{0,6}$, we have

$$L(s,\chi_{-6}) = L(s,\chi_{-3})\left(1 - \frac{\chi_{-3}(2)}{2^s}\right)\left(1 - \frac{\chi_{-3}(3)}{3^s}\right) = L(s,\chi_{-3})\left(1 + \frac{1}{2^s}\right).$$

Lemma 3. (See [5], p. 303) *For* $\Delta > 0$*, the primitive L-series verifies the following formulas:*

i) $L(s,\chi_{-\Delta}) = 2^s\pi^{s-1}\Delta^{-s+1/2}\Gamma(1-s)\cos\left(\frac{s\pi}{2}\right)L_{-\Delta}(1-s,\chi_{-\Delta})$;

ii) $L(s,\chi_{\Delta}) = 2^s\pi^{s-1}\Delta^{-s+1/2}\Gamma(1-s)\sin\left(\frac{s\pi}{2}\right)L_{\Delta}(1-s,\chi_{\Delta})$.

Lemma 4. (See [5], p.294) *If* L *is the primitive series of Dirichlet, then*

$$L(1,\chi_\Delta) = \begin{cases} \pi/\left(3\sqrt{3}\right) & \text{if} \quad \Delta = -3 \\ \pi/4 & \text{if} \quad \Delta = -4 \\ \pi h(\Delta)/\sqrt{-\Delta} & \text{if} \quad \Delta < -4 \\ 2h(\Delta)\ln(\varepsilon)/\sqrt{\Delta} & \text{if} \quad \Delta > 1, \end{cases} \tag{2.6}$$

where, according to Zucker and Robertson for $\Delta > 0$,

$$h(-\Delta) = -\frac{1}{\Delta}\sum_{n=1}^{\Delta-1} n\cdot\chi_{-\Delta}(n) \tag{2.7}$$

and

$$h(\Delta)\ln(\varepsilon) = -\frac{1}{2}\sum_{n=1}^{\Delta-1}\chi_\Delta(n)\cdot\ln\sin\frac{n\pi}{2}. \tag{2.8}$$

In particular,

$$L(1,\chi_{-7}) = \frac{\pi}{\sqrt{7}}, \quad L(1,\chi_{-8}) = \frac{\pi}{2\sqrt{2}}, \quad L(1,\chi_{-11}) = \frac{\pi}{\sqrt{11}},$$
$$L(1,\chi_{5}) = \frac{2}{\sqrt{5}} \ln\left(\frac{1+\sqrt{5}}{2}\right), \qquad L(1,\chi_{8}) = \frac{1}{\sqrt{2}} \ln\left(1+\sqrt{2}\right),$$
$$L(1,\chi_{12}) = \frac{1}{\sqrt{3}} \ln\left(2+\sqrt{3}\right).$$

3 Main results

Theorem 1. *If* $\chi_{-\Delta}$ *is an odd primitive character* $(mod\,\Delta)$ *then*

$$\int_0^1 \frac{\sum\limits_{n=1}^{\Delta-1} \chi_{-\Delta}(n)\, x^{n-1}}{1-x^{\Delta}} \ln\left(\ln\frac{1}{x}\right) dx = \begin{cases} \frac{\pi}{\sqrt{3}} \ln\left(\frac{\sqrt{3}\cdot\Gamma^2(2/3)}{(2\pi)^{2/3}}\right) & if \quad \Delta = 3 \\ \pi \ln\left(\frac{\Gamma(3/4)}{\sqrt[4]{\pi}}\right) & if \quad \Delta = 4 \\ \frac{\pi}{\sqrt{\Delta}} \left\{\ln 2\pi - \sum\limits_{r=1}^{\Delta-1} \chi_{-\Delta}(r) \ln\Gamma\left(\frac{r}{\Delta}\right)\right\} & if \quad \Delta > 4 \end{cases}.$$

Proof: We start with some general considerations.

