Bull. Math. Soc. Sci. Math. Roumanie
Tome 50(98) No. 1, 2007, 3–31

The category of local algebras and points proches

by
Margherita Barile[1], Fioarella Barone[2] and
Wlodzimierz M. Tulczyjew[3*]

Abstract

Categorial methods for generating new local algebras from old ones are presented. A direct proof of the differential structure of the prolongations of a manifold is proposed.

Key Words: Local algebras. Points proches.
2000 Mathematics Subject Classification: Primary: 13H10, Secondary: 58A20.

1 Introduction.

In his paper on the theory of points proches [17], A. Weil, generalizing the notion of jet prolongation due to C. Ehresmann [5], introduced the idea that to any local algebra A corresponds a covariant functor which associates with a differential manifold P a fibre bundle AP over P. The bundle AP is called the prolongation of P of kind A and its elements the A-points, or points proches, on P.

From this paper, several lines of study followed. Among them, the study of the structures one can lift on the prolongations of a manifold ([2], [3], [6], [13], [14]); the theory of product preserving, or natural bundles ([4], [7], [8], [9], [10]); the study of differential invariants on sheaves of tangent vector fields ([15], [16]).

Since the starting point is, in any case, a local algebra, it seemed of some interest to study the category of local algebras and, in particular, the categorial methods for generating new local algebras from old ones. This is the content of the first section of the paper.

It is known that each local algebra is isomorphic to an algebra of (generalized) truncated polynomials. Thus, in the second section of the paper, we study these kind of local algebras, performing the constructions introduced in the previous part and finding, in particular, examples of local algebras which are isomorphic as A-modules but not as algebras.

*Supported by PRIN SINTESI.

Finally, in the third section, we study the differential structure of the prolongation bundles whose existence was first stated by Weil himself. We propose a proof which seems to be very direct and concise. An extension of this method to the more general framework of local algebra fibrations will be presented in a forthcoming paper.

We consider paracompact manifolds modelled over finite dimensional affine spaces. We make use of the notion of polynomial function on an affine space and of some related results presented in [1], which we recall, without proofs, in the Appendix.

We refer to [12] for all basic notions of category theory we use.

2 The Category of Local Algebras.

We will study the category of local algebras and, in particular, the categorial methods for generating new local algebras from old ones. For the sake of completeness, we will first recall the main definitions and basic properties.

2.1 Objects and Morphisms.

Let $\mathcal{C}$ be the category of associative, commutative, real algebras and algebra homomorphisms.

We will refer to them as *algebras* and *morphisms* respectively.

Definition 1. *A local algebra A is an algebra with identity such that*
(i) A is a finite dimensional vector space over $\mathbb{R}$;
(ii) A has a unique maximal ideal I_A and $A/I_A \simeq \mathbb{R}$. ∎

A morphism which preserves the identity will be called a *local algebra morphism*.

We denote by $\mathcal{A}$ the subcategory of $\mathcal{C}$ of local algebras and local algebra morphisms.

Unless otherwise noted, all commutative diagrams considered in this section will be assumed to be local algebra diagrams.

As a consequence of condition (ii) of Definition 1, we have an epimorphism of local algebras

$$0_A : A \to \mathbb{R}$$

whose kernel is I_A. If 1_A is the identity of A, it follows that $A = \mathbb{R}1_A + I_A$ since the subalgebra $\mathbb{R}1_A$ of A is mapped isomorphically onto $\mathbb{R}$ by 0_A, via $\lambda 1_A \mapsto \lambda$. This gives a natural way to identify $\mathbb{R}1_A$ with $\mathbb{R}$. Then we have that

$$A = \mathbb{R} + I_A\,, \tag{1}$$

where the sum is direct. This means that for every element a of A there is a unique decomposition $a = \alpha + \underline{a}$, where $\alpha \in \mathbb{R}$ and $\underline{a} \in I_A$; α will be called

the *finite part* of a. Conversely, for every object J of $\mathcal{C}$ without identity, we can endow the direct product

$$A = \mathbb{R} \times J \tag{2}$$

of real vector spaces with a local algebra structure by defining the product: $(\alpha_1, \underline{a}_1)(\alpha_2, \underline{a}_2) = (\alpha_1\alpha_2, \alpha_1\underline{a}_2 + \alpha_2\underline{a}_1 + \underline{a}_1\underline{a}_2)$.

Condition (i) of Definition 1 implies that all ideals of A are finitely generated as vector spaces over $\mathbb{R}$. A fortiori they are finitely generated as modules over A. Hence A is a Noetherian ring. Furthermore, finite dimension over $\mathbb{R}$ forces the descending sequence of powers of I_A to be stationary. By Nakayama's Lemma (cf. [11], Th.2.2), this implies that $I_A^{\ell+1} = (0)$ for some integer ℓ. The least integer ℓ with this property is called the *height* of A.

Every local algebra morphism $\varphi : A \to B$ maps $\mathbb{R}$ identically onto $\mathbb{R}$. In particular, φ preserves the finite part of every element; therefore, for all $\underline{a} \in I_A$, we have that $\varphi(\underline{a}) \in I_B$, i.e., $\varphi(I_A) \subset I_B$. It follows that 0_A is the only local algebra morphism from A to $\mathbb{R}$, and that the inclusion mapping of $\mathbb{R}$ into A is the only local algebra morphism from $\mathbb{R}$ to A. Hence, in the category $\mathcal{A}$, $\mathbb{R}$ is both the final and the initial object, i.e., it is the zero object of $\mathcal{A}$.

2.2 Subobjects and Quotient Objects.

(i) A local algebra B is a subobject of the local algebra A in $\mathcal{A}$ if there is a local algebra monomorphism $\iota : B \to A$. Up to an isomorphism, B can be identified with $\iota(B) = \mathbb{R} + \iota(I_B)$. Since $\iota(I_B)$ is a subspace of I_A which is closed under multiplication, $\iota(B)$ is an object of $\mathcal{A}$. We will call it a *local subalgebra* of A. Conversely, every subspace J of I_A which is closed under multiplication gives rise to a local subalgebra $\mathbb{R} + J$ of A, and this correspondence defines all local subalgebras of A.

The above arguments show that for every local algebra morphism $\varphi : A \to A'$, $\varphi(A) = \mathbb{R} + \varphi(I_A)$ is a local subalgebra of A': it is the image of φ in both categories $\mathcal{C}$ and $\mathcal{A}$.

Similarly one sees that $\mathbb{R} + \varphi^{-1}(I_{A'})$ is the inverse image of φ in both categories.

On the contrary, the kernel of φ in $\mathcal{C}$ ($\ker\varphi$) and the kernel of φ in $\mathcal{A}$ ($\operatorname{Ker}\varphi$) are distinct, since

$$\operatorname{Ker}\varphi = \mathbb{R} + \ker\varphi\ .$$

We will call *normal subalgebra* of A any subalgebra $\mathbb{R} + J$, where J is a proper ideal of A. The normal subalgebras of A are the kernels in $\mathcal{A}$ of all local algebra morphisms defined on A.

(ii) A quotient in $\mathcal{A}$ of the local algebra A is a local algebra C for which there is a local algebra epimorphism $\pi : A \to C$. Then C is isomorphic to the quotient algebra $A/\ker\pi = \mathbb{R} + I_A/\ker\pi$, which is a local algebra with maximal ideal $I_A/\ker\pi$.

This allows us to identify the quotient objects of A in $\mathcal{A}$ with the local algebras of the type $A/J = \mathbb{R} + I_A/J$, where J is any proper ideal of A.

Given a local algebra morphism $\varphi : A \longrightarrow A'$, the cokernel of φ in both $\mathcal{C}$ and $\mathcal{A}$ is

$$\operatorname{Cok}\varphi = \mathbb{R} + I_{A'}/\bigcap_{\substack{J \text{ proper ideal,} \\ \varphi(I_A)\subset J}} J\ .$$

2.3 Products and Coproducts.

Let A_1 and A_2 be local algebras.

(i) The usual product in $\mathcal{C}$ does not yield a product in $\mathcal{A}$, because the direct product $A_1 \times A_2$ is not a local algebra. In fact it has two maximal ideals, $A_1 \times I_{A_2}$ and $I_{A_1} \times A_2$.

If we consider the subalgebra without identity $I_{A_1} \times I_{A_2}$, following (2) we can construct the local algebra

$$\mathbb{R} \times I_{A_1} \times I_{A_2}$$

which is isomorphic to the following subalgebra of $A_1 \times A_2$:

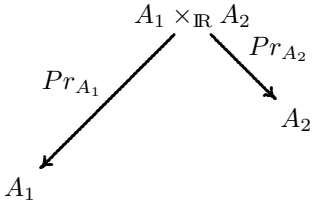
$$\mathbb{R}(1_{A_1}, 1_{A_2}) + I_{A_1} \times I_{A_2}\ .$$

We denote it by

$$A_1 \times_{\mathbb{R}} A_2 := \mathbb{R} + I_{A_1} \times I_{A_2}\ ,$$

where we have used the convention introduced in (1).

