Some classes of pseudo-MTL algebras

by

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Abstract

Pseudo-MTL algebras or weak pseudo-BL algebras are non-commutative fuzzy structures which arise from pseudo-t-norms, namely, pseudo-BL algebras without the pseudo-divisibility condition. The aim of this paper is to investigate the properties of pseudo-BL algebras that also hold for pseudo-MTL algebras. We will also study some classes of pseudo-MTL algebras such as good, local and Archimedean pseudo-MTL algebras and we show that, generally, an Archimedean pseudo-MTL algebra is not commutative. We prove that any locally finite pseudo-MTL algebra is Archimedean.

Key Words: Pseudo-MTL algebra, Local pseudo-MTL algebra, Good pseudo-MTL algebra, Perfect pseudo-MTL algebra, Archimedean pseudo-MTL algebra, Hyperarchimedean element.

2000 Mathematics Subject Classification: Primary 03G10, Secondary: 03G25, 06D35.

1 Introduction

In order to formalize the many-valued logics induced by continuous t-norms on the real unit interval \([0,1]\), in 1998 P. Hájek introduced a very general many-valued logic, called Basic Logic ([14]). It is well known the result that a t-norm has residuum if and only if the t-norm is left-continuous, so this shows that the Basic Logic is not the most general t-norm based logic. In fact, a logic weaker than the Basic Logic, called Monoidal t-norm based logic (MTL for short) was defined by Esteva and Godo in [8] and proved in [17] to be the logic of left-continuous t-norms and their residua. Thus, the MTL is indeed the most general t-norm based logic and MTL algebra is an algebraic counterpart of this logic. Pseudo-BL algebras were introduced by G. Georgescu and A. Iorgulescu in [11] as a non-commutative extension of Hájek’s BL-algebras. Pseudo-BL algebras are bounded non-commutative residuated lattices \((A, \land, \lor, \odot, \rightarrow, \leftrightarrow, 0, 1)\) which satisfy the conditions:
Depending on the above conditions, there are two directions to extend pseudo-BL algebras. One direction investigates the (bounded) non-commutative residuated lattices satisfying the pseudo-divisibility condition which were studied under the name (bounded) divisible pseudo-residuated lattices in [15] or (bounded) \(R\)-\ell-monomoids ([7], [19], [21]) and examples in the bounded case are given in [16]. The second direction deals with (bounded) non-commutative residuated lattices with the pseudo-prelinearity condition, that is pseudo-MTL algebras. Pseudo-MTL algebras were introduced in [9], under the name weak pseudo-BL algebras in order to obtain a structure on \([0, 1]\), since there are not pseudo-BL algebras on \([0, 1]\). They were studied in [15] (including the not-bounded case and the good pseudo-MTL algebras) and examples in the bounded case of finite good and not good pseudo-MTL algebras are given in [16].

In this paper we will study some properties for pseudo-BL algebras proved in [5] and [6] which are valid in the case of pseudo-MTL algebras, in other words, some properties of pseudo-BL algebras whose proofs don’t need the pseudo-divisibility condition. We will also investigate some special classes of pseudo-MTL algebras, such as good, local and Archimedean pseudo-MTL algebras. It was proved that every locally finite pseudo-MV algebra is commutative ([20]) and that every locally finite pseudo-BL algebra is an MV algebra, so it is commutative ([12]).

We show that in the case of pseudo-MTL algebras this fact is not true, namely we will give an example of locally finite pseudo-MTL algebra which is not commutative. Finally, we prove that any locally finite pseudo-MTL algebra is Archimedean.

2 Pseudo-MTL algebras and their basic properties

Definition 2.1. A pseudo-MTL algebra is an algebra \( A = (A, \land, \lor, \odot, \to, \multimap, 0, 1) \) of the type \((2, 2, 2, 2, 0, 0)\) satisfying the following conditions:

\(M_1\) \((A, \land, \lor, 0, 1)\) is a bounded lattice;
\(M_2\) \((A, \odot, 1)\) is a monoid;
\(M_3\) \(x \odot y \leq z \iff x \leq y \to z \iff y \leq x \multimap z\) for any \(x, y, z \in A\);
\(M_4\) \((x \to y) \lor (y \to x) = (x \multimap y) \lor (y \multimap x) = 1\) (pseudo-prelinearity).

Remark 2.2. (1) If additionally for any \(x, y \in A\) the structure \(A\) satisfies the axiom:
\(M_5\) \((x \to y) \odot x = x \odot (x \multimap y) = x \land y\) (pseudo-divisibility)
then \(A\) is a pseudo-BL algebra.

(2) If \(A\) satisfies the conditions \((M_1), (M_2), (M_3)\) and \((M_5)\), then it is a bounded divisible residuated lattice ([15], [16]). These structures were also studied under the name bounded \(R\)-\ell-monomoids ([7], [19], [21]).

\(A\) is called commutative if the operation \(\odot\) is commutative. In this case \(\to = \multimap\) and thus, a commutative pseudo-MTL algebra is a MTL algebra.
A totally ordered (linear ordered) pseudo-MTL algebra is called chain.

In the sequel we will agree that the operations $\land, \lor, \circ$ have higher priority than the operations $\rightarrow, \neg$.

A pseudo-MTL algebra $\mathcal{A}$ will be also referred by its universe $A$.

**Example 2.3.** Let’s consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\circ, \rightarrow, \neg$ given by the following tables:

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Then $\mathcal{A} = (A, \land, \lor, \circ, \rightarrow, \neg, 0, 1)$ is a pseudo-MTL chain. One can easily prove that $\mathcal{A}$ is not a pseudo-BL algebra, because $(b \rightarrow a) \circ b \not= b \circ (b \neg a)$, so $(M_3)$ doesn’t hold.

In a pseudo-MTL algebra $\mathcal{A} = (A, \land, \lor, \circ, \rightarrow, \neg, 0, 1)$ we define for all $x \in A$: $x^- = x \rightarrow 0$ and $x^+ = x \neg$.

The following proposition provides some rules of calculus in a pseudo-MTL algebra (see [5], [6], [9], [15]).

**Proposition 2.4.** In any pseudo-MTL algebra $\mathcal{A}$ the following rules of calculus hold:

1. $x \rightarrow (y \rightarrow z) = (x \circ y) \rightarrow z$ and $x \neg(y \neg z) = (y \circ x) \neg z$;
2. $x \leq y$ iff $x \rightarrow y = 1$ iff $x \neg y = 1$;
3. $x \rightarrow x = x \neg x = 1$ and $x \rightarrow 1 = x \neg 1 = 1$;
4. $0 \rightarrow x = 0 \neg x = 1$;
5. $x \circ 0 = 0 \circ x = 0$;
6. $x \circ y \leq x \land y$;
7. $(x \rightarrow y) \circ x \leq y$ and $x \circ (x \neg y) \leq y$;
8. $x \leq y \rightarrow (x \circ y)$ and $x \leq y \rightarrow (y \circ x)$;
9. $x \leq y$ implies $x \circ z \leq y \circ z$ and $z \circ x \leq z \circ y$ for any $z \in A$;
10. $(x \rightarrow y) \circ x \leq x \land y$ and $x \circ (x \neg y) \leq x \land y$;
11. $(x \rightarrow y) \circ x \leq x \leq y \rightarrow (x \circ y)$ and $(x \rightarrow y) \circ x \leq y \leq x \rightarrow (y \circ x)$;
12. $x \circ (x \neg y) \leq x \neg x \rightarrow (y \circ x)$ and $x \circ (y \rightarrow x) \leq x \leq y \rightarrow (y \circ x)$;
13. if $x \leq y$ then $z \rightarrow x \leq z \rightarrow y$ and $z \rightarrow x \leq z \neg y$;
14. if $x \leq y$ then $y \rightarrow z \leq x \rightarrow z$ and $y \rightarrow z \leq x \rightarrow z$;
15. $1 \rightarrow x = x$ and $1 \neg x = x$;
16. $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ and $x \neg y \leq (y \neg z) \rightarrow (x \neg z)$;
17. $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$ and $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$;
18. $x \rightarrow (y \neg z) = y \neg (x \rightarrow z)$ and $x \neg (y \rightarrow z) = y \rightarrow (x \neg z)$;
19. $x \rightarrow (y \neg z) = x \neg (x \rightarrow y)$ and $x \rightarrow (x \rightarrow y) = x \rightarrow (x \neg y)$;
(c20) $x \rightarrow y = x \rightarrow (x \land y)$ and $x \rightsquigarrow y = x \rightsquigarrow (x \land y)$;

(c21) $y \leq x \rightarrow y$ and $y \leq x \rightsquigarrow y$;

(c22) if $x \leq y$, then $x \leq z \rightarrow y$ and $x \leq z \rightsquigarrow y$;

(c23) $z \circ (x \land y) \leq (z \circ x) \land (z \circ y)$ and $(x \land y) \circ z \leq (x \circ z) \land (y \circ z)$;

(c24) $x \rightarrow y \leq (x \circ z) \rightarrow (y \circ z)$ and $x \rightsquigarrow y \leq (z \circ x) \rightarrow (z \circ y)$;

(c25) $(y \rightarrow z) \circ (x \rightarrow y) \leq x \rightarrow z$ and $(x \rightarrow y) \circ (y \rightsquigarrow z) \leq x \rightarrow z$;