The change of variable $t = nz$ in the integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

gives us

$$\Gamma(s) = \int_0^\infty e^{-nx} (nz)^{s-1}\, n dz = n^s \int_0^\infty e^{-nz} z^{s-1} dz,$$

so that

$$\Gamma(s) L(s,\chi) = \Gamma(s) \sum_{n=1}^\infty \frac{\chi(n)}{n^s} = \sum_{n=1}^\infty \frac{\chi(n)}{n^s} n^s \int_0^\infty e^{-nz} z^{s-1} dz =$$
$$= \sum_{n=1}^\infty \chi(n) \int_0^\infty e^{-nz} z^{s-1} dz = \int_0^\infty \left(\sum_{n=1}^\infty \chi(n)\, e^{-nz}\right) z^{s-1} dz.$$

A new change of variable $(x = e^{-z})$ leads us to the formula

$$\Gamma(s) L(s,\chi) = \int_0^1 \left(\sum_{n=1}^\infty \chi(n)\, x^n\right) \left(\ln\frac{1}{x}\right)^{s-1} \frac{dx}{x}.$$

Assuming that the main period of χ is q, then for $|x|<1$ we have

$$\sum_{n=1}^{\infty}\chi(n)\,x^n=\sum_{m=0}^{\infty}\sum_{n=1}^{q-1}\chi(mq+n)\,x^{mq+n}=\sum_{n=1}^{q-1}\sum_{m=0}^{\infty}\chi(mq+n)\,x^{mq+n}$$
$$=\sum_{n=1}^{q-1}\chi(n)\,x^n\sum_{m=0}^{\infty}x^{mq}=\frac{\sum\limits_{n=1}^{q-1}\chi(n)\,x^n}{1-x^q}.$$

Therefore

$$\Gamma(s)\,L(s,\chi)=\int_0^1\frac{\sum\limits_{n=1}^{q-1}\chi(n)\,x^n}{1-x^q}\left(\ln\frac{1}{x}\right)^{s-1}\frac{dx}{x}.$$

Now it is worth to notice that

$$\int_0^1\frac{\sum\limits_{n=1}^{q-1}\chi(n)\,x^n}{1-x^q}\left(\ln\frac{1}{x}\right)^{s-1}\frac{dx}{x}=\frac{d}{ds}\Gamma(s)\,L(s,\chi)\bigg|_{s=1}. \tag{3.1}$$

In our case, $\chi=\chi_{-\Delta}$ is an odd primitive character $(\operatorname{mod}\Delta)$, so by Lemma 3 i) we have

$$L(s,\chi_{-\Delta})=2^s\pi^{s-1}\Delta^{-s+1/2}\Gamma(1-s)\cos\left(\frac{s\pi}{2}\right)L(1-s,\chi_{-\Delta}).$$

By Lemma 4,

$$L(0,\chi_{-\Delta})=\begin{cases}1/3 & \text{if}\quad \Delta=3\\ 1/2 & \text{if}\quad \Delta=4\\ \frac{\sqrt{\Delta}}{\pi}\cdot\frac{\pi}{-\Delta\sqrt{\Delta}}\sum\limits_{n=1}^{\Delta-1}n\chi_{-\Delta}(n) & \text{if}\quad \Delta>4\end{cases}$$
$$=\begin{cases}1/3 & \text{if}\quad \Delta=3\\ 1/2 & \text{if}\quad \Delta=4\\ 1 & \text{if}\quad \Delta>4.\end{cases}$$

According to Remark 1,

$$L(s,\chi_{-\Delta})=\frac{1}{\Delta^s}\sum_{r=1}^{\Delta-1}\chi_{-\Delta}(r)\,\zeta\left(s,\frac{r}{\Delta}\right).$$

Since

$$\zeta'(0,a)=\ln\frac{\Gamma(a)}{\sqrt{2\pi}} \tag{3.2}$$

(see [15], p. 271) we obtain

$$L'(0,\chi_{-\Delta})=\begin{cases}-\frac{1}{3}\ln 3+\ln\frac{\Gamma(1/3)}{\Gamma(2/3)} & \text{if}\quad \Delta=3\\ -\ln 2+\ln\frac{\Gamma(1/4)}{\Gamma(3/4)} & \text{if}\quad \Delta=4\\ -\ln\Delta+\sum\limits_{r=1}^{\Delta-1}\chi_{-\Delta}(r)\ln\Gamma\left(\frac{r}{\Delta}\right) & \text{if}\quad \Delta>4.\end{cases}$$

