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# On the boundedness of solutions of certain nonlinear vector differential equations of third order

by
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**Abstract**

This paper is concerned with the boundedness of all solutions of nonlinear vector differential equations of the form:

$$\dddot{X} + A(t)F(\ddot{X}) + B(t)G(\dot{X}) + C(t)H(X) = 0.$$

The Lyapunov's second (or direct) method is used as a basic tool in obtaining the criteria for the boundedness of all solutions of the equation.

**Key Words**: Boundedness, Lyapunov's second method, differential equations of third order.
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## 1 Introduction

In the recent years there have been intensive studies on the qualitative behavior of solutions of certain scalar differential equations of third order; see for instance [2-12], [14], [15], [17-25] and the references cited therein for some works on the subject. In this connection, Swick [18] investigated the asymptotic behavior of solutions of the differential equation

$$\dddot{x} + a(t)\ddot{x} + b(t)g(\dot{x}) + h(x) = e(t). \tag{1.1}$$

Hara [11] studied the uniform ultimate boundedness of the solutions of the nonlinear differential equations

$$\dddot{x} + a(t)f(x,\dot{x})\ddot{x} + b(t)g(x,\dot{x}) + c(t)h(x) = p(t) \tag{1.2}$$

and

$$\dddot{x} + a(t)f(x,\dot{x})\ddot{x} + b(t)g(x,\dot{x}) + c(t)h(x) = p(t,x,\dot{x},\ddot{x}). \tag{1.3}$$

In 1999, Mehri&Shadman [12] discussed the boundedness of the solutions of the scalar differential equation

$$\dddot{x} + a(t)f(\ddot{x}) + b(t)g(\dot{x}) + c(t)h(x) = e(t). \tag{1.4}$$

However, for the case $n = 1$, not much is known about the boundedness of solutions of certain nonlinear differential equations of the form (1.1)-(1.4), (namely, when $a(t) \neq 1, b(t) \neq 1$ and $c(t) \neq 1$ in (1.1)-(1.4)). It is worth mentioning that the author of this paper (see [22], [23]), more recently, established some similar results on the same topic for the third order nonlinear scalar differential equations as follows:

$$\dddot{x} + a(t)f(x, \dot{x}, \ddot{x})\,\ddot{x} + b(t)g(x, \dot{x}) + c(t)h(x) = p(t)$$

and

$$\dddot{x} + a(t)f(x, \dot{x}, \ddot{x})\,\ddot{x} + b(t)g(x, \dot{x}) + c(t)h(x) = p(t, x, \dot{x}, \ddot{x}),$$

and

$$\dddot{x} + \psi(x, \dot{x}, \ddot{x})\,\ddot{x} + f(x, \dot{x}) = p(t, x, \dot{x}, \ddot{x}),$$

respectively.

This paper is interested in the boundedness of all solutions of the third-order nonlinear vector differential equations of the form:

$$\dddot{X} + A(t)F(\ddot{X}) + B(t)G(\dot{X}) + C(t)H(X) = 0, \tag{1.5}$$

in which $t \in \mathbb{R}^+$, $\mathbb{R}^+ = (0, \infty)$ and $X \in \mathbb{R}^n$; $A, B$ and $C$ are continuous $n \times n$ -symmetric matrices; $F : \mathbb{R}^n \to \mathbb{R}^n, G : \mathbb{R}^n \to \mathbb{R}^n$, $H : \mathbb{R}^n \to \mathbb{R}^n$ and $F(0) = G(0) = H(0) = 0$. It is supposed that the functions $F, G$ and $H$ are continuous. Let $J_F(\ddot{X}), J_G(\dot{X}), J_H(X)$ and $\dot{B}(t)$ denote the Jacobian matrices corresponding to $F(\ddot{X}), G(\dot{X}), H(X)$ and $B(t)$,respectively, that is,

$$\begin{aligned} J_F(\ddot{X}) &= \left(\frac{\partial f_i}{\partial \ddot{x}_j}\right), J_G(\dot{X}) = \left(\frac{\partial g_i}{\partial \dot{x}_j}\right), J_H(X) = \left(\frac{\partial h_i}{\partial x_j}\right), \\ \dot{B}(t) &= \frac{d}{dt}\left(b_{ij}(t)\right), (i, j = 1, 2, ..., n), \end{aligned}$$

