Bull. Math. Soc. Sci. Math. Roumanie
Tome 49(97) No. 3, 2006, 285–290

A Refinement of Popoviciu's Inequality

by
Constantin P. Niculescu and Florin Popovici

To the memory of Tiberiu Popoviciu (*1906-1975*)

Abstract

We give a new proof to Popoviciu's inequality, which yields a better result. Some applications are included.

Key Words: Convex function, right derivative, Leibniz-Newton formula
2000 Mathematics Subject Classification: Primary 26A51, Secondary 26D05.

Proving inequalities is rarely an easy task, but under certain circumstances some algorithms are available. For example, most inequalities for convex functions of one variable can be reduced to the case of continuous convex functions on compact intervals, and then to the case of affine functions and of translates of the absolute value function. In fact, any continuous convex function $f : [a, b] \to \mathbb{R}$ is the uniform limit of a sequence of functions of the form

$$f_n(x) = \lambda x + \mu + \sum_{k=0}^{N(n)} c_k |x - x_k|$$

where $\lambda, \mu \in \mathbb{R}$ and $c_k \geq 0$, $x_k \in [a, b]$ for $k = 0, ..., N(n)$. This nice result on the structure of convex functions was discovered by T. Popoviciu [7], who used it in a subsequent paper to prove the following characterization of continuous convex functions:

Theorem 1. (Popoviciu's inequality *[8]*). *Let f be a real-valued continuous function on an interval I. Then f is convex if and only if*

$$\frac{f(x_1) + f(x_2) + f(x_3)}{3} + f\left(\frac{x_1 + x_2 + x_3}{3}\right) \geq$$
$$\geq \frac{2}{3}\left[f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right) + f\left(\frac{x_3 + x_1}{2}\right)\right]$$

for all $x_1, x_2, x_3 \in I$.

In the variant of strictly convex functions this inequality is strict except for $x_1 = x_2 = x_3$.

According to the aforementioned result on the structure of convex functions, the necessity part of Popoviciu's inequality needs an argument only in the case of functions of the form $|x - x_0|$. It turns out that this is equivalent to the one-dimensional case of Hlawka's inequality (see [4], p. 171), that is,

$$|x_1| + |x_2| + |x_3| + |x_1 + x_2 + x_3| \geq |x_1 + x_2| + |x_2 + x_3| + |x_3 + x_1|,$$

for all $x_1, x_2, x_3 \in \mathbb{R}$.

The special structure of convex functions has many others spectacular consequences. For example, see L. Hörmander [3], p. 25.

The purpose of the present note is to give a new proof of Theorem 1, which yields a better inequality. It also covers the more general case of unequal weights, which can be easily established by adapting the above argument for Theorem 1. See [6], pp. 171-172.

The weighted form of Popoviciu's inequality was first noticed by J. C. Burkill [1], who used a different technique.

We start the proof of Theorem 1 by establishing the sufficiency part. Here the assumption on continuity is essential.

In fact, the Popoviciu inequality yields (for $x_1 = x$ and $x_2 = x_3 = y$) the following substitute for the classical condition of midpoint convexity:

$$\frac{1}{4} f(x) + \frac{3}{4} f\left(\frac{x + 2y}{3}\right) \geq f\left(\frac{x+y}{2}\right) \quad \text{for all } x, y \in I. \tag{Pop}$$

If f is not convex, then would exist a subinterval $[a, b] \subset I$ such that the graph of $f|_{[a,b]}$ is not under the chord joining $(a, f(a))$ and $(b, f(b))$. Equivalently, the function

$$\varphi(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a), \quad x \in [a, b]$$

verifies $\gamma = \sup\{\varphi(x) \mid x \in [a, b]\} > 0$. Notice that φ is continuous and $\varphi(a) = \varphi(b) = 0$. Also, an easy computation shows that φ also verifies (Pop). Put $c = \inf\{x \mid \varphi(x) = \gamma\}$; then necessarily $\varphi(c) = \gamma$ and $c \in (a, b)$. By the definition of c, if $h > 0$ is small enough such that the points $x = c - h$ and $y = c + h$ belong to (a, b), then

$$\varphi(c - h) < \varphi(c) \text{ and } \varphi(c + h/3) \leq \varphi(c),$$

which yields

$$\varphi(c) > \frac{1}{4}\varphi(c - h) + \frac{3}{4}\varphi\left(c + \frac{h}{3}\right)$$

in contradiction with (Pop).