By replacing s by $1-s$ in formula i) of Lemma 3, we get

$$\Gamma(s)\,L(s,\chi_{-\Delta})=\frac{\sqrt{\Delta}}{2}\left(\frac{2\pi}{\Delta}\right)^{s}\csc\left(\frac{s\pi}{2}\right)L_{-\Delta}(1-s,\chi_{-\Delta}),$$

which leads to

$$\left.\frac{d}{ds}\Gamma(s)\,L(s,\chi_{-\Delta})\right|_{s=1}=\begin{cases}\frac{\pi}{\sqrt{3}}\left\{\frac{1}{3}\ln\frac{2\pi}{3}+\frac{1}{3}\ln 3+\ln\frac{\Gamma(2/3)}{\Gamma(1/3)}\right\} & \text{if}\quad \Delta=3\\ \frac{\pi}{2}\left\{\frac{1}{2}\ln\frac{\pi}{2}+\ln 2+\ln\frac{\Gamma(3/4)}{\Gamma(1/4)}\right\} & \text{if}\quad \Delta=4\\ \frac{\pi}{\sqrt{\Delta}}\left\{\ln 2\pi-\sum_{r=1}^{\Delta-1}\chi_{-\Delta}(r)\ln\Gamma\left(\frac{r}{\Delta}\right)\right\} & \text{if}\quad \Delta>4.\end{cases}$$

The final result is now an easy consequence of the formula for the function Γ,

$$\Gamma(s)\,\Gamma(1-s)=\frac{\pi}{\sin s\pi},\quad 0<s<1, \tag{3.3}$$

see [15], p. 239. □

Theorem 2. *If χ_{-q} is an odd character $(mod\ q)$ induced by an odd primitive character $\chi_{-\Delta}$ $(mod\,\Delta)$, then*

$$I_{-q}=\int_0^1\frac{\sum\limits_{n=1}^{q-1}\chi_{-q}(n)\,x^{n-1}}{1-x^q}\ln\left(\ln\frac{1}{x}\right)dx$$
$$=\left(I_{-\Delta}+L(1,\chi_{-\Delta})\sum_{\substack{p\mid q\\ p\ prime}}^{q-1}\frac{\chi_{-\Delta}(p)\ln p}{p-\chi_{-\Delta}(p)}\right)\prod_{\substack{p\mid q\\ p\ prime}}\left(1-\frac{\chi_{-\Delta}(p)}{p}\right),$$

where

$$L(1,\chi_{-\Delta})=\begin{cases}\pi/\left(3\sqrt{3}\right) & \textit{if}\quad \Delta=3\\ \pi/4 & \textit{if}\quad \Delta=4\\ \frac{\pi}{-\Delta\sqrt{\Delta}}\sum\limits_{n=1}^{\Delta-1}n\cdot\chi_{-\Delta}(n) & \textit{if}\quad \Delta>4.\end{cases}$$

Proof: The basic remark is formula (3.1), established above. By Lemma 2,

$$L(s,\chi_{-q})=L(s,\chi_{-\Delta})\prod_{\substack{p\mid q\\ p\ prime}}\left(1-\frac{\chi_{-\Delta}(p)}{p^s}\right)$$

which leads to

$$\frac{d}{ds}\Gamma(s)L_{-q}(s) = \frac{d}{ds}\left(\Gamma(s)L(s,\chi_{-\Delta})\prod_{\substack{p|q\\ p\ prime}}\left(1-\frac{\chi_{-\Delta}(p)}{p^s}\right)\right)$$

$$= \frac{d}{ds}\left(\Gamma(s)L(s,\chi_{-\Delta})\right)\prod_{\substack{p|q\\ p\ prime}}\left(1-\frac{\chi_{-\Delta}(p)}{p^s}\right)+$$

$$+\Gamma(s)L(s,\chi_{-\Delta})\prod_{\substack{p|q\\ p\ prime}}\left(1-\frac{\chi_{-\Delta}(p)}{p^s}\right)\sum_{\substack{p|q\\ p\ prime}}^{q-1}\frac{\chi_{-\Delta}(p)\ln p}{p-\chi_{-\Delta}(p)}.$$