where $(x_1, x_2, ..., x_n), (\dot{x}_1, \dot{x}_2, ..., \dot{x}_n), (\ddot{x}_1, \ddot{x}_2, ..., \ddot{x}_n), (f_1, f_2, ..., f_n), (g_1, g_2, ..., g_n)$, $(h_1, h_2, ..., h_n)$ and $(b_{ij}(t))$ are components of $X$, $\dot{X}$, $\ddot{X}$, $F$, $G$, $H$ and $B$, respectively. Other than these, it is also assumed, as basic throughout what follows, that the Jacobian matrices $J_F(\ddot{X}), J_G(\dot{X}), J_H(X)$ and $\dot{B}(t)$ exist and are symmetric and continuous, and that all matrices given in the pairs $A(t), J_F(\ddot{X}); B(t), J_G(\dot{X}); C(t), J_H(X)$ and $\dot{B}(t), J_G(\dot{X})$ commute with each others. Equation (1.5) represents a system of real third-order differential equations of the form:

$$\dddot{x}_i + \sum_{k=1}^{n} a_{ik}(t)f_k(\ddot{x}) + \sum_{k=1}^{n} b_{ik}(t)g_k(\dot{x}) + \sum_{k=1}^{n} c_{ik}(t)h_k(x) = 0, (i = 1, 2, ..., n).$$

We consider through in what follows, in place of equation (1.5), the equivalent differential system:

$$\dot{X} = Y, \dot{Y} = Z,$$
$$\dot{Z} = -A(t)F(Z) - B(t)G(Y) - C(t)H(X) \tag{1.6}$$

was obtained as usual by setting $\dot{X} = Y, \ddot{X} = Z$ in (1.5).

The symbol $\langle X, Y\rangle$ will be used to denote the usual scalar product in $\mathbb{R}^n$ for given $X, Y$ in $\mathbb{R}^n$, that is, $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$; thus $\langle X, X\rangle = \|X\|^2$. It should be noted that the matrix $A$ is said to be negative-definite, when $\langle AX, X\rangle < 0$ for all non-zero $X$ in $\mathbb{R}^n$, and $\lambda_i(A)$, $(i = 1, 2, ..., n)$, are eigenvalues of the $n \times n$-matrix $A$.

The motivation for the present study has come from the paper of Mehri and Shadman [12], especially, and the papers mentioned above. Our aim is to extend and improve the first result established in [12]. It should also be noted that, to the best of our knowledge, there is not found any research on the boundedness of solutions of certain nonlinear vector differential equations of the form (1.5) or more general form of that equation, in the relevant literature.The present work is the first attempt to obtain sufficient conditions for the boundedness of solutions of certain nonlinear vector differential equations of the form (1.5).

## 2 The main result

The main result of this paper is the following theorem.

**Theorem :** In addition to the basic assumptions imposed on $A, B, C, F, G$ and $H$ that appeared in (1.6), we assume that the following conditions are satisfied:

(i) $\lambda_i(A(t)) \geq 0$, $\lambda_i(B(t)) \geq 0$, $\lambda_i(C(t)) \geq c_0(t) \geq 0$, and $\lambda_i(\dot{B}(t)) \leq 0$ for all $t \in \mathbb{R}^+$.

(ii) $\lambda_i(J_F(Z)) \geq 0$ for all $Z \in \mathbb{R}^n$.

(iii) $\lambda_i(J_G(Y)) \geq 0$ for all $Y \in \mathbb{R}^n$.

(iv) $|\lambda_i(J_H(X))| \leq K$ for all $X \in \mathbb{R}^n$, where $K$ is a positive constant.

(v) There are arbitrary continuous functions $\alpha_0, \alpha_1$ and $\beta$ on $\mathbb{R}^+ = (0, \infty)$ such that $\alpha_0$ and $\alpha_1$ are positive and decreasing and $\beta$ is positive and increasing for all $t \in \mathbb{R}^+$, $\mathbb{R}^+ = (0, \infty)$ and

$$\left(\frac{\alpha_0}{\alpha_1}\right)^{\frac{1}{2}}, \left(\frac{\alpha_1}{\beta}\right)^{\frac{1}{2}}, c_0\left(\frac{\beta}{\alpha_0}\right)^{\frac{1}{2}} \in L^1(0, \infty),$$

where $L^1(0, \infty)$ is space of integrable functions Lebesque.