(c26) $x \circ (y \rightarrow z) \leq y \rightarrow (x \circ z)$ and $(y \rightsquigarrow z) \circ x \leq (z \circ y)$;

(c27) $(x_{n-1} \rightarrow x_n) \circ (x_{n-2} \rightarrow x_{n-1}) \circ \ldots \circ (x_2 \rightarrow x_3) \circ (x_1 \rightarrow x_2) \leq x_1 \rightarrow x_n$ and

$$
(x_1 \rightsquigarrow x_2) \circ (x_2 \rightsquigarrow x_3) \circ \ldots \circ (x_{n-1} \rightsquigarrow x_n) \leq x_1 \rightsquigarrow x_n;
$$

(c28) $(x \rightarrow y) \circ (x' \rightarrow y') \leq (x \lor x') \rightarrow (y \lor y')$ and $(x \rightarrow y) \circ (x' \rightsquigarrow y') \leq (x \lor x') \rightsquigarrow (y \lor y')$;

(c29) $(x \rightarrow y) \circ (x' \rightarrow y') \leq (x \land x') \rightarrow (y \land y')$ and $(x \rightarrow y) \circ (x' \rightsquigarrow y') \leq (x \land x') \rightsquigarrow (y \land y')$;

(c30) $1^\sim = 0^\sim = 0$ and $0^\sim = 0^\sim = 1$;

(c31) $x^\sim \circ x = 0$ and $x \circ x^\sim = 0$;

(c32) $x \leq y^\sim$ iff $x \circ y = 0$ and $x \leq y^\sim$ iff $y \circ x = 0$;

(c33) $x \leq x^\sim^\sim$ and $x \leq x^\sim$;

(c34) $x \rightarrow y^\sim = (x \circ y)^\sim$ and $x \rightsquigarrow y^\sim = (y \circ x)^\sim$;

(c35) $x \leq y^\sim$ iff $y \leq x^\sim$;

(c36) if $x \leq y$, then $y^\sim \leq x^\sim$ and $y^\sim \leq x^\sim$;

(c37) $x \leq x^\sim \rightarrow y$ and $x \leq x^\sim \rightarrow y$;

(c38) $x \rightarrow y^\sim \rightarrow x^\sim$ and $x \rightarrow y \leq x^\sim$;

(c39) $x \rightarrow y^\sim = x^\sim \rightarrow x$ and $x \rightarrow y^\sim = y \rightarrow x^\sim$;

(c40) $x^\sim \rightarrow x^\sim = x^\sim$ and $x^\sim \rightarrow x^\sim = x^\sim$;

(c41) $x \rightarrow x^\sim = x^\sim \rightarrow x^\sim$;

(c42) $x \circ (\lor_{i \in I} y_i) = \lor_{i \in I} (x \circ y_i)$ and $(\lor_{i \in I} y_i) \circ x = \lor_{i \in I} (y_i \circ x)$;

(c43) $(\lor_{i \in I} x_i) \rightarrow y = \lor_{i \in I} (x_i \rightarrow y)$ and $(\lor_{i \in I} x_i) \rightsquigarrow y = \lor_{i \in I} (x_i \rightsquigarrow y)$;

(c44) $y \rightarrow (\lor_{i \in I} x_i) = \lor_{i \in I} (y \rightarrow x_i)$ and $y \rightsquigarrow (\lor_{i \in I} x_i) = \lor_{i \in I} (y \rightsquigarrow x_i)$;

(c45) $(x \lor y) \rightarrow (x \land y) = (x \rightarrow y) \land (y \rightarrow x)$ and $(x \lor y) \rightarrow (x \land y) = (x \rightarrow y) \land (y \rightarrow x)$;

(c46) $(x \land y)^\sim = x^\sim \land y^\sim$ and $(x \lor y)^\sim = x^\sim \lor y^\sim$;

(c47) $(x \land y)^\sim = x^\sim \land y^\sim$ and $(x \lor y)^\sim = x^\sim \lor y^\sim$;

(c48) $(x \land y)^\sim = x^\sim \land y^\sim$ and $(x \lor y)^\sim = x^\sim \lor y^\sim$;

(c49) $y^\sim \rightarrow x^\sim = x^\sim \rightarrow y^\sim = x \rightarrow y^\sim$;

(c50) $y^\sim \rightarrow x^\sim = x^\sim \rightarrow y^\sim = x \rightarrow y^\sim$.

Corollary 2.5. In any pseudo-MLT algebra we have:

(c51) $z \circ (x_1 \land x_2 \land \ldots \land x_n) \leq (z \circ x_1) \land (z \circ x_2) \land \ldots \land (z \circ x_n)$ and $(x_1 \land x_2 \land \ldots \land x_n) \circ z \leq (x_1 \circ z) \land (x_2 \circ z) \land \ldots \land (x_n \circ z)$.

Proposition 2.6. ([15]) For any $g, h, k \in A$ we have

(c52) $g \lor (h \circ k) \geq (g \lor h) \circ (g \lor k)$.

Corollary 2.7. (c53) $g \lor (h_1 \circ h_2 \circ \ldots h_n) \geq (g \lor h_1) \circ (g \lor h_2) \circ \ldots \circ (g \lor h_n)$. 

Some classes of pseudo-MTL algebras

Proposition 2.8. ([9]) In any pseudo-MTL algebra the following properties hold:

(c54) $x \lor y = [(x \multimap y) \rightarrow y] \land [(y \rightarrow x) \multimap x]$;

c55) $x \lor y = [(x \multimap y) \multimap y] \land [(y \rightarrow x) \multimap x]$.

Corollary 2.9. If $x \lor y = 1$, then $x \rightarrow y = x \multimap y = y$.

For any $x \in A$ we put $x^0 = 1$ and $x^{n+1} = x^n \bowtie x^n$. The order of $x \in A$, denoted $\text{ord}(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If there is no such $n$, then $\text{ord}(x) = \infty$.

Proposition 2.10. In any pseudo-MTL algebra the following hold:

(1) if $x \lor y = 1$, then for each $n \in \mathbb{N}, n \geq 1$, $x^n \lor y^n = 1$;

(2) $(x \multimap y)^n \lor (y \rightarrow x)^n = 1$ and $(x \multimap y)^n \lor (y \multimap x)^n = 1$;

(3) $x \lor y^n \geq (x \lor y)^n$.

Proof: (1) Similarly as in [5], Lemma 2.16;

(2) It follows from (1) and the pseudo-prelinearity condition;

(3) It follows from Corollary 2.7.

Definition 2.11. A pseudo-MTL algebra $A$ is locally finite if for any $x \in A$, $x \neq 1$ implies $\text{ord}(x) < \infty$.

Example 2.12. (1) Consider the pseudo-MTL chain $A$ from Example 2.3. Since $\text{ord}(c) = \infty$, it follows that $A$ is not locally finite.

(2) Let’s consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\bowtie, \rightarrow, \multimap$ given by the following tables:

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Then $A = (A, \land, \lor, \bowtie, \rightarrow, \multimap, 0, 1)$ is a pseudo-MTL chain. Because $(b \rightarrow a) \bowtie b = c \bowtie b = a$ and $b \bowtie (b \multimap a) = c \bowtie c = 0$, it follows that $(b \rightarrow a) \bowtie b \neq b \bowtie (b \multimap a)$, so $A$ is not a pseudo-BL algebra. We have:

$\text{ord}(0) = 1$, $\text{ord}(a) = 2$, $\text{ord}(b) = 2$, $\text{ord}(c) = 3$.

so $A$ is a locally finite pseudo-MTL algebra.

Remark 2.13. (1) In [20] is proved that every locally finite pseudo-MV algebra is commutative.

(2) In [12] is proved that every locally finite pseudo-BL algebra is an MV algebra, so it is commutative.

(3) By the above example we proved that there exist locally finite pseudo-MTL algebras which are not MTL algebras.
Let \((L, \vee, \wedge, 0, 1)\) be a bounded lattice. Recall (see \([13]\)) that an element \(a \in L\) is called \textit{complemented} if there is an element \(b \in L\) such that \(a \vee b = 1\) and \(a \wedge b = 0\); if such element \(b\) exists it is called a \textit{complement} of \(a\). We will denote \(b = a'\) and the set of all complemented elements in \(L\) by \(B(L)\). Complements are generally not unique, unless the lattice is distributive. In pseudo-\(\text{MTL}\) algebras however, although the underlying lattices need not be distributive, the complements are unique.

The next result is proved in \([18]\) for the case of commutative residuated lattices, but the proof is also valid for the non-commutative case.

\begin{lemma} ([18]) \label{lemma:complemented} Let \(A\) be a pseudo-\(\text{MTL}\) algebra. Suppose that \(a \in A\) has a complement \(b \in A\). Then the following hold:
\begin{enumerate}[\upshape (1)]
\item If \(c\) is another complement of \(a\) in \(A\), then \(c = b\);
\item \(a' = b\) and \(b' = a\);
\item \(a^2 = a\).
\end{enumerate}
\end{lemma}

Let \(B(A)\) the set of all complemented elements of the lattice \(L(A) = (A, \wedge, \vee, 0, 1)\).

\begin{lemma} \label{lemma:complements}
Let \(A\) be a pseudo-\(\text{MTL}\) algebra. Then the following are equivalent:
\begin{enumerate}[\upshape (a)]
\item \(x \in B(A)\);
\item \(x \vee x^- = 1\) and \(x \wedge x^- = 0\);
\item \(x \vee x^\sim = 1\) and \(x \wedge x^\sim = 0\).
\end{enumerate}
\end{lemma}

\begin{proof}
(a) \(\Rightarrow\) (b). Since \(x \in B(A)\), there exists \(y \in A\) such that \(x \vee y = 1\) and \(x \wedge y = 0\). Hence, \(x^- = x^- \circ 1 = x^- \circ (x \vee y) = (x^- \circ x) \vee (x^- \circ y) = x^- \circ y\), so \(y \geq x^- \circ y = x^-\).