The necessity part is an immediate consequence of the following result, which represents a refinement of the aforementioned results due to Popoviciu and Burkill:

Lemma 1. *Let $f : I \to \mathbb{R}$ be a convex function, let $x_1 < x_2 < x_3$ be points in the interval I, and let $\lambda_1, \lambda_2, \lambda_3$ be positive numbers such that $\sum_{i=1}^{3} \lambda_i = 1$. Then*

$$\sum_{k=1}^{3} \lambda_k f(x_k) + f\left(\sum_{k=1}^{3} \lambda_k x_k\right) - \sum_{1 \le j < k \le 3} (\lambda_j + \lambda_k) f\left(\frac{\lambda_j x_j + \lambda_k x_k}{\lambda_j + \lambda_k}\right)$$

is greater than or equal to

$$\begin{cases} \lambda_3 f(x_3) + \lambda_2 f(\frac{\lambda_1 x_1 + \lambda_3 x_3}{\lambda_1 + \lambda_3}) - (\lambda_2 + \lambda_3) f(\frac{\lambda_1 \lambda_2 x_1 + \lambda_3 x_3}{\lambda_1 \lambda_2 + \lambda_3}), & \text{if } x_2 \in [x_1, \frac{\lambda_1 x_1 + \lambda_3 x_3}{\lambda_1 + \lambda_3}] \\ \lambda_1 f(x_1) + \lambda_2 f(\frac{\lambda_1 x_1 + \lambda_3 x_3}{\lambda_1 + \lambda_3}) - (\lambda_1 + \lambda_2) f(\frac{\lambda_1 x_1 + \lambda_2 \lambda_3 x_3}{\lambda_1 + \lambda_2 \lambda_3}), & \text{if } x_2 \in [\frac{\lambda_1 x_1 + \lambda_3 x_3}{\lambda_1 + \lambda_3}, x_3]. \end{cases}$$

Notice that this lower bound is nonnegative (due to the convexity of f).

Proof: Clearly, we may assume that x_1 and x_3 are interior points of I. In this case f is absolutely continuous on $[x_1, x_3]$. See [9], Theorem A, p. 4. The derivative of f is Lebesgue integrable and well defined except for an enumerable subset of points.

If $x_2 \in [x_1, (\lambda_1 x_1 + \lambda_3 x_3) / (\lambda_1 + \lambda_3)]$, we consider the function

$$g(t) = \lambda_3 f(x_3) + f(\lambda_1 x_1 + \lambda_2 t + \lambda_3 x_3) - \\ - (\lambda_2 + \lambda_3) f\left(\frac{\lambda_2 t + \lambda_3 x_3}{\lambda_2 + \lambda_3}\right) - (\lambda_1 + \lambda_3) f\left(\frac{\lambda_1 x_1 + \lambda_3 x_3}{\lambda_1 + \lambda_3}\right),$$

for $t \in [x_1, (\lambda_1 x_1 + \lambda_3 x_3) / (\lambda_1 + \lambda_3)]$. Since f is convex, it admits a right derivative at all points of $[x_1, (\lambda_1 x_1 + \lambda_3 x_3) / (\lambda_1 + \lambda_3))$ and moreover, this derivative is nondecreasing. See [9], Theorem B, p. 5. Then

$$g'_+(t) = \lambda_2 f'_+(\lambda_1 x_1 + \lambda_2 t + \lambda_3 x_3) - \lambda_2 f'_+\left(\frac{\lambda_2 t + \lambda_3 x_3}{\lambda_2 + \lambda_3}\right),$$

is ≤ 0 on $[x_1, (\lambda_1 x_1 + \lambda_3 x_3) / (\lambda_1 + \lambda_3))$ since

$$\lambda_1 x_1 + \lambda_2 t + \lambda_3 x_3 \le \frac{\lambda_2 t + \lambda_3 x_3}{\lambda_2 + \lambda_3}.$$

According to a well known generalization of the Leibniz-Newton formula (see [2], Exercise 18.41, pp. 298-299), we get

$$g\left(\frac{\lambda_1 x_1 + \lambda_3 x_3}{\lambda_1 + \lambda_3}\right) - g(x_2) = \int_{x_2}^{(\lambda_1 x_1 + \lambda_3 x_3)/(\lambda_1 + \lambda_3)} g'(t) dt \\ = \int_{x_2}^{(\lambda_1 x_1 + \lambda_3 x_3)/(\lambda_1 + \lambda_3)} g'_+(t)\, dt \le 0,$$

that is,

$$g(x_2) \geq \lambda_3 f(x_3) + \lambda_2 f\left(\frac{\lambda_1 x_1 + \lambda_3 x_3}{\lambda_1 + \lambda_3}\right) - (\lambda_2 + \lambda_3) f\left(\frac{\lambda_1 \lambda_2 x_1 + \lambda_3 x_3}{\lambda_1 \lambda_2 + \lambda_3}\right),$$

which yields the conclusion of our lemma for $x_2 \in [x_1, (\lambda_1 x_1 + \lambda_3 x_3) / (\lambda_1 + \lambda_3)]$.