Then, every solution $(X(t), Y(t), Z(t))$ of system (1.6) is bounded for all $t \in \mathbb{R}^+$.

Now, we dispose of some well-known algebraic results which will be required in the proof. The first of these is quite standard one:

**Lemma 1:** Let $D$ be a real symmetric $n \times n-$ matrix. Then for any $X \in \mathbb{R}^n$

$$\delta_d \|X\|^2 \leq \langle DX, X \rangle \leq \Delta_d \|X\|^2 ,$$

where $\delta_d$ and $\Delta_d$ are the least and greatest eigenvalues of $D$, respectively.

**Proof:** See ([13]).

Secondly, we require the following lemma.

**Lemma 2.** Let $Q, D$ be any two real $n \times n-$ commuting symmetric matrices. Then,

(i) The eigenvalues $\lambda_i(QD), (i = 1, 2, ..., n)$, of the product matrix $QD$ are real and satisfy

$$\max_{1 \leq j,k \leq n} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1 \leq j,k \leq n} \lambda_j(Q)\lambda_k(D).$$

(ii) The eigenvalues $\lambda_i(Q + D), (i = 1, 2, ..., n)$, of the sum of matrices $Q$ and $D$ are real and satisfy

$$\left\{ \max_{1 \leq j \leq n} \lambda_j(Q) + \max_{1 \leq k \leq n} \lambda_k(D) \right\} \geq \lambda_i(Q + D) \geq \left\{ \min_{1 \leq j \leq n} \lambda_j(Q) + \min_{1 \leq k \leq n} \lambda_k(D) \right\},$$

where $\lambda_j(Q)$ and $\lambda_k(D)$ are, respectively, the eigenvalues of matices $Q$ and $D$ .

**Proof:** See ([1]).

In the proof of the theorem, our main tool is the continuous differentiable Lyapunov function $V = V(t, X, Y, Z)$ defined by:

$$V := \frac{\alpha_0(t)}{\beta(t)} \langle X, X \rangle + \frac{\alpha_1(t)}{\beta(t)} \langle Y, Y \rangle + \langle Z, Z \rangle + 2 \int_0^1 \langle B(t)G(\sigma Y), Y \rangle \, d\sigma. \qquad (2.1)$$

**Proof of the theorem:** By considering $G(0) = 0$, it follows that $V(0, 0, 0, 0) = 0$. Further, since $G(0) = 0$ and $\frac{\partial}{\partial \sigma} G(\sigma Y) = J_G(\sigma Y)Y$, then

$$G(Y) = \int_0^1 J_G(\sigma Y) Y \, d\sigma.$$

Hence, by using assumptions (i) and (iii) of the theorem and Lemma 1 and Lemma 2, it follows that

$$\int_0^1 \langle B(t)G(\sigma Y), Y\rangle \, d\sigma = \int_0^1 \int_0^1 \langle \sigma_1 B(t) J_G(\sigma_1\sigma_2 Y)Y, Y\rangle \, d\sigma_1 d\sigma_2 \geq 0. \tag{2.2}$$

In view of the positive definiteness of the function $\alpha_0, \alpha_1$ and $\beta, V(0,0,0,0) = 0$ and the inequality (2.2), then it is clear that the function $V$ defined by (2.1) is positive definite. Now, let $(X(t), Y(t), Z(t))$ be any solution of the differential system (1.6) and the function $v = v(t)$ be defined by $v(t) = V\left(t, X(t), Y(t), Z(t)\right).$ An easy calculation from (2.1) and (1.6) shows that

$$\begin{aligned}
\dot{v}\,(t) \equiv \tfrac{d}{dt} V(t,X,Y,Z) &= \left(\tfrac{\alpha_0'(t)}{\beta(t)} - \tfrac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\right)\langle X, X\rangle + \tfrac{2\alpha_0(t)}{\beta(t)}\langle X, Y\rangle \\
&+ \left(\tfrac{\alpha_1'(t)}{\beta(t)} - \tfrac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\right)\langle Y, Y\rangle + \tfrac{2\alpha_1(t)}{\beta(t)}\langle Y, Z\rangle \\
&- 2\langle A(t)F(Z), Z\rangle - 2\langle B(t)G(Y), Z\rangle \\
&- 2\langle C(t)H(X), Z\rangle + 2\tfrac{d}{dt}\int_0^1 \langle B(t)G(\sigma Y), Y\rangle \, d\sigma.
\end{aligned} \tag{2.3}$$