Therefore

$$\left.\frac{d}{ds}\Gamma(s)L_{-q}(s)\right|_{s=1}$$

equals

$$\left(\left.\frac{d}{ds}\Gamma(s)L(s,\chi_{-\Delta})\right|_{s=1}+\Gamma(1)L(1,\chi_{-\Delta})\sum_{\substack{p|q\\ p\ prime}}^{q-1}\frac{\chi_{-\Delta}(p)\ln p}{p-\chi_{-\Delta}(p)}\right)$$

$$\times\prod_{\substack{p|q\\ p\ prime}}\left(1-\frac{\chi_{-\Delta}(p)}{p}\right)$$

$$=\left(I_{-\Delta}+L(1,\chi_{-\Delta})\sum_{\substack{p|q\\ p\ prime}}^{q-1}\frac{\chi_{-\Delta}(p)\ln p}{p-\chi_{-\Delta}(p)}\right)\prod_{\substack{p|q\\ p\ prime}}\left(1-\frac{\chi_{-\Delta}(p)}{p}\right)$$

and the proof is done. ∎

4 Applications

Our first application is the formula

$$\int_0^1\frac{1+x^2}{1+x^4}\ln\left(\ln\frac{1}{x}\right)dx=\frac{\pi}{\sqrt{2}}\ln\left(\frac{\Gamma\left(\frac{5}{8}\right)\cdot\Gamma\left(\frac{7}{8}\right)}{2^{1/4}\cdot\sqrt{\pi}}\right). \tag{4.1}$$

To prove this, we have to remark that

$$\int_0^1\frac{1+x^2}{1+x^4}\ln\left(\ln\frac{1}{x}\right)dx=\int_0^1\frac{\sum_{n=1}^{7}\chi_{-8}(n)x^{n-1}}{1-x^8}\ln\left(\ln\frac{1}{x}\right)dx=I_{-8},$$

where χ_{-8} is the odd primitive character. By Theorem 1,

$$\begin{aligned}I_{-8} &= \frac{\pi}{\sqrt{8}}\left\{\ln 2\pi - \sum_{r=1}^{7}\chi_{-8}(r)\ln\Gamma\left(\frac{r}{8}\right)\right\}\\ &= \frac{\pi}{\sqrt{8}}\left\{\ln 2\pi - \ln\Gamma\left(\frac{1}{8}\right) - \ln\Gamma\left(\frac{3}{8}\right) + \ln\Gamma\left(\frac{5}{8}\right) + \ln\Gamma\left(\frac{7}{8}\right)\right\}\\ &= \frac{\pi}{\sqrt{8}}\left\{\ln 2\pi + \ln\frac{\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)}{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{1}{8}\right)}\right\}\end{aligned}$$

and it remains to take into account the formula 3.3.

A second application concerns the formula

$$\int_0^1 \frac{1+2x+x^2+2x^3+x^4}{1+x+x^3+x^4+x^5+x^6}\ln\left(\ln\frac{1}{x}\right)dx = \frac{\pi}{\sqrt{7}}\left\{\ln 2\pi + \ln\frac{\Gamma\left(\frac{3}{7}\right)\Gamma\left(\frac{5}{7}\right)\Gamma\left(\frac{6}{7}\right)}{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)}\right\}. \tag{4.2}$$

In this case,

$$\begin{aligned}&\int_0^1 \frac{1+2x+x^2+2x^3+x^4}{1+x+x^3+x^4+x^5+x^6}\ln\left(\ln\frac{1}{x}\right)dx\\ &\qquad = \int_0^1 \frac{\sum_{n=1}^{6}\chi_{-7}(n)\,x^{n-1}}{1-x^7}\ln\left(\ln\frac{1}{x}\right)dx = I_{-7}.\end{aligned}$$

By Theorem 1,

$$\begin{aligned}I_{-7} &= \frac{\pi}{\sqrt{7}}\left\{\ln 2\pi - \sum_{k=1}^{6}\chi_{-7}(k)\ln\Gamma\left(\frac{k}{7}\right)\right\}\\ &= \frac{\pi}{\sqrt{7}}\left\{\ln 2\pi - \ln\Gamma\left(\frac{1}{7}\right) - \ln\Gamma\left(\frac{2}{7}\right) + \ln\Gamma\left(\frac{3}{7}\right) - \ln\Gamma\left(\frac{4}{7}\right)\right.\\ &\qquad\left. + \ln\Gamma\left(\frac{5}{7}\right) + \ln\Gamma\left(\frac{6}{7}\right)\right\}\\ &= \frac{\pi}{\sqrt{7}}\left\{\ln 2\pi + \ln\frac{\Gamma\left(\frac{3}{7}\right)\Gamma\left(\frac{5}{7}\right)\Gamma\left(\frac{6}{7}\right)}{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)}\right\}.\end{aligned}$$