On the other hand, because \(y \circ x \leq x \wedge y = 0\) it follows that \(y \circ x = 0\), so \(y \leq x^-\). Thus, \(x^- = y\), that is \(x \vee x^- = 1\) and \(x \wedge x^- = 0\).

(b) \(\Rightarrow\) (a). Obviously.

(a) \(\Leftrightarrow\) (c). Similarly as (a) \(\Leftrightarrow\) (b).
\end{proof}

\begin{proposition} \label{proposition:complemented}
Let \(A\) be a pseudo-\(\text{MTL}\) algebra, \(x \in B(A)\) and \(n \in \mathbb{N}, n \geq 1\). Then the following are equivalent:
\begin{enumerate}[\upshape (a)]
\item \(x^n \in B(A)\);
\item \(x \vee (x^n)^\sim = 1\) and \(x \wedge (x^n)^\sim = 1\).
\end{enumerate}
\end{proposition}

\begin{proof}
(a) \(\Rightarrow\) (b). Let \(x^n \in B(A)\). By Lemma 2.15 we have \(x^n \vee (x^n)^\sim = 1\). Since \(x^n \leq x\), we get \(1 = x^n \vee (x^n)^\sim \leq x \vee (x^n)^\sim\), so \(x \vee (x^n)^\sim = 1\). Similarly, \(x \wedge (x^n)^\sim = 1\).

(b) \(\Rightarrow\) (a). Since \(x \vee (x^n)^\sim = 1\), by Proposition 2.10(1) we have \(x^n \vee ((x^n)^\sim)^n = 1\). Because \(((x^n)^\sim)^n \leq (x^n)^\sim\), we get \(1 = x^n \vee ((x^n)^\sim)^n \leq x^n \vee (x^n)^\sim\), so \(x^n \vee (x^n)^\sim = 1\).
\end{proof}
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$(x^n)^- = 1.$ Similarly, $x^n \lor (x^n)^\sim = 1$, so $(x^n)^- \land (x^n)^\sim = 0.$

Because $x^n \leq (x^n)^\sim$ we get $(x^n)^- \land x^n \leq (x^n)^- \land (x^n)^\sim = 0$, so $(x^n)^- \land x^n = 0.$ From $x^n \lor (x^n)^- = 1$ and $x^n \land (x^n)^- = 0$ it follows that $x^n \in B(A).$

Proposition 2.17. ([4]) If $x \in A$, $n \in \mathbb{N}$, $n \geq 1$ such that $x^n \in B(A)$ and $x^n \geq x^- \lor x^\sim$, then $x = 1.$

3 Lattice of filters of a pseudo-MTL algebra

Recall that a nonempty subset $F$ of a lattice $L$ is a filter of $L$ if it satisfies the conditions: (i) $x, y \in F$ implies $x \land y \in F$ and (ii) $x \in F$, $y \in L$, $x \leq y$ implies $y \in F$.

Definition 3.1. Let $A$ be a pseudo-MTL algebra. A nonempty set $F$ of $A$ is called filter of $A$ if the following conditions hold:

(F$_1$) If $x, y \in F$, then $x \circ y \in F$;
(F$_2$) If $x \in F$, $y \in A$, $x \leq y$, then $y \in F$.

We will denote by $\mathcal{F}(A)$ the set of all filters of $A$.

Remark 3.2. If $F$ is a filter of $A$, then:

(F$_3$) $1 \in F$;
(F$_4$) If $x \in F$, $y \in A$, then $y \to x \in F$, $y \leftarrow x \in F$;
(F$_5$) If $x, y \in F$, then $x \land y \in F$.

Example 3.3. The subset $F = \{c, 1\}$ of the pseudo-MTL chain $A$ from Example 2.3 is a filter of $A$.

Remark 3.4. Any filter of $A$ is a filter for the lattice $(A, \lor, \land)$, but the converse is not true. Indeed, let $F$ be a filter of a pseudo-MTL algebra $A$ and $x, y \in A$. Since $x \circ y \in F$ and $x \circ y \leq x \land y$, we get $x \land y \in F$, so $F$ is a filter of the lattice $(A, \lor, \land)$.

Let’s consider the Example 2.12 and $F = \{c, 1\}$. One can easily prove that $F$ is a filter of the lattice $(A, \lor, \land)$, but $F$ is not a filter of the pseudo-MTL algebra $A$.

Proposition 3.5. ([5]) For a subset $F$ of $A$ the following are equivalent:

(a) $F$ is a filter of $A$;
(b) $1 \in A$ and if $x, x \to y \in A$, then $y \in A$;
(c) $1 \in F$ and if $x, x \leftarrow y \in A$, then $y \in A$.

Definition 3.6. A filter $F$ of $A$ is proper if $F \neq A$.

Remark 3.7. If $F$ is a proper filter, then $0 \notin F$.
Proposition 3.8. ([12]) If $A$ is a pseudo-\(\text{MTL}\) algebra, then the sets

\[ A^- = \{ x \in A \mid x^- = 0 \} \text{ and } A^0 = \{ x \in A \mid x^0 = 0 \} \]

are proper filters of $A$.

Definition 3.9. For every subset $X \subseteq A$, the smallest filter of $A$ containing $X$ (i.e. the intersection of all filters $F \in \mathcal{F}(A)$ such that $X \subseteq F$) is called the filter generated by $X$ and will be denoted by $[X]$.

Lemma 3.10. ([12]) Let $A$ be a pseudo-\(\text{MTL}\) algebra and $x, y \in A$. Then:

(1) $[x]$ is proper iff $\text{ord}(x) = \infty$;
(2) if $x \leq y$ and $\text{ord}(y) < \infty$, then $\text{ord}(x) < \infty$;
(3) if $x \leq y$ and $\text{ord}(x) = \infty$, then $\text{ord}(y) = \infty$.

Proposition 3.11. ([5]) If $X \subseteq A$, then

$[X] = \{ y \in A \mid y \geq x_1 \odot x_2 \odot \cdots \odot x_n \text{ for some } n \geq 1 \text{ and } x_1, x_2, \ldots, x_n \in X \}$.

Proposition 3.12. ([5]) If $X \subseteq A$, then

$[X] = \{ y \in A \mid x_1 \multimap (x_2 \multimap (\ldots (x_n \multimap y) \ldots)) = 1$ 
for some $n \geq 1$ and $x_1, x_2, \ldots, x_n \in X \} = \{ y \in A \mid x_1 \multimap (x_2 \multimap (\ldots (x_n \multimap y) \ldots)) = 1$ 
for some $n \geq 1$ and $x_1, x_2, \ldots, x_n \in X \}$.

Remark 3.13. ([5]) (1) If $X$ is a filter of $A$, then $[X] = X$;
(2) If $X = \{x\}$ we write $[x]$ instead of $\{\{x\}\}$ and $[x] = \{y \in X \mid y \geq x^n \text{ for some } n \geq 1\}$. $[x]$ is called principal filter.
(3) If $F$ is a filter of $A$ and $x \in A$, then

$F(x) = [F \cup \{x\}] = \{ y \in A \mid y \geq (f_1 \odot x^{n_1}) \odot (f_2 \odot x^{n_2}) \odot \cdots \odot (f_m \odot x^{n_m}) \text{ for some } m \geq 1, n_1, n_2, \ldots, n_m \geq 0, f_1, f_2, \ldots, f_m \in F \}$.

If $F_1$ and $F_2$ are filters of $A$, we define $F_1 \land F_2 = F_1 \cap F_2$ and $F_1 \lor F_2 = [F_1 \cup F_2]$.

Proposition 3.14. ([5],[4]) In any pseudo-\(\text{MTL}\) algebra $A$ the following hold:

(1) If $F$ is a filter of $A$ and $x \in A \setminus F$, then $F(x) = F \lor [x]$;
(2) $[x]$ is a proper filter iff $\text{ord}(x) = \infty$;
(3) $[x] \lor y = [x] \cap [y]$;
(4) If $x \leq y$, then $[y] \subseteq [x]$;
(5) $[x] \lor [y] = [x \lor y] = [x \odot y]$;
(6) $[x \odot y] = [y \odot x]$;
(7) $[x \multimap y] \lor [x] = [x \multimap y] \lor [x]$.