The case where $x_2 \in [(\lambda_1 x_1 + \lambda_3 x_3) / (\lambda_1 + \lambda_3), x_3]$ can be treated in a similar manner. ☐

Corollary 1. *If $f : I \to \mathbb{R}$ is a convex function and $x_1 < x_2 < x_3$ are three points in the interval I, then*

$$\frac{f(x_1) + f(x_2) + f(x_3)}{3} + f\left(\frac{x_1 + x_2 + x_3}{3}\right) \geq$$
$$\geq \frac{2}{3}\left[f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_2 + x_3}{2}\right) + f\left(\frac{x_3 + x_1}{2}\right)\right] +$$
$$+ \begin{cases} \frac{1}{3}\left(f(x_3) + f\left(\frac{x_1+x_3}{2}\right) - 2f\left(\frac{x_1+3x_3}{4}\right)\right), & \text{if } x_2 \in [x_1, (x_1 + x_3)/2] \\ \frac{1}{3}\left(f(x_1) + f\left(\frac{x_1+x_3}{2}\right) - 2f\left(\frac{3x_1+x_3}{4}\right)\right), & \text{if } x_2 \in [(x_1 + x_3)/2, x_3]. \end{cases}$$

In the case of strictly convex functions the above inequality is strict.

Remarks. Clearly, Popoviciu's inequality covers Hlawka's inequality. However, Corollary 1 yields a better result:

$$|x_1| + |x_2| + |x_3| + |x_1 + x_2 + x_3| \geq |x_1 + x_2| + |x_2 + x_3| + |x_3 + x_1| +$$
$$+ \begin{cases} |x_3| + \frac{1}{2}|x_1 + x_3| - \frac{1}{2}|x_1 + 3x_3|, & \text{if } x_2 \in [x_1, (x_1 + x_3)/2] \\ |x_1| + \frac{1}{2}|x_1 + x_3| - \frac{1}{2}|3x_1 + x_3|, & \text{if } x_2 \in [(x_1 + x_3)/2, x_3], \end{cases}$$

for all triplets $x_1 \leq x_2 \leq x_3$.

In turn, this refinement of Hlawka's inequality allows us easily to recover the conclusion of Lemma 1 (in exactly the same way Popoviciu has proved Theorem 1). Moreover, we can built up a direct proof of it based on the order structure of the real line, but is that the natural way to arrive to Lemma 1? Certainly, the matter is delicate. Formulating a mathematical result is considerably more involving than finding an argument once it is known to be true.

We can avoid the dichotomy stated in Theorem 1 by considering a lower bound, the minimum of the two alternatives. This remark, when applied to the exponential function, yields the following inequality:

$$\frac{x_1 + x_2 + x_3}{3} + \sqrt[3]{x_1 x_2 x_3} > \frac{2}{3}\left[\sqrt{x_1 x_2} + \sqrt{x_2 x_3} + \sqrt{x_3 x_1}\right] + \frac{1}{3}\sqrt{x_1}\left(\sqrt[4]{x_3} - \sqrt[4]{x_1}\right)^2,$$

for all positive numbers $x_1 \leq x_2 \leq x_3$ (not all equal). Equivalently,

$$\frac{x_1^{12}+x_2^{12}+x_3^{12}}{3}+x_1^4x_2^4x_3^4 > \frac{2}{3}\left(x_1^6x_2^6+x_2^6x_3^6+x_3^6x_1^6\right)+ \\ +\frac{1}{3}\left[\min_{1\leq k\leq 3} x_k^6\right]\cdot\left[\max_{1\leq j,k\leq 3}(x_j^3-x_k^3)^2\right]$$

for all positive numbers x_1, x_2, x_3 (not all equal).

Last, but not the least, we should mention that Lemma 1 has obvious analogues for the algebraic variants of convexity (such as log-convexity, multiplicative convexity etc.). See [5] for a description of these classes of convex like functions.

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Received: 22.12.2005.

University of Craiova, Department of Mathematics,
Street A. I. Cuza 13,
Craiova 200585, Romania
E-mail: cniculescu@central.ucv.ro

College Nicolae Titulescu,
Braşov 500435, Romania
E-mail: popovici.florin@yahoo.com