We can now define a functor $\times_{\mathbb{R}}$ which associates with every pair of local algebras A_1 and A_2 the diagram



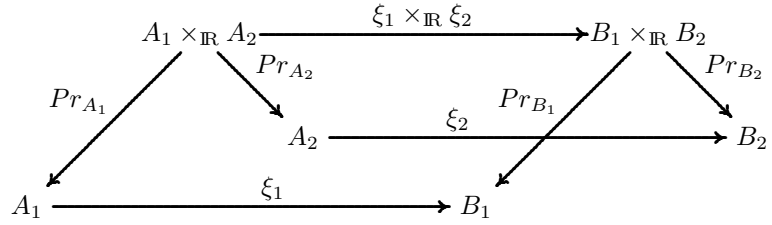
(3)

where

$$Pr_{A_i} : A_1 \times_{\mathbb{R}} A_2 \longrightarrow A_i$$

$$\alpha + (\underline{a}_1, \underline{a}_2) \mapsto \alpha + \underline{a}_i \qquad (i = 1, 2),$$

are the natural projections, and with every pair of local algebra morphisms $\xi_1 : A_1 \longrightarrow B_1$ and $\xi_1 : A_2 \longrightarrow B_2$ the diagram



where, for any $\alpha + (\underline{a}_1, \underline{a}_2) \in A_1 \times_{\mathbb{R}} A_2$,

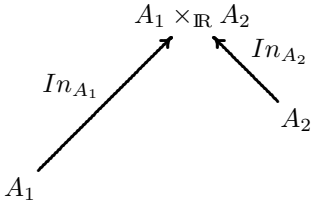
$$\xi_1 \times_{\mathbb{R}} \xi_2\big(\alpha + (\underline{a}_1, \underline{a}_2)\big) = \alpha + \big(\xi_1(\underline{a}_1), \xi_2(\underline{a}_2)\big).$$

Since diagram (3) fulfills the universal property of products, we will call $\times_{\mathbb{R}}$ the *product functor* in $\mathcal{A}$ and we will refer to $A_1 \times_{\mathbb{R}} A_2$ as the product of A_1 and A_2.

Generalizing the above construction, we can consider the product of any finite number of local algebras.

The associative property is trivially fulfilled.

(ii) Similarly, the *coproduct functor* associates with A_1, A_2 the diagram:



(4)

where

$$In_{A_1} : A_1 \longrightarrow A_1 \times_{\mathbb{R}} A_2$$
$$\alpha + \underline{a}_1 \mapsto \alpha + (\underline{a}_1, 0)\ ,$$

and

$$In_{A_2} : A_2 \longrightarrow A_1 \times_{\mathbb{R}} A_2$$
$$\alpha + \underline{a}_2 \mapsto \alpha + (0, \underline{a}_2)$$

are the natural injections. It acts in an obvious way on pairs of local algebra morphisms.

It is easy to verify that diagrams (3) and (4) determine a biproduct of A_1 and A_2 in $\mathcal{A}$.

2.4 Relative Products.

Definition 2. *Let $\pi_{A_1} : A_1 \to B$ and $\pi_{A_2} : A_2 \to B$ be local algebra epimorphisms. The following triple product of local algebras*

$$A_1 \underset{(\pi_{A_1},\pi_{A_2})}{\circ} A_2 := B \times_{\mathbb{R}} \operatorname{Ker}\pi_{A_1} \times_{\mathbb{R}} \operatorname{Ker}\pi_{A_2}$$

$$= \mathbb{R} + I_B \times \ker\pi_{A_1} \times \ker\pi_{A_2}$$

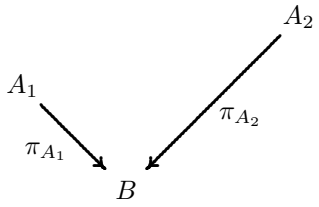
will be called the product of A_1 and A_2 relative to π_{A_1} and π_{A_2}. ∎

Note that, since $\ker 0_{A_i} = I_{A_i}$ for $i = 1, 2$, we have that

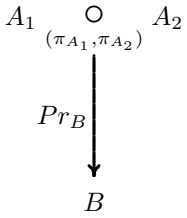
$$A_1 \underset{(0_{A_1},0_{A_2})}{\circ} A_2 = A_1 \times_{\mathbb{R}} A_2 \,.$$

This shows that Definition 2 generalizes the product introduced in 2.3.

We can now define a functor $\underset{(\cdot,\cdot)}{\circ}$ which associates with every diagram of epimorphisms

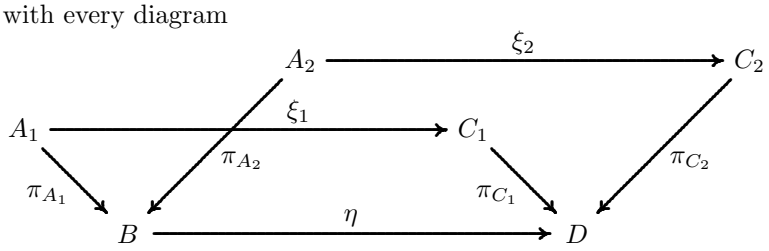


the diagram

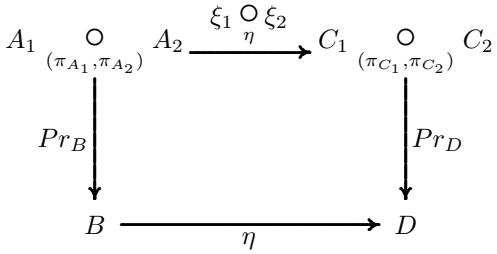


(5)

and with every diagram



the diagram



where, for any $\alpha + (\underline{b}, \underline{a}_1, \underline{a}_2) \in A_1 \underset{(\pi_{A_1},\pi_{A_2})}{\circ} A_2$,

$$\xi_1 \circ \xi_2\big(\alpha + (\underline{b}, \underline{a}_1, \underline{a}_2)\big) = \alpha + \big(\eta(\underline{b}), \xi_1(\underline{a}_1), \xi_2(\underline{a}_2)\big).$$

Definition 2 can be extended to any finite number of local algebra epimorphisms. For example, for $n = 3$, consider the product

$$B \times_{\mathbb{R}} \operatorname{Ker} \pi_{A_1} \times_{\mathbb{R}} \operatorname{Ker} \pi_{A_2} \times_{\mathbb{R}} \operatorname{Ker} \pi_{A_3}\,. \tag{6}$$

In view of diagram (5), we can also consider the product

$$\left(A_1 \underset{(\pi_{A_1},\pi_{A_2})}{\circ} A_2\right) \underset{(Pr_B,\pi_{A_3})}{\circ} A_3\,. \tag{7}$$

Since

$$\operatorname{Ker} Pr_B = \operatorname{Ker} \pi_{A_1} \times_{\mathbb{R}} \operatorname{Ker} \pi_{A_2}$$

and $\times_{\mathbb{R}}$ is associative, we have that (6) and (7) are equal. Similarly one can prove that (6) is also equal to

$$A_1 \underset{(\pi_{A_1},Pr_B)}{\circ} \left(A_2 \underset{(\pi_{A_2},\pi_{A_3})}{\circ} A_3\right).$$

This shows the associativity of functor $\underset{(\cdot,\cdot)}{\circ}$.