Proposition 3.15. ([4]) If $F_1, F_2$ are nonempty subsets of $A$ such that $1 \in F_1 \cap F_2$, then $F_1 \lor F_2 = [F_1 \cup F_2] = \{ x \in A \mid x \geq (f_1 \odot f'_1) \odot (f_2 \odot f'_2) \odot \cdots \odot (f_n \odot f'_n) \text{ for some } n \geq 1, f_1, f_2, \ldots, f_n \in F_1, f'_1, f'_2, \ldots, f'_n \in F_2 \}$.
Definition 3.16. ([3]) Let $L = (L, \wedge, \vee)$ be a lattice. 

(i) For every $y, z \in L$, the relative pseudocomplement of $y$ with respect to $z$, provided it exists, is the greatest element $x$ such that $x \wedge y \leq z$. It is denoted by $y \Rightarrow z$ (i.e. $y \Rightarrow z = \max\{x \mid x \wedge y \leq z\}$).

(ii) $L$ is said to be relatively pseudocomplemented provided the relative pseudocomplement $y \Rightarrow z$ exists for every $y, z \in L$.

(iii) A Heyting algebra is a relatively pseudocomplemented lattice with $0$, i.e a bounded one.

If $L$ is a relatively pseudocomplemented lattice, then $\Rightarrow$ can be viewed as a binary operation on $L$ and there exists the greatest element, $1$, of the lattice $1 = x \Rightarrow x$ for all $x \in L$. Consequently, we have the following equivalent definition, with $\odot = \wedge$:

Definition 3.17. (1) A relatively pseudocomplemented lattice is an algebra $L = (L, \wedge, \vee, \Rightarrow, 1)$, where $(L, \wedge, \vee, 1)$ is a lattice with greatest element and the binary operation $\Rightarrow$ on $L$ verifies : for all $x, y, z \in L, x \leq y \Rightarrow z$ if and only if $x \wedge y \leq z$.

(1') A Heyting algebra is a duplicate name for bounded relatively pseudocomplemented lattice. For any $x \in L$, the element $x^* = x \Rightarrow 0$ is called the pseudocomplement of $x$.

Remark 3.18. (1) ([1], [3]) A Brouwer algebra is the dual of a Heyting algebra ($\vee$ instead of $\wedge$, $\geq$ instead of $\leq$, $y \rightarrow z = \min\{x \mid z \leq x \vee y\}$ instead of $y \Rightarrow z$).

(2) Recall that Gödel algebras are Heyting algebras verifying the condition $(x \Rightarrow y) \lor (y \Rightarrow x) = 1$ and that the Gödel t-norm and its associated residuum (implication) on $[0,1]$ are:

$$x \odot_G y = \min(x, y) = x \wedge y, \quad x \rightarrow_G y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y, \end{cases} \quad (\text{Gödel implication}).$$

Note also that a proper Heyting algebra (i.e. which is not a Gödel algebra) is not linearly ordered.

Theorem 3.19. ([2]) A complete lattice is a Heyting algebra if and only if it satisfies the identity

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i).$$

Theorem 3.20. ([4]) $(\mathcal{F}(A), \wedge, \vee, \Rightarrow, \{1\}, A)$ is a complete Heyting algebra, that is

$$F \wedge (\bigvee_{i \in I} G_i) = \bigvee_{i \in I} (F \wedge G_i)$$

for any filter $F$ and for any family of filters $\{G_i\}_{i \in I}$ of $A$.

Definition 3.21. A filter $H$ of $A$ is called normal if for any $x, y \in A$

$$(N) \quad x \rightarrow y \in H \text{ iff } x \hookrightarrow y \in H.$$
We will denote by \( F_n(A) \) the set of all normal filters of \( A \).

**Remark 3.22.** It is obvious that for any pseudo-\( MTL \) algebra \( A \):
1. \( \{1\} \) and \( A \) are normal filters of \( A \);
2. \( F_n(A) \subseteq F(A) \).

**Example 3.23.** Let us consider the filter \( F = \{c, 1\} \) of the pseudo-\( MTL \) chain \( A \) from Example 2.3. Since \( b \to a = c \in F \) and \( b \not\vDash a = b \notin F \), it follows that \( F \) is not a normal filter of \( A \).

**Example 3.24.** ([16]) Let’s consider \( A = \{0, a, b, c, 1\} \) with \( 0 < a < b < c < 1 \) and the operations \( \odot, \to, \vDash \) given by the following tables:

<table>
<thead>
<tr>
<th>( \odot )</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>a</td>
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<td>a</td>
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<td>a</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
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<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>a</td>
<td>a</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \to )</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>a</td>
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<td>b</td>
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<tr>
<td>c</td>
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<td>a</td>
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<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>1</td>
</tr>
</tbody>
</table>

Then \( A = (A, \wedge, \vee, \odot, \to, \vDash, 0, 1) \) is a pseudo-\( MTL \) chain. Since \( (c \to b) \odot c = b \odot c = b \neq a = c \odot b = c \odot (c \vDash b) \), it follows that \( A \) is not a pseudo-\( BL \) algebra.

It is easy to check that \( H = \{a, b, c, 1\} \) is a normal filter of \( A \).

**Remark 3.25.** ([6]) Let \( H \) be a normal filter of \( A \). Then:
1. \( x^- \in H \) iff \( x^- \in H \);
2. \( x \in H \) implies \( (x^-)^- \in H \) and \( (x^-)^- \in H \).

**Remark 3.26.** In the case of a pseudo-\( BL \) algebra \( A \), it is proved that a filter \( H \) is normal if and only if \( x \odot H = H \odot x \) for any \( x \in A \) ([6]). This equality doesn’t hold in the case of pseudo-\( MTL \) algebras as we can see from Example 3.24. Indeed, in this case we have \( c \odot H = \{a, c\} \) and \( H \odot c = \{a, b, c\} \), so \( c \odot H \neq H \odot c \).

**Lemma 3.27.** Let \( H \) be a normal filter of \( A \). Then:
1. For any \( x \in A \) and \( h \in H \) there is \( h' \in H \) such that \( x \odot h \geq h' \odot x \);
2. For any \( x \in A \) and \( h \in H \) there is \( h' \in H \) such that \( h \odot x \geq x \odot h' \).

**Proof:**
1. Let \( y = x \odot h \). Then \( x \odot h = y = x \wedge y \geq (x \to y) \odot x \). But \( h \leq x \iff x \odot h = x \to y \). Since \( h \in H \), it follows that \( x \to y \in H \).
   Because \( H \) is a normal filter we have \( h' = x \to y \in H \). Thus \( x \odot h \geq h' \odot x \).
2. Let \( y = h \odot x \). Then \( h \odot x = y = x \wedge y \geq x \odot (x \vDash y) \). But \( h \leq x \iff h \odot x = x \vDash y \). Since \( h \in H \), it follows that \( x \vDash y \in H \). Because \( H \) is a normal filter we have \( h' = x \vDash y \in H \). Thus \( h \odot x \geq x \odot h' \).
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Proposition 3.28. Let $H$ be a normal filter of $A$ and $x \in A$. Then

$$H(x) = [H \cup \{x\}] = \{y \in A \mid y \geq h \odot x^n \text{ for some } n \in \mathbb{N}, h \in H\}$$

$$= \{y \in A \mid y \geq x^n \odot h \text{ for some } n \in \mathbb{N}, h \in H\}$$

$$= \{y \in A \mid x^n \rightarrow y \in H \text{ for some } n \geq 1\}$$

$$= \{y \in A \mid x^n \leadsto y \in H \text{ for some } n \geq 1\}.$$

Proof: Let $y \in H(x)$. Then $y \geq (h_1 \odot x^{n_1}) \odot (h_2 \odot x^{n_2}) \odot \cdots \odot (h_m \odot x^{n_m})$ for some $m \geq 1$, $n_1, n_2, \ldots, n_m \geq 0, h_1, h_2, \ldots, h_m \in H$, by Remark 3.13(3).

If $m = 1$, then $y \geq h_1 \odot x^{n_1}$ and we take $h = h_1$ and $n = n_1$.

If $m = 2$, then $y \geq (h_1 \odot x^{n_1}) \odot (h_2 \odot x^{n_2}) = h_1 \odot (x^{n_1} \odot h_2) \odot x^{n_2}$.

According to Lemma 3.27, there is $h_2' \in H$ such that $x^{n_1} \odot h_2 \geq h_2' \odot x^{n_1}$.

Hence, $y \geq h_1 \odot (h_2' \odot x^{n_1}) \odot x^{n_2} = (h_1 \odot h_2') \odot x^{n_1+n_2}$ and we take $h = h_1 \odot h_2'$ and $n = n_1 + n_2$. By induction we get $y \geq h \odot x^n$ for some $n \in \mathbb{N}, h \in H$.

Similarly, $y \geq x^n \odot h$ for some $n \in \mathbb{N}, h \in H$. Thus,

$$H(x) = \{y \in A \mid y \geq h \odot x^n \text{ for some } n \in \mathbb{N}, h \in H\} = \{y \in A \mid y \geq x^n \odot h \text{ for some } n \in \mathbb{N}, h \in H\}.$$ 

If $y \in H(x)$, then $h \odot x^n \leq y$ for some $n \geq 1, h \in H$. Thus, $h \leq x^n \rightarrow y$, hence $x^n \rightarrow y \in H$.

Conversely, assume that $h = x^n \rightarrow y \in H$ for some $n \geq 1$.

We also have $(h \odot x^n) \rightarrow y = h \rightarrow (x^n \rightarrow y) = h \rightarrow h = 1$, hence $h \odot x^n \leq y$.