A third application is provided by the formula

$$\int_0^1 \frac{1+x^2}{(1+x+x^2)(1-x+x^2)}\ln\left(\ln\frac{1}{x}\right)dx = \frac{\pi}{\sqrt{3}}\ln\left(\frac{3^{3/4}\Gamma^3\left(\frac{2}{3}\right)}{\pi\cdot 2^{7/6}}\right). \tag{4.3}$$

In fact,

$$\int_0^1 \frac{1+x^2}{(1+x+x^2)(1-x+x^2)} \ln\left(\ln\frac{1}{x}\right) dx = \int_0^1 \frac{\sum\limits_{n=1}^{5} \chi_{-6}(n)\, x^{n-1}}{1-x^6} \ln\left(\ln\frac{1}{x}\right) dx = I_{-6},$$

where χ_{-6} is the odd character induced by odd primitive character χ_{-3}. So we can apply the Theorem 2. By Theorem 1,

$$I_{-3} = \frac{\pi}{\sqrt{3}} \ln\left(\frac{\sqrt{3}\Gamma^2\left(\frac{2}{3}\right)}{(2\pi)^{2/3}}\right),$$

so that

$$\begin{aligned} I_{-6} &= \left(I_{-3} + L(1,\chi_{-3})\left(\frac{\chi_{-3}(2)\ln 2}{2-\chi_{-3}(2)}\right)\left(\frac{\chi_{-3}(3)\ln 3}{3-\chi_{-3}(3)}\right)\right) \\ &\quad \times \left(1-\frac{\chi_{-3}(2)}{2}\right)\left(1-\frac{\chi_{-3}(3)}{3}\right) \\ &= \left(I_{-3} + L(1,\chi_{-3})\left(-\frac{\ln 2}{2+1}\right)\right)\left(1+\frac{1}{2}\right) \\ &= \frac{3}{2}\left(I_{-3} - \frac{\pi\ln 2}{9\sqrt{3}}\right) \\ &= \frac{\sqrt{3}\pi}{2}\ln\left(\frac{\sqrt{3}\Gamma^2\left(\frac{2}{3}\right)}{(2\pi)^{\frac{2}{3}}}\right) - \frac{\pi\sqrt{3}\ln 2}{18} \\ &= \frac{\pi}{\sqrt{3}}\ln\left(\frac{3^{3/4}\Gamma^3\left(\frac{2}{3}\right)}{\pi\cdot 2^{7/6}}\right). \end{aligned}$$

A final application is provided by the formula

$$\int_0^1 \frac{1+x^2}{1-x+x^2} \ln\left(\ln\frac{1}{x}\right) dx = \frac{\pi}{\sqrt{3}} \ln\left(\frac{3\Gamma^4\left(\frac{2}{3}\right)}{(2\pi)^{4/3}\cdot 2^{1/3}}\right), \tag{4.4}$$

that appears in the book of I. S. Gradzhteyn and I. M. Ryzhyk [11], p. 572, under the following equivalent form:

$$\int_0^1 \ln\left(\ln\frac{1}{x}\right)\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\left[\frac{5}{6}\ln 2\pi - \ln\Gamma\left(\frac{1}{6}\right)\right]. \tag{4.5}$$

In fact

$$\begin{aligned} I &= \int_0^1 \frac{1+x^2}{1-x+x^2} \ln\left(\ln\frac{1}{x}\right) dx \\ &= 2\int_0^1 \frac{1+x^2}{(1+x+x^2)(1-x+x^2)} \ln\left(\ln\frac{1}{x}\right) dx - \int_0^1 \frac{1}{1+x+x^2} \ln\left(\ln\frac{1}{x}\right) dx. \end{aligned}$$