Recall that

$$\begin{aligned}
\tfrac{d}{dt}\int_0^1 \langle B(t)G(\sigma Y), Y\rangle \, d\sigma &= \int_0^1 \left\langle \dot{B}\,(t)G(\sigma Y), Y\right\rangle d\sigma \\
&+ \int_0^1 \sigma \langle B(t)J_G(\sigma Y)Z, Y\rangle \, d\sigma + \int_0^1 \langle B(t)G(\sigma Y), Z\rangle \, d\sigma \\
&= \int_0^1 \left\langle \dot{B}\,(t)G(\sigma Y), Y\right\rangle d\sigma + \int_0^1 \sigma \tfrac{\partial}{\partial\sigma}\langle B(t)G(\sigma Y), Z\rangle \, d\sigma \\
&+ \int_0^1 \langle B(t)G(\sigma Y), Z\rangle \, d\sigma \\
&= \int_0^1 \left\langle \dot{B}\,(t)G(\sigma Y), Y\right\rangle d\sigma + \sigma\langle B(t)G(\sigma Y), Z\rangle \Big|_0^1 \\
&= \int_0^1 \left\langle \dot{B}\,(t)G(\sigma Y), Y\right\rangle d\sigma + \langle B(t)G(Y), Z\rangle.
\end{aligned} \tag{2.4}$$

On substituting the estimate (2.4) into (2.3) we obtain

$$\begin{aligned}\dot{v} = &\left(\tfrac{\alpha_0'(t)}{\beta(t)} - \tfrac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\right)\langle X, X\rangle + \tfrac{2\alpha_0(t)}{\beta(t)}\langle X, Y\rangle \\ &+ \left(\tfrac{\alpha_1'(t)}{\beta(t)} - \tfrac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\right)\langle Y, Y\rangle + \tfrac{2\alpha_1(t)}{\beta(t)}\langle Y, Z\rangle \\ &-2\langle A(t)F(Z), Z\rangle - 2\langle C(t)H(X), Z\rangle \\ &+2\int_0^1 \left\langle \dot{B}\,(t)G(\sigma Y), Y\right\rangle d\sigma.\end{aligned}$$

Now, clearly, the assumptions imposed on the functions $\alpha_0, \alpha_1$ and $\beta$ show that

$$\left(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\right)\langle X, X\rangle < 0$$

and

$$\left(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\right)\langle Y, Y\rangle < 0.$$

Hence

$$\begin{aligned}\dot{v} \leq &\tfrac{2\alpha_0(t)}{\beta(t)}\langle X, Y\rangle + \tfrac{2\alpha_1(t)}{\beta(t)}\langle Y, Z\rangle \\ &-2\langle A(t)F(Z), Z\rangle - 2\langle C(t)H(X), Z\rangle \\ &+2\int_0^1 \left\langle \dot{B}\,(t)G(\sigma Y), Y\right\rangle d\sigma.\end{aligned} \tag{2.5}$$

Making the use of the assumptions of the theorem, (2.1) and the inequality $2\|X\|\|Y\| \leq \|X\|^2 + \|Y\|^2$, it follows that

$$\frac{2\alpha_0(t)}{\beta(t)}\langle X, Y\rangle \leq \frac{2\alpha_0(t)}{\beta(t)}\|X\|\|Y\| \leq \left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{1/2} v, \tag{2.6}$$

$$\frac{2\alpha_1(t)}{\beta(t)}\langle Y, Z\rangle \leq \frac{2\alpha_1(t)}{\beta(t)}\|Y\|\|Z\| \leq \left(\frac{\alpha_1(t)}{\beta(t)}\right)^{1/2} v. \tag{2.7}$$