Therefore, $y \in H(x)$ and we conclude that

$$H(x) = \{y \in L \mid x^n \rightarrow y \in H \text{ for some } n \geq 1\}.$$ 

Similarly, $H(x) = \{y \in A \mid x^n \leadsto y \in H \text{ for some } n \geq 1\}$.

Note that the last two equalities are also proved in [6], Lemma 1.12 for the case of pseudo-BL algebras.

Proposition 3.29. If $F_1, F_2 \in \mathcal{F}_n(A)$ then,

$$F_1 \lor F_2 = [F_1 \cup F_2] = \{x \in A \mid x \geq u \odot v \text{ for some } u \in F_1, v \in F_2\}.$$ 

Proof: By Proposition 3.15 we have:

$$F_1 \lor F_2 = [F_1 \cup F_2] = \{x \in A \mid x \geq (f_1 \odot f'_1) \odot (f_2 \odot f'_2) \odot \cdots \odot (f_n \odot f'_n) \text{ for some } n \geq 1, f_1, f_2, \ldots, f_n \in F_1, f'_1, f'_2, \ldots, f'_n \in F_2\}.$$ 

Put $f = (f_1 \odot f'_1) \odot (f_2 \odot f'_2) \odot \cdots \odot (f_n \odot f'_n) = f_1 \odot (f'_1 \odot f_2) \odot \cdots \odot (f'_{n-1} \odot f_n) \odot f'_n$.

By Lemma 3.27, there is $f'' \in F_2$ such that $f_1' \odot f_2 \geq f_2 \odot f''$. Hence,

$$f \geq f_1 \odot f_2 \odot (f'' \odot f_3) \odot \cdots \odot (f_n \odot f'_n).$$

Similarly, there is $f'' \in F_2$ such that $f'_2 \odot f_3 \geq f_3 \odot f''$, so

$$f \geq f_1 \odot f_2 \odot f_3 \odot (f'' \odot f_4) \odot \cdots \odot (f_n \odot f'_n).$$
Finally, $f \geq f_1 \circ f_2 \circ f_3 \cdots \circ f_n \circ f''$ with $f_1, f_2, \ldots, f_n \in F_1, f'' \in F_2$. Taking $u = f_1 \circ f_2 \circ f_3 \cdots \circ f_n, v = f''$, we get $x \geq f \geq u \circ v$ with $u \in F_1, v \in F_2$. □

**Proposition 3.30.** If $F_1, F_2 \in \mathcal{F}_n(A)$ then:
1. $F_1 \wedge F_2 \in \mathcal{F}_n(A)$;
2. $F_1 \vee F_2 \in \mathcal{F}_n(A)$.

**Proof:** (1) We have $F_1 \wedge F_2 = F_1 \cap F_2$. Consider $x, y \in A$ such that $x \rightarrow y \in F_1 \cap F_2$, that is $x \rightarrow y \in F_1$ and $x \rightarrow y \in F_2$. It follows that $x \rightarrow y \in F_1 \cap F_2$. Similarly, $x \rightarrow y \in F_1 \cap F_2$ implies $x \rightarrow y \in F_1 \cap F_2 = F_1 \wedge F_2$.

(2) Let $x, y \in A$ such that $x \rightarrow y \in F_1 \vee F_2$. By Proposition 3.29, there are $u \in F_1, v \in F_2$ such that $u \circ v \leq x \rightarrow y$. Hence, $(u \circ v) \circ x \leq y$ so $u \circ (v \circ x) \leq y$. Since there is $v' \in F_2$ such that $v \circ x \geq v'$, we get $y \geq (u \circ v) \circ v'$. Similarly, there is $u' \in F_1$ such that $u \circ x \geq u' \circ v$, so $y \geq x \circ (u' \circ v')$. We get $u' \circ v' \leq x \rightarrow y$, hence $x \rightarrow y \in F_1 \vee F_2$. Similarly, $x \rightarrow y \in F_1 \vee F_2$ implies $x \rightarrow y \in F_1 \vee F_2$. □

**Proposition 3.31.** If $(F_i)_{i \in I}$ is a family of normal filters of $A$, then:
1. $\bigwedge_{i \in I} F_i \in \mathcal{F}_n(A)$;
2. $\bigvee_{i \in I} F_i \in \mathcal{F}_n(A)$;

**Proof:** Similarly as above. □

As a consequence of the above result we get:

**Theorem 3.32.** (\cite{4}) $\mathcal{F}_n(A)$ is a complete sublattice of $(\mathcal{F}(A), \subseteq)$.

For any normal filter $H$ of $A$ we associate a binary relation $\equiv_H$ on $A$ by defining $x \equiv_H y$ iff $x \rightarrow y, y \rightarrow x \in H$ iff $x \sim y, y \sim x \in H$.

**Proposition 3.33.** (\cite{6}) For a given normal filter $H$ of $A$ the relation $\equiv_H$ is a congruence relation on $A$.

For any $x \in A$, let $x/H$ be the equivalence class $x/\equiv_H$ and $A/H = \{x/H \mid x \in A\}$. $A/H$ becomes a pseudo-MTL algebra with the natural operations induced from those of $A$. If $x, y \in A$, then $x/H \leq y/H$ iff $x \sim y \in H$ iff $x \sim y \in H$.

**Definition 3.34.** A proper (normal) filter of $A$ is called maximal (normal) filter or (normal) ultrafilter if it is not contained in any other proper (normal) filter of $A$. 

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Denote:

\[ \text{Max}(A) = \{ F \mid F \text{ is maximal filter of } A \} \]

and

\[ \text{Max}_n(A) = \{ F \mid F \text{ is maximal normal filter of } A \}. \]

Clearly, \( \text{Max}_n(A) \subseteq \text{Max}(A) \).

**Example 3.35.** Let’s consider the pseudo-MTL algebra \( A \) from Example 2.3. It is obvious that \( H_1 = \{ c, 1 \} \) is a maximal filter of \( A \), but \( H_2 = \{ 1 \} \) is not a maximal filter.

**Theorem 3.36.** ([4]) If \( F \) is a proper filter of \( A \), then the following are equivalent:

1. \( F \in \text{Max}(A) \);
2. For any \( x \notin F \) there is \( f \in F \), \( n, m \in \mathbb{N} \), \( n, m \geq 1 \) such that \( (f \odot x^n)^m = 0 \).

**Theorem 3.37.** ([6]) If \( H \) is a proper normal filter of \( A \), then the following are equivalent:

(a) \( H \in \text{Max}_n(A) \);
(b) For any \( x \in A \), \( x \notin H \) iff \( (x^n)^\sim \in H \) for some \( n \in \mathbb{N} \);
(c) For any \( x \in A \), \( x \notin H \) iff \( (x^n)^\sim \in H \) for some \( n \in \mathbb{N} \).

**Proposition 3.38.** ([6]) If \( H \) is a proper normal filter of \( A \), then the following are equivalent:

(a) \( H \in \text{Max}_n(A) \);
(b) \( A/H \) is locally finite.

**Proposition 3.39.** Let \( F \) be a maximal filter of a pseudo-MTL algebra \( A \) and \( x, y \in A \). Then:

1. \( y \notin F \) and \( y \odot x = x \) implies \( x = 0 \);
2. \( y \notin F \) and \( x \odot y = x \) implies \( x = 0 \).

**Proof:** Let’s consider \( y \in A \setminus F \) such that \( y \odot x = x \).

1. Assume \( x \in A \), \( x > 0 \) and consider \( E = \{ z \in A \mid z \odot x = x \} \). First we prove that \( E \) is a proper filter. Obviously, \( 1, y \in E \) and \( 0 \notin E \). Consider \( z \in A \) such that \( y \rightarrow z \in E \), so \( (y \rightarrow z) \odot x = x \). Since \( (y \rightarrow z) \odot y \odot x = (y \rightarrow z) \odot x = x \), it follows that

\[
   x = [(y \rightarrow z) \odot y] \odot x \leq (y \land z) \odot x \leq (y \odot x) \land (z \odot x) = x \land (z \odot x) = z \odot x \leq x
\]

Thus \( z \odot x = x \), hence \( z \in E \). Therefore \( E \) is a proper filter. Since \( y \in E \) and \( F \) is maximal, it follows that \( y \in F \), a contradiction. Thus, \( x = 0 \).
2. Similarly as in (1). \( \square \)
Example 3.42. Let’s consider tables: are incomparable. Consider also the operations $c$ Spec $x, y$ if for all $P$.

Proposition 3.40. ([5]) For a given filter $F$, the relations $\equiv_{L(F)}$ and $\equiv_{R(F)}$ are equivalence relations on $A$.

We also define two order relations $\leq_{L(F)}$ on $A/L(F)$ and $\leq_{R(F)}$ on $A/R(F)$ by:

$$x/L(F) \leq_{L(F)} y/L(F) \iff x \to y \in F \text{ and } x/R(F) \leq_{R(F)} y/R(F) \iff x \sim y \in F.$$ 

Definition 3.41. A proper (normal) filter $P$ of $A$ is called (normal) prime filter if for all $x, y \in A$, $x \lor y \in P$ implies $x \in P$ or $y \in P$.