and it was already noticed that

$$\int_0^1 \frac{1}{1+x+x^2} \ln\left(\ln\frac{1}{x}\right) dx = I_{-3}.$$

Then

$$\begin{aligned} I = 2I_{-6} - I_{-3} &= \frac{2\pi}{\sqrt{3}} \ln\left(\frac{3^{3/4}\Gamma^3\left(\frac{2}{3}\right)}{\pi \cdot 2^{7/6}}\right) - \frac{\pi}{\sqrt{3}} \ln\left(\frac{\sqrt{3}\Gamma^2\left(\frac{2}{3}\right)}{(2\pi)^{2/3}}\right) \\ &= \frac{\pi}{\sqrt{3}} \ln\left(\frac{3\Gamma^4\left(\frac{2}{3}\right)}{(2\pi)^{4/3}\cdot 2^{1/3}}\right). \end{aligned}$$

By using the duplication formula

$$2^{2s-1}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = \pi^{1/2}\Gamma(2s). \tag{4.6}$$

(see [15], p. 240) and the formula 3.3 cited above, we can establish the equivalence of the formulas (4.4) and (4.5).

5 How computers can help us

In our Computer Era, the interest for formulas of any kind aroused considerably in an attempt to replace human been by machine. Trying to avoid some controversial issues, we will mention here the existence of the powerful *PSLQ algorithm* (developed in 1991 by H.R.P. Ferguson), that is able to detect integer relations of special constants (see [12]).

For example, the numerical methods give us

$$\begin{aligned} J_1 = \int_0^1 \frac{\sqrt{2}\left(1+x^2\right)}{\pi\left(1+x^4\right)} \ln\left(\ln\frac{1}{x}\right) dx \\ \approx -.29896353535041190973869843644880630771 54. \end{aligned}$$

and the PLSQ algorithm based program offers for the right hand side the guess

$$J_1 = \ln\left(\frac{\Gamma\left(\frac{5}{8}\right)\cdot\Gamma\left(\frac{7}{8}\right)}{2^{1/4}\cdot\sqrt{\pi}}\right).$$

We proved rigorously this formula as application (4.1).

Similarly, in the case of

$$J_2 = \int_0^1 \frac{\sqrt{7}\left(1+2x+x^2+2x^3+x^4\right)}{\pi\left(1+x+x^3+x^4+x^5+x^6\right)} \ln\left(\ln\frac{1}{x}\right) dx$$
$$\approx -.56157997490781248164988933873175376879 73.$$

the PLSQ algorithm based program offers the guess

$$J_2 = \ln\left(2\pi \cdot \frac{\Gamma\left(\frac{3}{7}\right)\Gamma\left(\frac{5}{7}\right)\Gamma\left(\frac{6}{7}\right)}{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)}\right).$$

A rigorous proof is offered by application (4.2).

For

$$J_3 = \int_0^1 \frac{\sqrt{3}\left(1+x^2\right)}{\pi\left(1+x+x^2\right)\left(1-x+x^2\right)} \ln\left(\ln\frac{1}{x}\right) dx$$
$$\approx -.21999155455901672722284244657967065559 26.$$

the PLSQ algorithm based program offers the equivalent guesses:

$$\ln\left(\frac{3^{1/4}\cdot\Gamma\left(\frac{2}{3}\right)\cdot\Gamma\left(\frac{5}{6}\right)}{\sqrt{2\pi}}\right); \quad \ln\left(\frac{3^{3/4}\cdot\Gamma^3\left(\frac{2}{3}\right)}{\pi\cdot 2^{7/6}}\right); \quad \ln\left(\frac{\Gamma^{3/2}\left(\frac{5}{6}\right)}{\pi^{1/4}\cdot 2^{1/6}}\right).$$

which are correct, see (4.3).

The same works for

$$J_4 = \int_0^1 \frac{\sqrt{3}\left(1+x^2\right)}{\pi\left(1-x+x^2\right)} \ln\left(\ln\frac{1}{x}\right) dx$$
$$\approx -.37033842614090511512126016449046938168 74.$$

The PLSQ algorithm based program indicates as possible value

$$\ln\left(\frac{3\Gamma^4\left(\frac{2}{3}\right)}{(2\pi)^{4/3}\cdot 2^{1/3}}\right) = \ln\left(\frac{\Gamma^2\left(\frac{5}{6}\right)}{(2\pi)^{1/3}}\right)$$

which is indeed the case, by our formula (4.4).

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