2.5 Pullbacks and Pushouts.

(i) Let $\varphi_{A_1} : A_1 \to B$ and $\varphi_{A_2} : A_2 \to B$ be local algebra morphisms.
Let us introduce the local algebra

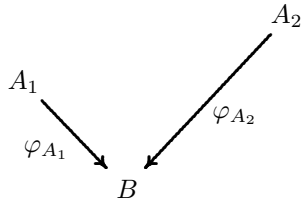
$$A_1 \underset{(\varphi_{A_1},\varphi_{A_2})}{\times} A_2 = \mathbb{R} + \{(\underline{a}_1, \underline{a}_2) \in I_{A_1} \times I_{A_2} \mid \varphi_{A_1}(\underline{a}_1) = \varphi_{A_2}(\underline{a}_2)\}. \tag{8}$$

and notice that it is isomorphic to the following subalgebra of $A_1 \times A_2$:

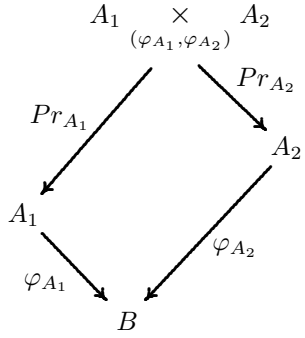
$$\{(a_1, a_2) \mid \varphi_{A_1}(a_1) = \varphi_{A_2}(a_2)\} =$$

$$= \{(\alpha_1 + \underline{a}_1, \alpha_2 + \underline{a}_2) \mid \alpha_1 = \alpha_2 \text{ and } \varphi_{A_1}(\underline{a}_1) = \varphi_{A_2}(\underline{a}_2)\}.$$

We can now define a functor $\underset{(\cdot,\cdot)}{\times}$ which associates with every diagram

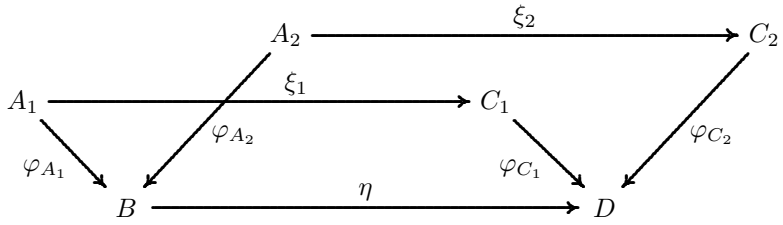


the diagram

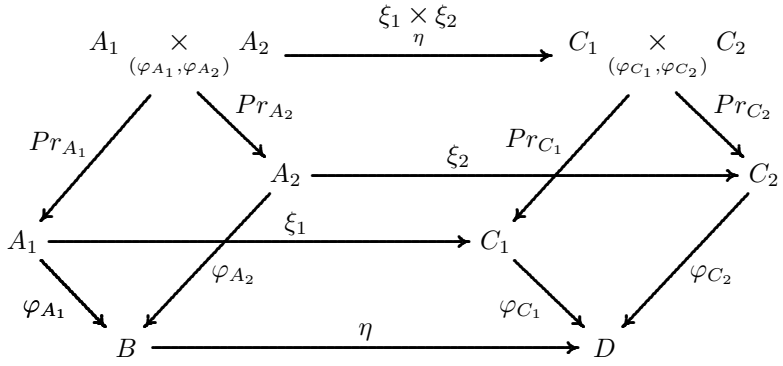


(9)

where Pr_{A_1} and Pr_{A_2}, are the restrictions of the natural projections defined in 2.3, and associates with every diagram



the diagram



where, for any $\alpha + (\underline{a}_1, \underline{a}_2) \in A_1 \underset{(\varphi_{A_1}, \varphi_{A_2})}{\times} A_2$,

$$\xi_1 \underset{\eta}{\times} \xi_2\big(\alpha + (\underline{a}_1, \underline{a}_2)\big) = \alpha + (\xi_1(\underline{a}_1), \xi_2(\underline{a}_2))$$

We will call $\underset{(\cdot,\cdot)}{\times}$ the *pullback functor* in $\mathcal{A}$, since diagram (9) fulfills the corresponding universal property (in both categories $\mathcal{A}$ and $\mathcal{C}$) and we will refer to $A_1 \underset{(\varphi_{A_1}, \varphi_{A_2})}{\times} A_2$ as the pullback of φ_{A_1} and φ_{A_2}.

It is possible to consider the pullback of an arbitrary number of local algebra morphisms following the usual categorial method. The associative property is trivially fulfilled.

It is worth pointing out that $A_1 \times_{\mathbb{R}} A_2 = A_1 \underset{(0_{A_1}, 0_{A_2})}{\times} A_2$. This shows that the pullback gives one more generalization of the product defined in 2.3.

(ii) Let $\pi_{A_1} : A \to A_1$ and $\pi_{A_2} : A \to A_2$ be local algebra epimorphisms.

Consider the quotient

$$A/(\ker \pi_{A_1} + \ker \pi_{A_2})$$

together with the natural epimorphism

$$\pi : A \longrightarrow A/(\ker \pi_{A_1} + \ker \pi_{A_2})\,.$$

It is well known that there are unique local algebra epimorphisms

$$Ep_{A_i} : A_i \longrightarrow A/(\ker \pi_{A_1} + \ker \pi_{A_2}) \qquad (i = 1, 2)$$

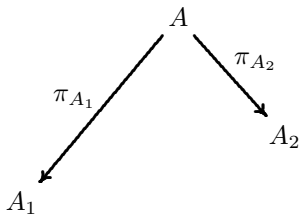
such that

$$Ep_{A_1} \circ \pi_{A_1} = \pi = Ep_{A_2} \circ \pi_{A_2}\,.$$

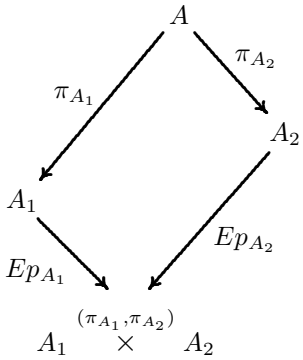
We introduce the local algebra

$$A_1 \overset{(\pi_{A_1}, \pi_{A_2})}{\times} A_2 := A/(\ker \pi_{A_1} + \ker \pi_{A_2}) \tag{10}$$

and define a functor $\overset{(\cdot,\cdot)}{\times}$ which associates with every diagram of epimorphisms

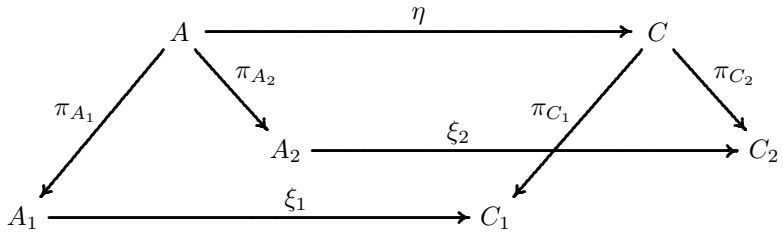


the diagram

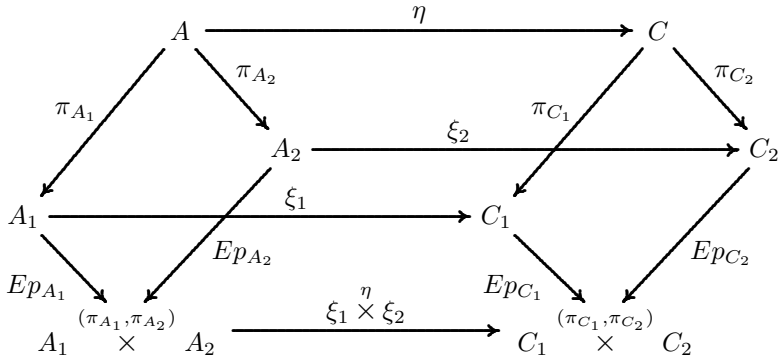


(11)

and with every diagram



the diagram

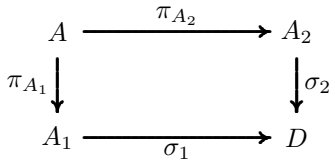


where, for any $a + (\ker \pi_{A_1} + \ker \pi_{A_2}) \in A_1 \overset{(\pi_{A_1},\pi_{A_2})}{\times} A_2$,

$$\xi_1 \overset{\eta}{\times} \xi_2\big(a + (\ker \pi_{A_1} + \ker \pi_{A_2})\big) = \eta(a) + (\ker \pi_{C_1} + \ker \pi_{C_2}).$$

Proposition 3. *Diagram (11) fulfills the universal property of pushouts in both categories $\mathcal{A}$ and $\mathcal{C}$.*

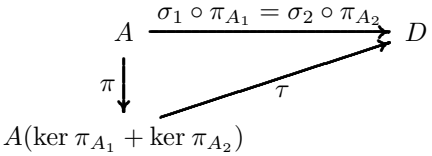
Proof: Suppose we have a diagram



where π_{A_1} and π_{A_2} are epimorphisms. We first observe that

$$\ker \pi = \ker \pi_{A_1} + \ker \pi_{A_2} \subset \ker(\sigma_1 \circ \pi_{A_1}) = \ker(\sigma_2 \circ \pi_{A_2})\,.$$

Then, from the Factorization Theorem, we conclude that there is a unique local algebra morphism τ for which the diagram



is commutative, i.e., $\tau \circ \pi = \sigma_1 \circ \pi_{A_1} = \sigma_2 \circ \pi_{A_2}$, or

$$\tau \circ Ep_{A_i} \circ \pi_{A_i} = \sigma_i \circ \pi_{A_i} \qquad (i = 1, 2)\,,$$

or, since each π_{A_i} is surjective,

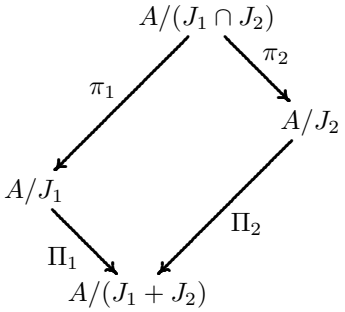
$$\tau \circ Ep_{A_i} = \sigma_i \qquad (i = 1, 2)\,.$$

This completes the proof. □

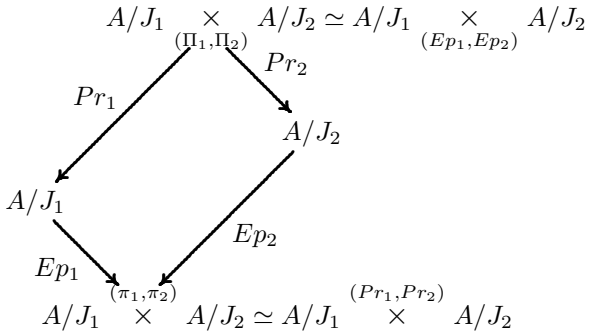
We will call $\overset{(\cdot,\cdot)}{\times}$ the *pushout functor* in $\mathcal{A}$ and we will refer to $A_1 \overset{(\pi_{A_1},\pi_{A_2})}{\times} A_2$ as the pushout of π_{A_1} and π_{A_2}.