The set of all prime filters of $A$ will be denoted by $\text{Spec}(A)$. We also denote by $\text{Spec}_n(A)$ the set of all prime normal filters of $A$. Clearly, $\text{Spec}_n(A) \subseteq \text{Spec}(A)$.

Example 3.42. Let’s consider $A = \{0, a, b, c, 1\}$ with $0 < a < b, c < 1$, but $b, c$ are incomparable. Consider also the operations $\odot, \to, \sim$ given by the following tables:

<table>
<thead>
<tr>
<th>$\odot$</th>
<th>0</th>
<th>a</th>
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<th>$\to$</th>
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<th>$\sim$</th>
<th>0</th>
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</tbody>
</table>

Then $A = (A, \land, \lor, \odot, \to, \sim, 0, 1)$ is a pseudo-MTL algebra. Because $(b \to a) \odot b = c \odot b = a$ and $b \odot (b \sim a) = b \odot c = 0$, it follows that $(b \to a) \odot b \neq b \odot (b \sim a)$, so $A$ is not a pseudo-BL algebra. Clearly, $A$ is not a pseudo-MTL chain. Obviously, $P = \{c, 1\}$ is a prime filter of $A$. Consider the filter $F = \{1\}$ of $A$. Since $b \lor c = 1 \in F$, but $b, c \notin F$, it follows that $F$ is not a prime filter of $A$.

Proposition 3.43. ([5]) If $P$ is a proper filter of $A$, then the following properties are equivalent:

(a) $P$ is prime;
(b) For all $x, y \in A$, $x \to y \in P$ or $y \to x \in P$;
(c) For all $x, y \in A$, $x \sim y \in P$ or $y \sim x \in P$;
(d) $A/L(P)$ is a chain;
(e) $A/R(P)$ is a chain.

Corollary 3.44. ([5]) If $P$ is a prime filter and $Q$ is a proper filter such that $P \subseteq Q$, then $Q$ is a prime filter.

Proposition 3.45. If $P$ is a prime filter of $A$, then $x \odot y \in P$ implies $x^2 \in P$ or $y^2 \in P$. 
Some classes of pseudo-MTL algebras

Proposition 3.53. A prime and y.

Thus, Proposition 3.53. A prime and y.

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Thus, y.

Thus, y.

Thus, y.

Thus, y.

Thus, y.

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Thus, y.

Thus, y.

Thus, y.

Thus, y.

Thus, y.

Corollary 3.47. (5) If x, y ∈ A, then there is a prime filter P of A such that F ⊆ P and P ∩ F = ∅.

Corollary 3.48. (5) If x ∈ A, x ≠ 1, then there is a prime filter P of A such that x ∉ P.

Corollary 3.49. (5) Every proper filter F is the intersection of those prime filters which contain F. In particular, ∩Spec(A) = {1}.

Corollary 3.50. (5) Max(A) ⊆ Spec(A).

Proposition 3.51. (5) Any proper filter can be extended to a maximal filter.

Proposition 3.52. (5) The set of proper filters including a prime filter P of A is a chain.

Proposition 3.53. (5) A pseudo-MTL algebra A is a chain if and only if any proper filter of A is prime.

Example 3.54. (1) Consider the pseudo-MTL chain A from Example 3.24. It is obvious that: F(A) = {{1}, {c, 1}, {a, b, c, 1}, A}, F_n(A) = {{1}, {a, b, c, 1}, A}, Max(A) = {{a, b, c, 1}, Max_n(A) = {{a, b, c, 1}, Spec(A) = {{1}, {c, 1}, {a, b, c, 1}}, Spec_n(A) = {{1}, {a, b, c, 1}}. Obviously, in this case we have Max_n(A) = Max(A).

(2) For the pseudo-MTL algebra A from Example 3.42 we have: F(A) = {{1}, {b, 1}, {c, 1}, A}, F_n(A) = {{1}, A}, Max(A) = {{b, 1}, {c, 1}}, Max_n(A) = ∅, Spec(A) = {{b, 1}, {c, 1}}, Spec_n(A) = ∅. The filter {1} is not prime and A is not a chain. This fact is in accordance with the assertion of Proposition 3.53.
Definition 3.55. An element $p < 1$ of a bounded lattice $(A, \wedge, \vee, 0, 1)$ is called meet-irreducible if $p = x \wedge y$ implies $p = x$ or $p = y$.

Theorem 3.56. ([4]) If $P$ is a proper filter of $A$, then the following are equivalent:

(a) $P$ is prime;
(b) $P$ is meet-irreducible in the lattice $\mathcal{F}(A)$;
(c) If $x, y \in A$ such that $x \vee y = 1$, then $x \in P$ or $y \in P$;
(d) For all $x, y \in A \setminus P$ there is $z \in A \setminus P$ such that $x \leq z$ and $y \leq z$;
(e) If $x, y \in A$ and $[x] \wedge [y] \subseteq P$, then $x \in P$ or $y \in P$.

Proposition 3.57. Any locally finite pseudo-MTL algebra $A$ is a chain.

Proof: Let $x, y \in A$ such that $x \vee y = 1$. Applying (c54), we get:

$$1 = x \vee y = [(x \rightarrow y) \leadsto y] \wedge [(y \leadsto x) \rightarrow x] \leq (x \rightarrow y) \leadsto y,$$

so $(x \rightarrow y) \leadsto y = 1$, that is $x \rightarrow y \leq y$. Taking in consideration that $y \leq x \rightarrow y$, we get $x \rightarrow y = y$. Let’s suppose that $x \neq 1$. Since $A$ is locally finite, there is $n \in \mathbb{N}$ such that $x^n = 0$. We have:

$$y = x \rightarrow y = x \rightarrow (x \rightarrow y) = x^2 \rightarrow y = \cdots = x^n \rightarrow y = 0 \rightarrow y = 1.$$

Thus, $x \vee y = 1$ iff $x = 1$ or $y = 1$. But, for all $x, y \in A$ we have $(x \rightarrow y) \vee (y \rightarrow x) = 1$, so, applying the above result we get $x \rightarrow y = 1$ or $y \rightarrow x = 1$. Hence, $x \leq y$ or $y \leq x$. We conclude that $A$ is a chain.

4 Good pseudo-MTL algebras

Definition 4.1. A pseudo-MTL algebra $A$ is called good if $x^{-} = x^{\sim}$ for any $x \in A$.

Example 4.2. (1) The pseudo-MTL chains from Examples 2.3 and 3.24 are good;
(2) The pseudo-MTL chain from Example 2.12 and the pseudo-MTL algebra from Example 3.42 are not good.

Proposition 4.3. If $F = A \setminus \{0\}$ is a maximal filter of a pseudo-MTL algebra $A$, then $A$ is good.

Proof: Obviously $(0^{-})^{-} = (0^{-})^{-} = 0$. Assume $x > 0$, that is $x \in F$. If $x^{-}, x^{\sim} \in F$ it follows that $x^{-} \odot x, x \odot x^{\sim} \in F$, that is $0 \in F$, a contradiction. Thus $x^{-} = x^{\sim} = 0$, hence $(x^{-})^{-} = (x^{\sim})^{-} = 1$. Therefore, $A$ is a good pseudo-MTL algebra.
Proposition 4.4. In any good pseudo-MTL algebra we have \((x^- \odot y^-)^- = (x^- \odot y^-)^-\).

Proof: Applying \((c_{34}), (c_{46}), (c_{50})\) we have:

\[
(x^- \odot y^-)^- = x^- \rightarrow y^- \odot y^- = y^- \rightarrow y^- = y^- \rightarrow x^- = (x^- \odot y^-)^-.
\]

(In the last equality we also applied \((c_{34})\)).

Proposition 4.5. In any good pseudo-MTL algebra \(A\) we have \(x^- \odot y^- \leq (x \odot y)^-\).

Proof: Because \(A\) is good, by \((c_{10})\), we have:

\[
(x \odot y)^- = (x \odot y)^- \leq (x \odot y)^- \land x^- \geq x^- \odot (x^- \rightarrow (x \odot y)^-)
\]

\[
= x^- \odot (x^- \rightarrow (x \odot y)^-) = x^- \odot (x^- \rightarrow (x \rightarrow y)^-).
\]

But, applying \((c_{34})\) and \((c_1)\) we have:

\[
x^- \rightarrow (x \rightarrow y)^- = x^- \rightarrow ((x \rightarrow y) \rightarrow 0) = (x^- \rightarrow y^-) \odot x^- \rightarrow 0
\]

\[
= ((x^- \rightarrow y^-) \odot x^-)^- \geq (x^- \land y^-)^- \geq x^- \land (y^-)^-
\]

\[
= x^- \lor y^-.
\]

By \((c_{34})\) we have \((x^- \rightarrow y^-) \odot x^- \leq (x^- \land y^-),\) so

\[
((x^- \rightarrow y^-) \odot x^-)^- \geq (x^- \land y^-)^-.
\]

It follows that

\[
(x \odot y)^- \geq x^- \odot (x^- \lor y^-) = (x^- \odot x^-) \lor (x^- \odot y^-) = 0 \lor (x^- \odot y^-) = x^- \odot y^- = x^- \odot y^-.
\]

(we applied \((c_{42})\) and \((c_{31})\)).