Since

$$F(0) = 0, \frac{\partial}{\partial\sigma}F(\sigma Z) = J_F(\sigma Z)Z$$

and

$$H(0) = 0, \frac{\partial}{\partial\sigma}H(\sigma X) = J_H(\sigma X)X,$$

then

$$F(Z) = \int_0^1 J_F(\sigma Z)Z d\sigma, H(X) = \int_0^1 J_H(\sigma X)X d\sigma \text{ and } G(Y) = \int_0^1 J_G(\sigma Y)Y d\sigma. \tag{2.8}$$

Therefore, by noting assumptions (i), (ii), (iv) of the theorem, the expressions in (2.8) and Lemma 1 and Lemma 2, we get that

$$-2\left\langle A(t)F(Z),Z\right\rangle=-2\int_0^1\left\langle A(t)J_F(\sigma Z)Z,Z\right\rangle d\sigma\le 0, \tag{2.9}$$

$$\begin{aligned}-2\left\langle C(t)H(X),Z\right\rangle= & -2\int_0^1\left\langle C(t)J_H(\sigma X)X,Z\right\rangle d\sigma\le 2c_0(t)K\left\|X\right\|\left\|Z\right\| \\ & \le Kc_0(t)\left(\tfrac{\beta(t)}{\alpha_0(t)}\right)^{1/2}v,\end{aligned} \tag{2.10}$$

$$\int_0^1\left\langle \dot{B}\,(t)G(\sigma Y),Y\right\rangle d\sigma=\int_0^1\int_0^1\left\langle \sigma_1\ \dot{B}\,(t)J_G(\sigma_1\sigma_2 Y)Y,Y\right\rangle d\sigma_2 d\sigma_1\le 0. \tag{2.11}$$

On gathering the estimates (2.6)-(2.7), (2.9)-(2.11) in (2.5) we obtain

$$\dot{v}\le\left[\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{1/2}+\left(\frac{\alpha_1(t)}{\beta(t)}\right)^{1/2}+Kc_0(t)\left(\frac{\beta(t)}{\alpha_0(t)}\right)^{1/2}\right]v. \tag{2.12}$$

Let

$$\phi(t)=\left[\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{1/2}+\left(\frac{\alpha_1(t)}{\beta(t)}\right)^{1/2}+Kc_0(t)\left(\frac{\beta(t)}{\alpha_0(t)}\right)^{1/2}\right]. \tag{2.13}$$

Then

$$\dot{v}\le\phi(t)v(t), \tag{2.14}$$

by (2.12) and (2.13).

Integrating inequality (2.14) from 0 to $t$, we obtain that

$$v(t)-v(0)\le\int_0^t v(s)\phi(s)ds.$$

By using the assumption (v) of the theorem and Gronwall-Reid-Bellman inequality, (see Rao [16]), we finally conclude that

$$v(t)\le D\exp\left(\int_0^t\phi(s)ds\right),$$

where $D=v(0)$. Assumption (v) of the theorem, that is, $\phi\in L^1(0,\infty)$ implies the boundedness of the function $v=v(t)$. This completes the proof of the theorem.

**Remark:** It should be noted that as mentioned above Mehri&Shadman [12] considered the scalar ordinary differential equation of the form

$$\dddot{x} + a(t)f(\ddot{x}) + b(t)g(\dot{x}) + c(t)h(x) = e(t).$$

However, the equation considered here,

$$\dddot{X} + A(t)F(\ddot{X}) + B(t)G(\dot{X}) + C(t)H(X) = 0$$

is an $n$-dimensional extension of the above equation (for the case $e(t) = 0$). Next, for the case $n = 1$, our assumptions reduce those established by Mehri&Shadman [12, Theorem 1] except some minor modifications. The case arises because of the Lyapunov's function used here as basic tool,

$$V := \frac{\alpha_0(t)}{\beta(t)} \langle X, X\rangle + \frac{\alpha_1(t)}{\beta(t)} \langle Y, Y\rangle + \langle Z, Z\rangle + 2 \int_0^1 \langle B(t)G(\sigma Y), Y\rangle \, d\sigma$$

which is different than that used in Mehri&Shadman [12], that is,

$$E := \frac{\alpha_0(t)}{\beta(t)} x^2 + \frac{\alpha_1(t)}{\beta(t)} y^2 + \frac{1}{b(t)} z^2 + 2 \int_0^y g(\tau) d\tau.$$

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