It is possible to consider the pushout of an arbitrary number of local algebra epimorphisms following the usual categorial method. The associative property is trivially fulfilled.

(iii) A Special Case. In this subsection we will apply the pullback and pushout functors to the natural epimorphisms between quotients of a given local algebra. We will thus establish a connection between the pullback and the intersection of ideals, and between the pushout and the sum of ideals in the sense that, if A is a local algebra, and J_1, J_2 are proper ideals of A, the diagram of natural epimorphisms



is isomorphic to the pullback–pushout diagram



In fact, according to (10), we have

$$A/J_1 \overset{(\pi_1,\pi_2)}{\times} A/J_2 = \frac{A/(J_1 \cap J_2)}{\ker \pi_1 + \ker \pi_2} =$$

$$= \frac{A/(J_1 \cap J_2)}{J_1/(J_1 \cap J_2) + J_2/(J_1 \cap J_2)} = \frac{A/(J_1 \cap J_2)}{(J_1 + J_2)/(J_1 \cap J_2)} \simeq A/(J_1 + J_2),$$

where the last isomorphism follows from the third isomorphism theorem for rings.

Moreover, according to (8), we have

$$A/J_1 \underset{(\Pi_1,\Pi_2)}{\times} A/J_2 =$$

$$\mathbb{R} + \{(\underline{a}_1 + J_1, \underline{a}_2 + J_2) \in I_A/J_1 \times I_A/J_2 \mid \underline{a}_1 + (J_1 + J_2) = \underline{a}_2 + (J_1 + J_2)\}$$

and, as a consequence of the first isomorphism theorem for rings, the mapping

$$A/(J_1 \cap J_2) \longrightarrow A/J_1 \underset{(\Pi_1,\Pi_2)}{\times} A/J_2$$

$$a + (J_1 \cap J_2) \mapsto (a + J_1, a + J_2)$$

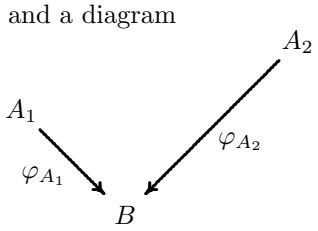
is a local algebra isomorphism.

2.6 Composition of Functors.

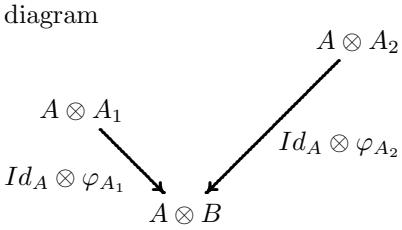
We conclude our study of the category of local algebras by examining the behaviour under composition of the functors introduced above together with the tensor product functor, which was considered by A. Weil [17]. We just recall that for any local algebras A_1 and A_2, the tensor product $A_1 \otimes A_2$ is a local algebra, whose maximal ideal is $I_{A_1 \otimes A_2} = I_{A_1} \otimes A_2 + A_1 \otimes I_{A_2}$.

We will only take into consideration the cases involving $\underset{(\cdot,\cdot)}{\times}$ with $\otimes$ and $\underset{(\cdot,\cdot)}{\circ}$ with $\otimes$.

Given a local algebra A and a diagram



we can consider the diagram



Proposition 4. *The functor $\otimes$ is distributive with respect to the functor $\underset{(\cdot,\cdot)}{\times}$, in the sense that we have a natural local algebra isomorphism*

$$A \otimes A_1 \underset{(Id_A \otimes \varphi_{A_1}, Id_A \otimes \varphi_{A_2})}{\times} A \otimes A_2 \simeq A \otimes \left(A_1 \underset{(\varphi_{A_1}, \varphi_{A_2})}{\times} A_2 \right) \tag{12}$$

Proof: As we already noticed in 2.3(i), the left-hand side of (12) is (isomorphic to) a subalgebra of $(A \otimes A_1) \times (A \otimes A_2)$.

Let $(e_1, \ldots, e_d)$ be a basis in A. We have that, for all $a_{i1} \in A_1$ and $a_{i2} \in A_2$ $(i = 1, \ldots, d)$,

$$\left(\sum_{i=1}^{d} e_i \otimes a_{i1}, \sum_{i=1}^{d} e_i \otimes a_{i2} \right) \in A \otimes A_1 \underset{(Id_A \otimes \varphi_{A_1}, Id_A \otimes \varphi_{A_2})}{\times} A \otimes A_2$$

if and only if

$$\sum_{i=1}^{d} e_i \otimes \varphi_{A_1}(a_{i1}) = \sum_{i=1}^{d} e_i \otimes \varphi_{A_2}(a_{i2})$$

which is equivalent to

$$\varphi_{A_1}(a_{i1}) = \varphi_{A_2}(a_{i2}) \qquad \forall\, i = 1, \ldots, d$$

i.e., to

$$(a_{i1}, a_{i2}) \in A_1 \underset{(\varphi_{A_1}, \varphi_{A_2})}{\times} A_2 \qquad \forall\, i = 1, \ldots, d$$

and, finally to

$$\sum_{i=1}^{d} e_i \otimes (a_{i1}, a_{i2}) \in A \otimes \left(A_1 \underset{(\varphi_{A_1}, \varphi_{A_2})}{\times} A_2 \right) .$$

It can be easily shown that the resulting local algebra isomorphism is independent of the choice of the basis in A. □

Recall that $\times_{\mathbb{R}}$ is a special case of $\underset{(\cdot,\cdot)}{\times}$ and that $\underset{(\cdot,\cdot)}{\bigcirc}$ is obtained by means of triple products. Therefore, as a trivial consequence of the above proposition, we have

Corollary 5. *The functor $\otimes$ is distributive with respect to both the functors $\times_{\mathbb{R}}$ and $\underset{(\cdot,\cdot)}{\bigcirc}$.* ∎

It is easy to check that functor $\times_{\mathbb{R}}$ is distributive with respect to functor $\underset{(\cdot,\cdot)}{\times}$.

3 Algebras of Truncated Polynomials.

After studying the category of local algebras in general, the next natural step is to look for concrete examples. The model of the most general local algebra is a finite dimensional quotient of an algebra of polynomials over a local algebra A, a so-called algebra of 'generalized truncated polynomials'. We will apply some of the most significant functors introduced in the previous section in order to obtain special algebras of this kind. In particular we will find examples of local algebras which are isomorphic as A–modules but not as algebras.

Let A be a d-dimensional local algebra.

Consider the algebra $A[x]$ of all polynomials with coefficients in A in the indeterminate x.

Let $\mathbb{I} = \langle x \rangle$ be the ideal of $A[x]$ generated by x; then, for all non negative integers k, $\mathbb{I}^k = \langle x^k \rangle$.

The proof of the next proposition is reported for the only sake of completeness.

Proposition 6. *The set of all residue classes mod x^{k+1} of polynomials in $A[x]$, i.e.,*

$$\mathbb{P}_k A[x] := A[x]/\mathbb{I}^{k+1},$$

is a local algebra.

Proof: Since $\dim_{\mathbb{R}}(\mathbb{P}_k A[x]) = (k+1)d$, condition (i) of Definition 1 is fulfilled.