If \(A\) is a good pseudo-MTL algebra, then we will denote

\[
M(A) = \{x \in A \mid x^- = x^- = x\}.
\]

Proposition 4.6. ([12]) Let \(A\) be a good pseudo-MTL algebra. Then:

1. \(0, 1 \in M(A)\);
2. \(x^-, x^- \in M(A)\) for all \(x \in A\);
3. if \(x, y \in M(A)\), then \(x \rightarrow y = y^- \rightarrow x^-\) and \(x \rightarrow y = y^- \rightarrow x^-\);
4. if \(x, y \in M(A)\), then \((x^- \odot y^-)^- = (x^- \odot y^-)^- = x^- \rightarrow y = y^- \rightarrow x\).

Definition 4.7. ([10]) If \(A\) is a good pseudo-MTL algebra we say that two elements \(x, y \in A\) are orthogonals, denoted \(x \perp y\), if \(x^- \leq y^-\).
Remark 4.8. ([10]) If $A$ is a good pseudo-MTL algebra, then the following are equivalent:

(a) $x \perp y$;
(b) $y^\sim \leq x^\sim$;
(c) $y^\sim \odot x^\sim = 0$.

Remark 4.9. ([10]) Let $A$ be a good pseudo-MTL algebra. For all $x, y \in A$ we have:
(1) if $x \leq y$, then $x \perp y^\sim$ and $y^\sim \perp x$;
(2) $x \perp x^\sim$ and $x^\sim \perp x$.

5 Local pseudo-MTL algebras

Definition 5.1. A pseudo-MTL algebra is called local if it has a unique ultrafilter.

If $A$ is a pseudo-MTL algebra, we will denote by:

$D(A) = \{x \in A \mid \text{ord}(x) = \infty\}$

$D(A)^* = \{x \in A \mid \text{ord}(x) < \infty\}$.

Obviously, $D(A) \cap D(A)^* = \emptyset$ and $D(A) \cup D(A)^* = A$.

We also can remark that $1 \in D(A)$ and $0 \in D(A)^*$.

Let $A$ be a pseudo-MTL algebra and $F$ a filter of $A$. We will use the following notations:

$F^+_x = \{x \in A \mid x \leq y^\sim \text{ for some } y \in F\}$;

$F^\sim_x = \{x \in A \mid x \leq y^\sim \text{ for some } y \in F\}$.

Remark 5.2. ([12]) Let $A$ be a pseudo-MTL algebra. Then:

(1) $F^+_x = \{x \in A \mid y \odot x = 0 \text{ for some } y \in F\}$;

(2) $F^\sim_x = \{x \in A \mid x \odot y = 0 \text{ for some } y \in F\}$;

(3) $F^+_x = \{x \in A \mid x^\sim \in F\}$;

(4) $F^\sim_x = \{x \in A \mid x^\sim \in F\}$.

Proposition 5.3. ([12]) Let $A$ be a local pseudo-MTL algebra. Then:

(1) any proper filter of $A$ is included in the unique maximal filter of $A$;

(2) $A^\circ$ and $A^\circ_0$ are included in the unique maximal filter of $A$.

Proposition 5.4. ([12]) Let $A$ be a pseudo-MTL algebra. Then the following are equivalent:

(a) $D(A)$ is a filter of $A$;

(b) $D(A)$ is a proper filter of $A$;

(c) $A$ is local;

(d) $D(A)$ is the unique ultrafilter of $A$;

(e) for all $x, y \in A$, $\text{ord}(x \odot y) < \infty$ implies $\text{ord}(x) < \infty$ or $\text{ord}(y) < \infty$. 

Definition 5.12. Let Corollary 5.11. a chain.

Open problem 2. Find an example of perfect pseudo-MTL algebra which is not a chain.

Example 3.42 are not perfect, since they are not good.

The pseudo-MTL chain from Example 2.12 and the pseudo-MTL algebra from $A$ of $\text{Rad}$ and it is denoted by $\text{Rad}$. The intersection of all maximal normal filters of $A$ is called the normal radical of $A$ and it is denoted by $\text{Rad}_n(A)$. It is obvious that $\text{Rad}(A)$ and $\text{Rad}_n(A)$ are filters of $A$ and $\text{Rad}(A) \subseteq \text{Rad}_n(A)$.

Proposition 5.6. ([12]) Any pseudo-MTL chain is a local pseudo-MTL algebra.

Example 5.7. (1) The good pseudo-MTL chains from Examples 2.3, 2.12 and 3.24 are local;
(2) The pseudo-MTL algebra $A$ from Example 3.42 is not local. Indeed, $D(A) = \{b, c, 1\}$ is not a filter of $A$ ($b \odot c = 0 \notin A$).

Open problem 1. Find an example of local pseudo-MTL algebra which is not a chain.

Definition 5.8. ([12]) A pseudo-MTL algebra $A$ is called perfect if it satisfies the following conditions:
(1) $A$ is a local good pseudo-MTL algebra;
(2) for any $x \in A$, $\text{ord}(x) < \infty$ iff $\text{ord}(x^-) = \infty$ iff $\text{ord}(x^-) = \infty$.

Proposition 5.9. ([12]) Let $A$ be a local good pseudo-MTL algebra. Then the following are equivalent:
(a) $A$ is perfect;
(b) for any $x \in A$, $\text{ord}(x) < \infty$ implies $\text{ord}(x^-) = \infty$;
(c) for any $x \in A$, $\text{ord}(x) < \infty$ implies $\text{ord}(x^-) = \infty$;
(d) $D(A)^-_\text{ord} = D(A)^+$;
(e) $D(A)^-_\text{ord} = D(A)^+$.

Example 5.10. (1) Consider the good pseudo-MTL chain $A$ from Example 3.24. By Proposition 5.6 it is local. One can easily check that $\text{ord}(x) < \infty$ iff $\text{ord}(x^-) = \infty$ iff $\text{ord}(x^-) = \infty$ for all $x \in A$. Thus, $A$ is a perfect pseudo-MTL chain.
(2) The pseudo-MTL chain from Example 2.3 is not perfect ($\text{ord}(a) = 2 < \infty$, but $\text{ord}(a^-) = \text{ord}(b) = 2 < \infty$).
(3) The pseudo-MTL chain from Example 2.12 and the pseudo-MTL algebra from Example 3.42 are not perfect, since they are not good.

Open problem 2. Find an example of perfect pseudo-MTL algebra which is not a chain.

Corollary 5.11. If $A$ is a perfect pseudo-MTL algebra, then

$$D(A)^* = \{x^- \mid x \in D(A)\} = \{x^\sim \mid x \in D(A)\}.$$
Example 5.13. (1) In the case of the pseudo-MTL algebra $A$ from Example 3.42 we have $\text{Max}(A) = \{\{b, 1\}, \{c, 1\}\}$. It follows that $\text{Rad}(A) = \{1\}$. Note that $A$ is not a chain.

(2) In the case of the pseudo-MTL chain $A$ from Example 3.24 we have $\text{Max}(A) = \text{Max}_n(A) = \{\{a, b, c, 1\}\}$. Hence, $\text{Rad}(A) = \text{Rad}_n(A) = \{\{a, b, c, 1\}\}$.

Proposition 5.14. If $A$ is a local pseudo-MTL algebra, then $\text{Rad}(A) = D(A)$.

Proof: By Proposition 5.4 it follows that $D(A)$ is the unique maximal filter of $A$, so $\text{Rad}(A) = D(A)$. \hfill \Box

Remark 5.15. If $A$ is a local pseudo-MTL algebra and $x \in \text{Rad}(A)^*$, $y \in A$ such that $y \leq x$, then $y \in \text{Rad}(A)^*$.

Proposition 5.16. (44) For any $x, y \in \text{Rad}(A)$, $x^\sim \circ y^\sim = x^\sim \circ y^\sim = 0$.

Corollary 5.17. Let $A$ be a perfect pseudo-MTL algebra. If $x \in \text{Rad}(A)$ and $y \in \text{Rad}(A)^*$, then $x^\sim \leq y^\sim$ and $x^\sim \leq y^\sim$.

Proof: Since $x, y^\sim \in \text{Rad}(A)$, by Proposition 5.16 we get $x^\sim \circ y^\sim = 0$.

Because $y \leq y^\sim$, we have $x^\sim \circ y \leq x^\sim \circ y^\sim = 0$, so $x^\sim \circ y = 0$. Hence, $x^\sim \leq y^\sim$. Similarly, $x^\sim \leq y^\sim$. \hfill \Box

Proposition 5.18. If $A$ is a perfect pseudo-MTL algebra and $x, y \in \text{Rad}(A)^*$, then $x \perp y$ and $y \perp x$.

Proof: Since $x, y \in \text{Rad}(A)^*$, it follows that $y^\sim, x^\sim \in \text{Rad}(A)$. Hence, $y^\sim \circ x^\sim = 0$. By Proposition 4.8(c) we get $x \perp y$. Similarly, $y \perp x$. \hfill \Box

Theorem 5.19. If $A$ is a perfect pseudo-MTL algebra, then $\text{Rad}(A)$ is a normal filter of $A$.

Proof: We have to prove that $x \rightarrow y \in \text{Rad}(A)$ iff $x \sim y \in \text{Rad}(A)$ for all $x, y \in A$. Consider $x, y \in A$ such that $x \rightarrow y \in \text{Rad}(A)$ and suppose $x \sim y \notin \text{Rad}(A)$.