Now consider a maximal ideal J of $A[x]$ containing $\mathbb{I}^{k+1}$. Since J is prime and $x^{k+1} \in J$, we have that $x \in J$. Hence $\mathbb{I} \subset J$, so that $J = I + \mathbb{I}$ where I is the ideal of A formed by the constant terms of all the polynomials belonging to J. Therefore $J \subset I_A + \mathbb{I}$, whence equality follows by maximality. We have thus proved that $\mathbb{P}_k A[x]$ has $I_k = (I_A + \mathbb{I}/\mathbb{I}^{k+1})$ as its unique maximal ideal. It also holds that $\mathbb{P}_k A[x]/I_k \simeq \mathbb{R}$. Hence condition (ii) of Definition 1 is fulfilled, too. □

$\mathbb{P}_k A[x]$ can be identified with the set of all polynomials of degree at most k.

We can consider the following algebras:

$$\mathbb{P}_k A[x_1, \ldots, x_n] := A[x_1, \ldots, x_n]/\mathbb{I}^{k+1}, \tag{13}$$

where $\mathbb{I} = \langle x_1, \ldots, x_n \rangle$, and

$$\mathbb{P}_{k_1,\ldots,k_n} A[x_1, \ldots, x_n] := A[x_1, \ldots, x_n]/\mathbb{I}^{(k_1+1,\ldots,k_n+1)} \tag{14}$$

where

$$\mathbb{I}^{(k_1+1,\ldots,k_n+1)} := \langle x_1^{k_1+1}, \ldots, x_n^{k_n+1} \rangle .$$

By the same arguments as in Proposition 6, one can easily conclude that all the above algebras (13) and (14) are local. We will call them *algebras of generalized truncated polynomials*.

We now apply the categorial constructions described in the previous section to algebras of generalized truncated polynomials. In all the cases considered, the resulting local algebras will still be of the same kind.

Note that we have the following inclusions:

$$\mathbb{I}^k \subset \mathbb{I}^{k'} \iff k \geq k'$$

$$\mathbb{I}^{(k_1,\ldots,k_n)} \subset \mathbb{I}^{(k'_1,\ldots,k'_n)} \iff k_i \geq k'_i \qquad \text{for all } i = 1, \ldots, n$$

$$\mathbb{I}^k \subset \mathbb{I}^{(k_1,\ldots,k_n)} \iff k \geq k_1 + \cdots + k_n - n + 1$$

$$\mathbb{I}^{(k_1,\ldots,k_n)} \subset \mathbb{I}^k \iff k_i \geq k \qquad \text{for all } i = 1, \ldots, n$$

In each of the above cases we can consider the corresponding quotients of the local algebras defined in (13) and (14):

$$\mathbb{P}_{k-1}A[x_1,\dots,x_n]/(\mathbb{I}^{\,k'}/\mathbb{I}^{\,k}) \simeq \mathbb{P}_{k'-1}A[x_1,\dots,x_n]$$

$$\mathbb{P}_{(k_1-1,\dots,k_n-1)}A[x_1,\dots,x_n]/\left(\mathbb{I}^{\,(k'_1,\dots,k'_n)}/\mathbb{I}^{\,(k_1,\dots,k_n)}\right) \simeq$$
$$\simeq \mathbb{P}_{(k'_1-1,\dots,k'_n-1)}A[x_1,\dots,x_n]$$

$$\mathbb{P}_{k-1}A[x_1,\dots,x_n]/\left(\mathbb{I}^{\,(k_1,\dots,k_n)}/\mathbb{I}^{\,k}\right) \simeq \mathbb{P}_{(k_1-1,\dots,k_n-1)}A[x_1,\dots,x_n]$$

$$\mathbb{P}_{(k_1-1,\dots,k_n-1)}A[x_1,\dots,x_n]/\left(\mathbb{I}^{\,k}/\mathbb{I}^{\,(k_1,\dots,k_n)}\right) \simeq \mathbb{P}_{k-1}A[x_1,\dots,x_n]$$

All the above isomorphisms are a consequence of the third isomorphism theorem for rings.

We also have the natural isomorphisms

$$\bigotimes_{i=1}^{n} \mathbb{P}_{k_i}A[x_i] \simeq \mathbb{P}_{k_1,\dots,k_n}A[x_1,\dots,x_n]$$

Let $m \in \mathbb{N}^*$.

We will show that the free A–module A^m of rank m can be given different local algebra structures.

For any $r,s,t \in \mathbb{N}^*$ such that $r+t-s+1=m$ and $s \le \min(r,t)$, consider the diagram

$$\begin{array}{ccc} & & \mathbb{P}_r A[x] \\ & & \big\downarrow \tau_r^s \\ \mathbb{P}_t A[y] & \xrightarrow[\tau_t^s]{} & \mathbb{P}_s A[z] \end{array} \tag{15}$$

where $\tau_r^s\big(\sum_{i=0}^r a_i x^i\big) = \sum_{i=0}^s a_i z^i$, and τ_t^s is defined similarly (with a slight abuse of notation), and consider the corresponding (relative) product

$$\mathbb{P}_r A[x] \underset{(\tau_r^s,\tau_t^s)}{\circ} \mathbb{P}_t A[y] = \mathbb{P}_s A[z] \times_{\mathbb{R}} \operatorname{Ker}\tau_r^s \times_{\mathbb{R}} \operatorname{Ker}\tau_t^s\,. \tag{16}$$

We introduce the free A–module of rank m

$$C^s_{r,t} = \left\{ \left(a_0 + \sum_{i=1}^{s} a_i z^i \right) + \sum_{i=s+1}^{r} b_i x^i + \sum_{i=s+1}^{t} c_i y^i \right\},$$

which can be identified in an obvious way with a subset of

$$\mathbb{P}_{r,t,s}A[x,y,z]/\langle xy,xz,yz\rangle\,.$$

It is easy to see that $C^s_{r,t}$, endowed with the induced local algebra structure, is isomorphic to (16).

We now apply the pullback functor to diagram (15) and consider the corresponding local algebra

$$\mathbb{P}_r A[x] \underset{(\tau^s_r,\tau^s_t)}{\times} \mathbb{P}_t A[y] = \mathbb{R} + \left\{(\underline{p}_1,\underline{p}_2) \in I_{\mathbb{P}_r A[x]} \times I_{\mathbb{P}_t A[y]} \mid \tau^s_r(\underline{p}_1) = \tau^s_t(\underline{p}_2)\right\}. \tag{17}$$

We introduce the free A–module of rank m

$$B^s_{r,t} = \left\{ a_0 + \sum_{i=1}^{s} a_i(x^i + y^i) + \sum_{i=s+1}^{r} b_i x^i + \sum_{i=s+1}^{t} c_i y^i \right\},$$

which can be identified, in an obvious way, with a subset of

$$\mathbb{P}_{r,t}A[x,y]/\langle xy\rangle\,.$$

It is easy to see that $B^s_{r,t}$, endowed with the induced local algebra structure, is isomorphic to (17).

In both of the above constructions, different decompositions of integer m produce different A–modules (of the same rank m), which, in general, are pairwise non-isomorphic as local algebras. This yields different local algebra structures on A^m.

4 Points Proches.

We start this section by recalling the construction of the functor associated with a local algebra, introduced by A.Weil [17].

4.1 Functors associated with Local Algebras.

Let A be a local algebra. For any differential manifold P consider the set AP of all morphisms $u : C^\infty(P) \to A$ which preserve the identity. The elements in AP are called *points proches* associated with the local algebra A.

Let $u \in AP$. There is just one point $p \in P$ such that the composition

$$0_A \circ u : C^\infty(P) \longrightarrow \mathbb{R}$$

is the evaluation mapping ev_p at p (cf., e.g., [2], Prop. 0.7).

This defines a surjective mapping (*target*) $\alpha^T_P : AP \longrightarrow P$ such that

$$(\alpha^T_P)^{-1}(p) = A_pP = \left\{u \in AP | 0_A \circ u = ev_p\right\}.$$

Now consider a differentiable mapping $\varphi : P \longrightarrow Q$ and define

$$A\varphi : AP \longrightarrow AQ$$

by setting

$$A\varphi(u) : C^\infty(Q) \longrightarrow A : g \mapsto u(g \circ \varphi).$$

For each $p \in P$, this mapping induces by restriction a mapping

$$A_p\varphi : A_pP \longrightarrow A_{\varphi(p)}Q\,.$$

Proposition 7.