From $y \leq y^\sim$ we get $x \rightarrow y \leq x \rightarrow y^\sim$ (by (c33) and (c13)). Since $\text{Rad}(A)$ is a filter of $A$, it follows that $x \rightarrow y^\sim \in \text{Rad}(A)$, that is $(x \circ y^\sim)^\sim \in \text{Rad}(A)$ (by (c14)).

Hence, $x \circ y^\sim \in \text{Rad}(A)^*$. On the other hand, from $x \sim y \notin \text{Rad}(A)$, it follows that $x \sim y \in \text{Rad}(A)^*$.

Since $x \leq x^\sim$, by (c14) we get $x^\sim \sim y \leq x \sim y$, so $x^\sim \sim y \in \text{Rad}(A)^*$ (by Remark 5.15). By (c37) we have $x^\sim \leq x^\sim \sim y$, so $x^\sim \in \text{Rad}(A)^*$, that is $x \in \text{Rad}(A)$. But $y \leq x \sim y$, so $y \in \text{Rad}(A)^*$, that is $y^\sim \in \text{Rad}(A)$. Since $\text{Rad}(A)$ is a filter of $A$ and $x, y^\sim \in \text{Rad}(A)$, we get $x \circ y^\sim \in \text{Rad}(A)$ which is a contradiction. Thus, $x \rightarrow y \in \text{Rad}(A)$ implies $x \sim y \in \text{Rad}(A)$.

Similarly, $x \sim y \in \text{Rad}(A)$ implies $x \rightarrow y \in \text{Rad}(A)$ and we conclude that $\text{Rad}(A)$ is a normal filter of $A$. \hfill \Box
Remark 5.20. If the pseudo-MTL algebra $A$ is not perfect, then the above result is not always valid, as we can see in the following example.

Let’s consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\odot, \rightarrow, \Rightarrow$ given by the following tables:

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<th>$\odot$</th>
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<table>
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One can easily check that the pseudo-MTL chain $A$ is not perfect and $H = \{b, c, 1\}$ is the unique maximal filter of $A$, so $\text{Rad}(A) = \{b, c, 1\}$. Since $a \rightarrow 0 = c \in H$ and $a \Rightarrow 0 = a \notin H$, it follows that $\text{Rad}(A)$ is not a normal filter of $A$.

Remark 5.21. There exist pseudo-MTL algebras which are not perfect, but $\text{Rad}(A)$ is a normal filter, so it is not a necessary condition. Indeed, let’s consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\odot, \rightarrow, \Rightarrow$ given by the following tables:

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One can easily check that the pseudo-MTL chain $A$ is not perfect and $H = \{b, c, 1\}$ is the unique maximal filter of $A$, so $\text{Rad}(A) = \{b, c, 1\}$. One can easily check that $\text{Rad}(A)$ is also a normal filter of $A$.

6 Archimedean pseudo-MTL algebras

We will introduce the notion of Archimedean pseudo-MTL algebra in the same way as in the case of pseudo-BL algebras (see [4]).

Proposition 6.1. In any pseudo-MTL algebra the following are equivalent:

(a) $x^n \geq x^- \lor x^-$ for any $n \in \mathbb{N}$ implies $x = 1$;
(b) $x^n \geq y^- \lor y^-$ for any $n \in \mathbb{N}$ implies $x \lor y = 1$;
(c) $x^n \geq y^- \lor y^-$ for any $n \in \mathbb{N}$ implies $x \rightarrow y = x \Rightarrow y = y$.

Proof: (a) $\Rightarrow$ (b) Take $x, y \in A$ such that $x^n \geq y^- \lor y^-$ for any $n \in \mathbb{N}$.
By (c) and by the hypothesis we have:

$$(x \lor y)^- = x^- \lor y^- \leq y^- \lor y^- \leq x^n \leq (x \lor y)^n$$ and
\[(x \lor y)\sim = x\sim \lor y\sim \leq y\sim \leq y\sim \lor y\sim \leq x^n \leq (x \lor y)^n,\]
hence \((x \lor y)^n \geq (x \lor y)^\sim \lor (x \lor y)^\sim\) for any \(n \in \mathbb{N}\). Thus, by hypothesis we get \(x \lor y = 1\).

(b) \implies (c) By (c54) and (c55), for all \(x, y \in A\) we have:
\[
(x \lor y) = [(x \rightarrow y) \bowtie y] \land [(y \bowtie x) \rightarrow x]
\]
\[
(x \land y) = [(x \bowtie y) \rightarrow y] \land [(y \rightarrow x) \bowtie x].
\]

Since \(x \lor y = 1\), it follows that:
\[
[(x \rightarrow y) \bowtie y] \land [(y \bowtie x) \rightarrow x] = 1,
\]
\[
[(x \bowtie y) \rightarrow y] \land [(y \rightarrow x) \bowtie x] = 1,
\]
hence \((x \rightarrow y) \bowtie y = 1\) and \((x \bowtie y) \rightarrow y = 1\). From \((x \rightarrow y) \bowtie y = 1\) we have \(x \rightarrow y \leq y\) and taking in consideration that \(y \leq x \rightarrow y\), we obtain \(x \rightarrow y = y\). Similarly, \(x \bowtie y = y\).

(c) \implies (a) Consider \(x \in A\) such that \(x^n \geq x\sim \lor x\sim\) for any \(n \in \mathbb{N}\). Taking \(y = x\) in (c), we get \(x \rightarrow x = x\), hence \(x = 1\).

**Definition 6.2.** A pseudo-MTL algebra is called Archimedean if one of the equivalent conditions from the above proposition is satisfied.

**Definition 6.3.** An element \(x \in A\) is called Archimedean if there is \(n \in \mathbb{N}, n \geq 1\) such that \(x^n \in B(A)\). A pseudo-MTL algebra \(A\) is called hyperarchimedean if all its elements are Archimedean.

**Proposition 6.4.** Any locally finite pseudo-MTL algebra is hyperarchimedean.

**Proof:** Let \(A\) be a locally finite pseudo-MTL algebra and \(x \in A\). Hence, there exists \(n \in \mathbb{N}\) such that \(x^n = 0 \in B(A)\). It follows that any element \(x\) of \(A\) is Archimedean, so \(A\) is hyperarchimedean.

**Corollary 6.5.** Any hyperarchimedean pseudo-MTL algebra is Archimedean.

**Proof:** Let \(A\) be a hyperarchimedean pseudo-MTL algebra and \(x \in A\) such that \(x^n \geq x\sim \lor x\sim\) for any \(n \in \mathbb{N}\). Since \(A\) is hyperarchimedean, there exists \(m \in \mathbb{N}, m \geq 1\) such that \(x^m \in B(A)\). According to Proposition 2.17 if follows that \(x = 1\), so \(A\) is Archimedean.

**Corollary 6.6.** Any locally finite pseudo-MTL algebra is Archimedean.

**Proof:** It follows from Proposition 6.4 and Corollary 6.5.
Theorem 6.7. ([4]) For a pseudo-MTL algebra \( A \), the following are equivalent:
(a) \( A \) is hyperarchimedean;
(b) For any normal filter \( H \), the quotient pseudo-MTL algebra \( A/H \) is an Archimedean pseudo-MTL algebra.

Example 6.8. (1) The pseudo-MTL chains from Examples 2.3 and 3.24 are neither Archimedean, nor hyperarchimedean (for example, in the first case, \( c^n = c \geq c^- \lor c^- = 0 \lor 0 = 0 \) for all \( n \geq 2 \));
(2) The pseudo-MTL chain from Example 2.12 is locally finite, so it is hyperarchimedean and Archimedean;
(3) The pseudo-MTL algebra \( A \) from Example 3.42 is Archimedean, but it is not hyperarchimedean. Indeed:

\[
0^n = 0 \not\leq 0^- \lor 0^- = 1 \lor 1 = 1, \ n \geq 1
\]
\[
a^n = 0 \not\leq a^- \lor a^- = b \lor c = 1, \ n \geq 2
\]
\[
b^n = b \not\leq b^- \lor b^- = 0 \lor c = c, \ n \geq 1
\]
\[
c^n = c \not\leq c^- \lor c^- = b \lor 0 = b, \ n \geq 1
\]
\[
1^n = 1 \geq 1^- \lor 1^- = 0 \lor 0 = 0, \ n \geq 1.
\]

We conclude that, if \( x^n \geq x^- \lor x^- \) for all \( n \in \mathbb{N} \), \( n \geq 1 \), then \( x = 1 \). Hence, \( A \) is an Archimedean pseudo-MTL algebra. Since \( a^2 = 0 \in B(A) \), it follows that \( a \) is an Archimedean element. By contrary, \( b^n = b \not\in B(A) \) for all \( n \in \mathbb{N} \), \( n \geq 1 \), so \( b \) is not Archimedean element. Thus, \( A \) is not hyperarchimedean.

Remark 6.9. By Examples 6.8(2),(3) we proved that, generally, an Archimedean pseudo-MTL algebra is not commutative (i.e. a MTL algebra).

Acknowledgment

The author would like to thank Professor Afrodita Iorgulescu for her useful remarks and suggestions on the subject that helped improving the presentation.

References


Received: 10.11.2006.

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