(i) To any local algebra A corresponds the covariant functor

$$\begin{array}{c} A \\ {\scriptstyle \alpha^T}\big\downarrow \\ I \end{array}$$

which associates with any differential manifold P the mapping $\alpha^T_P : AP \longrightarrow P$ and with any differentiable mapping $\varphi : P \to Q$ the diagram

$$\begin{array}{ccc} AP & \xrightarrow{A\varphi} & AQ \\ {\scriptstyle \alpha^T_P}\big\downarrow & & \big\downarrow{\scriptstyle \alpha^T_Q} \\ P & \xrightarrow{\varphi} & Q \end{array}$$

(ii) To a morphism of local algebras $\kappa : A \longrightarrow B$ corresponds the functor

$$\begin{array}{ccc} A & \xrightarrow{\kappa} & B \\ {\scriptstyle \alpha^T}\big\downarrow & & \big\downarrow{\scriptstyle \beta^T} \\ I & = & I \end{array}$$

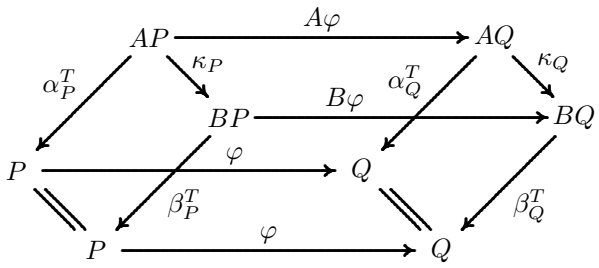
which associates with a manifold P the diagram

$$\begin{array}{ccc} AP & \xrightarrow{\kappa_P} & BP \\ {\scriptstyle \alpha^T_P}\big\downarrow & & \big\downarrow{\scriptstyle \beta^T_P} \\ P & = & P \end{array}$$

where

$$\kappa_P(u) = \kappa \circ u\,,$$

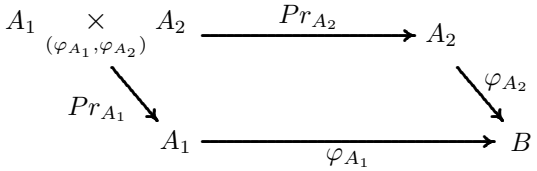
and with a morphism $\varphi : P \to Q$ the diagram



■

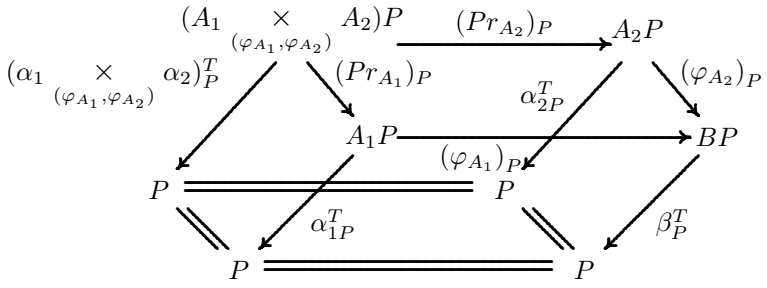
In the first part of the paper we introduced some basic categorial constructions in order to obtain new local algebras from old ones. In view of the above proposition, to each of these algebras corresponds a functor. Weil noticed that to the tensor product of local algebras corresponds the composition of the functors associated with the factors. Here we study what corresponds to the pullback of local algebras.

Let

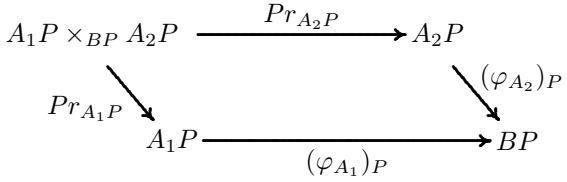


be the pullback diagram of two local algebra morphisms $\varphi_{A_1} : A_1 \to B$ and $\varphi_{A_2} : A_2 \to B$.

Then, for any manifold P, we have the diagram



On the other hand, consider the pullback diagram of $(\varphi_{A_1})_P$ and $(\varphi_{A_2})_P$



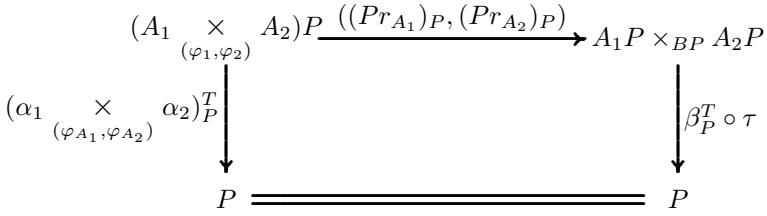
where

$$A_1P \times_{BP} A_2P = \{(u_1, u_2) \in A_1P \times A_2P | (\varphi_{A_1})_P(u_1) = (\varphi_{A_2})_P(u_2)\}.$$

If we put

$$\tau = (\varphi_{A_1})_P \circ Pr_{A_1P} = (\varphi_{A_2})_P \circ Pr_{A_2P}\,,$$

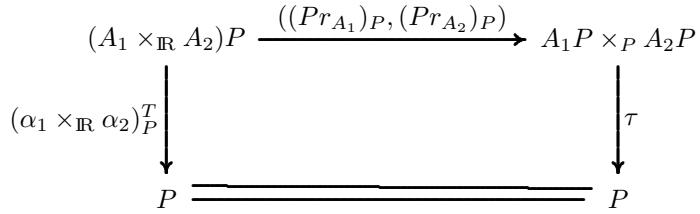
it is easy to prove that in the following diagram



the upper arrow is a bijection.

This allows us to conclude that the pullback commutes with the operation of associating with a local algebra the corresponding points proches.

Since the product of local algebras is a special case of pullback, we have the diagram



and the upper arrow is still bijective. In this case,

$$A_1P \times_P A_2P = \{(u_1, u_2) \in A_1P \times A_2P | \alpha^T_{1\,P}(u_1) = \alpha^T_{2\,P}(u_2)\}\,,$$

is the fibre product of the target mappings.

We conclude that the product of local algebras gives rise to the fibre product of the corresponding points proches.

4.2 The Local Character of Points Proches.

Let A be a local algebra and P a differential manifold.

Proposition 8. *Let $u \in A_pP$. Then $u(f) = 0$, for any function f vanishing in some neighbourhood of p.*

Proof: Let f be a function vanishing in the neighbourhood $\mathcal{U}$ of p. If β is a bump function at p with support in $\mathcal{U}$, we have

$$f = (1 - \beta)f$$

both on $\mathcal{U}$, where f vanishes, and on $P - \mathcal{U}$, where β vanishes. Therefore

$$u(f) = u(1 - \beta)u(f)\,. \tag{18}$$

Put

$$u(f) = f(p) + \underline{a} = \underline{a}$$
$$u(1 - \beta) = (1 - \beta)(p) + \underline{b} = \underline{b}\,.$$

Owing to (18) we have

$$\underline{a} = \underline{b}\,\underline{a} \iff (1 - \underline{b})\underline{a} = 0\,,$$

but, since $\underline{b} \in I_A$ and $1 \notin I_A$, it must be $1 - \underline{b} \notin I_A$. This implies that $1 - \underline{b}$ is invertible. Hence $u(f) = \underline{a} = 0$. □

The proof of Proposition 8 directly derives the local character of points proches from the local nature of algebra A (a different approach is followed in [2], Prop. 1.10). This result allows us to prove that the functor associated with a local algebra A preserves open inclusions. Let us apply this functor to the inclusion mapping

$$\iota_{\mathcal{U}} : \mathcal{U} \longrightarrow P$$

of the open subset $\mathcal{U}$ into P. Then for each $p \in \mathcal{U}$, we have the mapping

$$A_p\iota_{\mathcal{U}} : A_p\mathcal{U} \longrightarrow A_pP$$

given by

$$A_p\iota_{\mathcal{U}}(u)(f) := u(f|_{\mathcal{U}})\,.$$

On the other hand, the mapping

$$\eta : A_pP \longrightarrow A_p\mathcal{U}$$

given by

$$\eta(u)(g) := u(f)$$

for any differentiable prolongation f of g in P, is well defined, in view of the local character of u, and is the inverse of $A_p\iota_{\mathcal{U}}$. Hence $A_p\iota_{\mathcal{U}}$ is a bijection.

If we identify $A_p\mathcal{U}$ with A_pP and consequently regard $A\iota_{\mathcal{U}}$ as the inclusion $\iota_{A\mathcal{U}}$ of $A\mathcal{U}$ in AP, we have the following diagram:

$$\begin{array}{ccc} A\mathcal{U} & \xrightarrow{\iota_{A\mathcal{U}}} & AP \\ \alpha^T_{\mathcal{U}}\downarrow & & \downarrow\alpha^T_P \\ \mathcal{U} & \xrightarrow{\iota_{\mathcal{U}}} & P \end{array} \tag{19}$$

This result will be used for introducing the differential fibration structure on AP.

4.3 Points Proches in Affine Spaces.

In this section we study the special case where the functor associated with a local algebra is applied to an affine space. We will prove that the result is a fibre bundle whose total space has an affine structure. In addition, we will show that the functor transforms every differentiable mapping between affine spaces into a differentiable morphism of trivial bundles.

Proposition 9. *Let (M, V) be an affine space. Then, for each $x_0 \in M$, we have a natural bijection*

$$A_{x_0}M \longrightarrow \{x_0\} \times I_A \otimes V\,.$$

Proof: We denote by $K^s(M, x_0)$ the linear space of homogeneous polynomials in M, at x_0, of degree s (cf. Definition 13 in the Appendix). There is a natural isomorphism

$$K^1(M, x_0) \simeq V^*$$

which allows us to identify $I_A \otimes V$ with $\mathrm{Hom}_{\mathbb{R}}(K^1(M, x_0), I_A)$. We will prove that there is a natural bijection

$$A_{x_0}M \longrightarrow \{x_0\} \times \mathrm{Hom}_{\mathbb{R}}\big(K^1(M, x_0), I_A\big).$$

Let $(e_1, \ldots, e_m)$ be a basis of V, $(e^1, \ldots, e^m)$ its dual basis and

$$x^i : M \longrightarrow \mathbb{R} : x \mapsto e^i(x - x_0) \tag{20}$$

the corresponding Cartesian coordinate functions at x_0, for $i = 1, \ldots, m$.

Let us now consider the mapping

$$A_{x_0}M \longrightarrow \{x_0\} \times \mathrm{Hom}_{\mathbb{R}}\big(K^1(M, x_0), I_A\big)$$

$$u \mapsto \big(x_0, u|_{K^1(M,x_0)}\big)$$

We will prove that it admits an inverse, i.e., for each $\bar{u} \in \mathrm{Hom}_{\mathbb{R}}\big(K^1(M,x_0), I_A\big)$ there is a unique $u \in A_{x_0}M$ with $u|_{K^1(M,x_0)} = \bar{u}$. We are going to construct u starting from its action on polynomials. Let $h \in K^s(M,x_0)$. In view of Proposition 14 in the Appendix, we have

$$\begin{aligned} h(x) &= \frac{1}{s!}\Delta^s h(x_0; x - x_0) \\ &= \frac{1}{s!}\Delta^s h(x_0; x^i e_i) \\ &= \frac{1}{s!}\delta^s h(x_0; e_{i_1}, \ldots, e_{i_s}) x^{i_1} \cdots x^{i_s} \end{aligned}$$

where $\Delta^s h$ and $\delta^s h$ are the s-th unidirectional and (multidirectional) polarizations of h, respectively (cf. Definition 12 in the Appendix) and we have adopted the Einstein convention over repeated indices. We define

$$u(h) := \frac{1}{s!}\delta^s h(x_0; e_{i_1}, \ldots, e_{i_s})\bar{u}(x^{i_1}) \cdots \bar{u}(x^{i_s})\,.$$

Easy linear algebraic arguments show that this definition is independent of the choice of the basis. It follows that

$$u(h) = 0 \text{ if } s > \ell$$

where ℓ is the height of A, and that u preserves the product of polynomials.

We extend the definition of u from polynomials to arbitrary smooth functions $f \in C^\infty(M)$ by considering the Taylor polynomial of f of order ℓ at x_0 (cf. Note 18 in the Appendix) and setting

$$u(f) := f(x_0) + \sum_{i=1}^{\ell} \frac{1}{i!} u\big(D^i f(x_0; \cdot - x_0)\big)\,. \tag{21}$$

We recall that each $D^i f(x_0; \cdot - x_0)$ is a homogeneous polynomial at x_0 of degree i (cf. Proposition 16 in the Appendix).

The above formula defines u as a linear mapping. We show that u preserves products, too. In fact,

$$u(f)u(g) = \left[f(x_0) + \sum_{r=1}^{\ell} \frac{1}{r!} u\Big(D^r f(x_0; \cdot - x_0)\Big)\right] \times$$

$$\left[g(x_0) + \sum_{s=1}^{\ell} \frac{1}{s!} u\Big(D^s g(x_0; \cdot - x_0)\Big)\right].$$

Note that

$$u\Big(D^r(f)(x_0; \cdot - x_0)\Big)u\Big(D^s(g)(x_0; \cdot - x_0)\Big) = 0 \quad \text{if} \quad r + s > \ell.$$

Hence, by the Leibniz rule we deduce that

$$\begin{aligned} u(f)u(g) &= f(x_0)g(x_0) + \\ &\quad \sum_{r+s=1}^{\ell} \frac{1}{r!s!} u\Big(D^r(f)(x_0;\cdot - x_0)\Big) u\Big(D^s(g)(x_0;\cdot - x_0)\Big) \\ &= f(x_0)g(x_0) + \\ &\quad \sum_{i=1}^{\ell} \frac{1}{i!} u\left(\sum_{r+s=i} \frac{i!}{r!s!} D^r(f)(x_0;\cdot - x_0) D^s(g)(x_0;\cdot - x_0)\right) \\ &= fg(x_0) + \sum_{i=1}^{\ell} \frac{1}{i!}\, u\big(D^i(fg)(x_0;\cdot - x_0)\big) \\ &= u(fg). \end{aligned}$$

We have constructed a $u \in A_{x_0}M$ such that $u|_{K^1(M,x_0)} = \bar{u}$. It is the only element in $A_{x_0}M$ fulfilling this property because, on the one hand, the action of an algebra morphism on polynomials is uniquely determined by its action on $K^1(M, x_0)$ and, on the other hand, each element in $A_{x_0}M$ vanishes on all homogeneous polynomials of degree $\ell + 1$ at x_0; in particular it vanishes on the Lagrange remainders of order ℓ at x_0 (cf. Note 18 in the Appendix). As a consequence, its action on a function must be the one defined in (21). □

According to Proposition 9 we have a natural bijection

$$AM \longrightarrow M \times I_A \otimes V \tag{22}$$

which induces on AM a structure of $m(d-1)$–dimensional affine space, d being the dimension of A.

By virtue of (22) we will identify $\alpha_M^T : AM \to M$ with

$$Pr_M : M \times I_A \otimes V \longrightarrow M$$

Proposition 10. *Let (M, V) and (N, W) be affine spaces, and $\varphi : M \longrightarrow N$ a differentiable mapping. Then $A\varphi : AM \longrightarrow AN$ is a differentiable mapping.*

Proof: Let $(e) = (e_1, \ldots, e_m)$ and $(\varepsilon) = (\varepsilon_1, \ldots \varepsilon_n)$ be bases of V and W respectively. We can identify AM with

$$M \times \big(\underbrace{I_A \times \cdots \times I_A}_{m \text{ times}}\big)$$

via the affine isomorphism

$$AM \longrightarrow M \times \big(\underbrace{I_A \times \cdots \times I_A}_{m\,\text{times}}\big) : u \mapsto \big(x, u(x^1), \ldots, u(x^m)\big),$$

where $x = \alpha_M^T(u)$ and, $(x^1, \ldots, x^m)$ are the Cartesian coordinate functions at x relative to (e), defined as in (20). Similarly we identify AN with $N \times \big(\underbrace{I_A \times \cdots \times I_A}_{n\,\text{times}}\big)$.

Now consider $A\varphi : AM \to AN$ and recall that, for each $g \in C^\infty(N)$,

$$A\varphi(u)(g) = u(g \circ \varphi).$$

From the above identifications it follows that

$$u = \big(x, u(x^1), \ldots, u(x^m)\big)$$

and

$$A\varphi(u) = \big(\varphi(x), A\varphi(u)(y^1), \ldots, A\varphi(u)(y^n)\big),$$

where $(y^1, \ldots, y^n)$ are the Cartesian coordinate functions at $\varphi(x)$ relative to (ε). Moreover, for all $j = 1, \ldots, n$ we have:

$$\begin{aligned} A\varphi(u)(y^j) &= u(y^j \circ \varphi) \\ &= u(\varphi^j) \\ &= u\left[\varphi^j(x) + \sum_{k=1}^{\ell} \frac{1}{k!} D^k \varphi^j(x; \cdot - x)\right] \\ &= \varphi^j(x) + u\big(h^j[x^1, \ldots, x^m]\big) \end{aligned}$$

where h^j is a formal polynomial in m indeterminates, whose coefficients are differentiable functions of x. Hence

$$A\varphi(u)(y^j) = \varphi^j(x) + h^j[u(x^1), \ldots, u(x^m)]$$

which shows that $A\varphi(u)(y^j)$ is differentiable as a function of x and $u(x^i)$. □

The above arguments imply that

$$\begin{array}{ccc} AM & \xrightarrow{A\varphi} & AN \\ {\scriptstyle \alpha_M^T}\downarrow & & \downarrow{\scriptstyle \alpha_N^T} \\ M & \xrightarrow[\varphi]{} & N \end{array}$$

is a differentiable morphism of trivial bundles.

If φ is a diffeomorphism, the diagram is an isomorphism.

In view of the local character of the points proches, all the above results are still true if the affine spaces are replaced by open subsets.

4.4 The differential structure.

We now show that the functor associated with a local algebra transforms differential manifolds into differential fibre bundles.

Proposition 11. *Let P be a differential manifold modelled on an affine space (M, V). Then AP is a differential manifold modelled on the affine space $(AM, I_A \otimes V)$.*

Proof: Let $\xi : \mathcal{U} \subset P \to \xi(\mathcal{U}) \subset M$ be an admissible chart of P. We have the bijection

$$A\xi : A\mathcal{U} \subset AP \longrightarrow A\xi(\mathcal{U}) \subset AM\,.$$

where the inclusions are due to (19). Now let $\xi' : \mathcal{U}' \subset P \to \xi'(\mathcal{U}') \subset M$ be another admissible chart on P such that $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, then

$$\xi' \circ \xi^{-1} : \xi(\mathcal{U} \cap \mathcal{U}') \longrightarrow \xi'(\mathcal{U} \cap \mathcal{U}')$$

is a diffeomorphism between open subsets of M. By Proposition 10

$$A(\xi' \circ \xi^{-1}) = A\xi' \circ (A\xi)^{-1} : A\xi(\mathcal{U} \cap \mathcal{U}') \longrightarrow A\xi'(\mathcal{U} \cap \mathcal{U}')$$

is a diffeomorphism between open subsets of AM. This guarantees that the functor associated with local algebra A, when applied to a differential atlas of P, produces a differential atlas of AP.

□

An immediate consequence of the above proposition is that the functor associated with a local algebra A, applied to chart ξ of P, gives rise to a local trivialization of AP:

$$\begin{array}{ccc} A\mathcal{U} & \xrightarrow{A\xi} & \xi(\mathcal{U}) \times I_A \otimes V \\ \Big\downarrow{\scriptstyle \alpha^T_{\mathcal{U}}} & & \Big\downarrow{\scriptstyle \alpha^T_{\xi(\mathcal{U})}} \\ \mathcal{U} & \xrightarrow[\xi]{} & \xi(\mathcal{U}) \end{array}$$

so that we can conclude that

$$\alpha^T_P : AP \longrightarrow P$$

is a differentiable fibre bundle.

4.5 Appendix.

We report from [1] some notions and results on real functions defined on affine spaces.

Definition 12. *Let (Q, V) be a real affine space. The 0-th polarization of a function $f : Q \to \mathbb{R}$ is the function itself. For $n > 0$, the n-th polarization of f is the function*

$$\delta^n f \colon Q \times V^n \to \mathbb{R}$$

defined by

$$\delta^n f(q; v_1, \ldots, v_n) = (-1)^n \sum_{m=0}^{n} (-1)^m \sum_{1 \le i_1 < \cdots < i_m \le n} f(q + v_{i_1} + \cdots + v_{i_m}).$$

The term in the above sum corresponding to $m = 0$ is set equal to $f(q)$. ∎

We introduce the *n-th unidirectional polarization* of a function, defined by

$$\Delta^n f \colon Q \times V \to \mathbb{R} \colon (q; v) \mapsto \delta^n f(q; v, v, \ldots, v).$$

It is the restriction of the n-th polarization to the product of the space Q with the diagonal of V^n.

Definition 13. *A function $f : Q \to \mathbb{R}$ is said to be a homogeneous polynomial of degree n at $q \in Q$ if there is a function $F : Q \times V^n \to \mathbb{R}$, multilinear at q in its vectorial arguments, such that, for each $v \in V$,*

$$f(q + v) = \frac{1}{n!} F(q; \underbrace{v, \ldots, v}_{n \text{ times}}).$$ ∎

Polynomials are sums of homogeneous polynomials. The degree of a polynomial is the highest degree of its non zero homogeneous components.

We denote by $K^n(Q, q)$ the vector space of homogeneous polynomial functions of degree n at q.

Proposition 14. *If $f \in K^n(Q, q)$, then the polarization $\delta^n f$ is multilinear at q and for each $v \in V$,*

$$\Delta^n f(q; v) = n! f(q + v).$$ ∎

Definition 15. *The limit*

$$d^n f(q; v_1, v_2, \ldots, v_n) = \lim_{s \to 0} \frac{1}{s^n} \delta^n f(q; sv_1, sv_2, \ldots, sv_n),$$

if it exists, is called the n-th multidirectional derivative of f at the point $q \in Q$ in the multidirection $(v_1, v_2, \ldots, v_n) \in V^n$. ∎

The *n-th directional derivative* at q in the direction v is the restriction

$$D^n f(q; v) = d^n f(q; v, v, \ldots, v)$$

of the multidirectional derivative to Q times the diagonal of V^n.

Proposition 16. *If $f \in C^\infty(Q)$ then for each $i \in N$ and each $q_0 \in Q$,*

$$D^i f(q_0, \cdot - q_0) \in K^i(Q, q_0).$$ ∎

Theorem (Modified Taylor's theorem) 17. *If $f \in C^n(Q)$, then for each $q, q_0 \in Q$*

$$f(q) = \sum_{k=0}^{n} \frac{1}{k!} D^k f(q_0; q - q_0) + r(q, q_0) \tag{23}$$

where r satisfies

$$r(q,q) = 0 \quad \textit{and} \quad \lim_{(q,q_0)\to(\bar{q},\bar{q})} \frac{r(q,q_0)}{\|q - q_0\|^n} = 0$$

for each $\bar{q}$. ∎

Note 18.

In the formula (23) the function $\sum_{k=0}^{n} \frac{1}{k!} D^k f(q_0; \cdot - q_0)$ is the Taylor polynomial of f of order n at q' and the function $r(\cdot, q')$ is the remainder of f of order n at q_0. The latter can be given the usual form of a Lagrange remainder, i.e., the product of a homogeneous polynomial of order $n+1$ at q_0 and a suitable function belonging to $C^n(Q)$.

References

[1] M. Barile, F. Barone and W.M. Tulczyjew, *Polarizations and Differential Calculus in Affine Spaces*, to appear in Linear Multilinear Algebra.

[2] E. Dubuc, *Sur les modèles de la géométrie différentielle synthétique*, Cah. Topol. Géom. Différ. Catég., **XX**, 234–278 (1979).

[3] E. Dubuc and J. Zilber, *Weil Prolongations of Banach Manifolds in an Analytic Model of SDG*, Cah. Topol. Géom. Différ. Catég., **XLVI**, 83–98 (2005).

[4] D.J. Eck, *Product–Preserving Functors on Smooth Manifolds*, J. Pure Appl. Algebra, **42**, 133–140 (1986).

[5] C. Ehresmann, *Les prolongements d'une variété différentiable*, C.R. Acad. Sci., **233**, 598–600, 777–779 (1951).

[6] J. Gancarzewicz, W. Mikulski and Z. Pogoda, *Lifts of Some Tensor Fields and Connections to Product Preserving Functors*, Nagoya Math. J. **135**, 1–41 (1994).

[7] G. Kainz and P.W. Michor, *Natural Transformations in Differential Geometry*, Czech. Math. J., **37** (112), 584–607 (1987).

[8] I. Kolář, *On Natural Operators on Vector Fields*, Ann. Global Anal. Geom., **6**, 109–117 (1988).

[9] I. Kolář, P.W. Michor and J. Slovák, *Natural Operations in Differential Geometry*, Springer (1993).

[10] O.O. Luciano, *Categories of Multiplicative Functors and Weil's Infinitely Near Points*, Nagoya Math. J., **109**, 69–89 (1988).

[11] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press (1989).

[12] B. Mitchell, *Theory of Categories*, Academic Press, New York and London (1965).

[13] A. Morimoto, *Prolongations of Connections to Tangential Fibre Bundles of Higher Order*, Nagoya Math. J. **40**, 85–97 (1970).

[14] A. Morimoto, *Prolongations of G-Structure to Tangent Bundles of Higher Order*, Nagoya Math. J. **38**, 153–179 (1970).

[15] J. Muñoz, F.J. Muriel and J. Rodriguez, *The Contact System on the* (m, ℓ)*–Jet Spaces*, Arch. Math. Brno **37**, 291–300 (2001).

[16] J. Muñoz, F.J. Muriel and J. Rodriguez, *On the Finiteness of Differential Invariants*, J. Math. Anal. Appl., **284**, 266–282 (2003).

[17] A. Weil, *Théorie des points proches sur les variétés différentiables*, Colloques internat. Centre nat. Rech. Sci. **52**, 111–117 (1953).

Received: 16.10.2006.

[1]Università di Bari, Dipartimento di Matematica,
Via Orabona, 4
70125 Bari, Italy
E-mail: barile@dm.uniba.it

[2]Università di Bari, Dipartimento di Matematica,
Via Orabona, 4
70125 Bari, Italy
E-mail: barone@dm.uniba.it

[3]Università di Camerino, Dipartimento di Fisica,
Camerino, Italy
E-mail: tulczy@libero.it